

n = 7 verification (Lemma 2.7): $Z7(\alpha,t)=-1$

$$\ln[\ast]:= \omega = \sqrt{\frac{4 - 11 t + 8 t^2}{t}} ;$$

$$\ln[\ast]:= \alpha\alpha = \frac{-13 + 81 t^1 - 169 t^2 + 64 t^3 + 260 t^4 - 416 t^5 + 256 t^6 - 72 t^7 + 8 t^8}{8 (-1 + t)^3 t^2 (-1 + 5 t^1 - 6 t^2 + t^3)} + \frac{(-1 - 7 t^1 + 65 t^2 - 164 t^3 + 180 t^4 - 88 t^5 + 16 t^6) \omega\omega}{8 (-1 + t)^3 t^2 (-1 + 5 t^1 - 6 t^2 + t^3)} ;$$

$$\ln[\ast]:= \mathbf{q} = (2 t - 1)^{\wedge} 10 ;$$

$$p61 = 16 t^{\wedge} 5 (-10 + 27 t + 288 t^2 - 2022 t^3 +$$

$$5825 t^4 - 9477 t^5 + 9336 t^6 - 5494 t^7 + 1712 t^8 - 122 t^9 - 80 t^{10} + 16 t^{11});$$

$$p62 = (-16 t^{\wedge} 6) (-25 + 206 t - 726 t^2 + 1459 t^3 - 1927 t^4 + 1870 t^5 - 1394 t^6 + 724 t^7 - 210 t^8 + 24 t^9);$$

$$p41 = (-4 t^{\wedge} 3) (50 - 687 t + 4203 t^2 - 15695 t^3 + 41282 t^4 - 82204 t^5 + 125852 t^6 - 144150 t^7 + 117216 t^8 - 62520 t^9 + 18496 t^{10} - 1080 t^{11} - 960 t^{12} + 192 t^{13});$$

$$p42 = 4 t^{\wedge} 3 (10 - 121 t + 677 t^2 - 2521 t^3 + 7132 t^4 - 15412 t^5 + 24800 t^6 - 29742 t^7 + 27184 t^8 - 18840 t^9 + 9072 t^{10} - 2520 t^{11} + 288 t^{12});$$

$$p21 = (-2 + 40 t - 360 t^2 + 2245 t^3 - 11098 t^4 + 42924 t^5 - 126314 t^6 + 282304 t^7 - 483968 t^8 + 638896 t^9 - 640864 t^{10} + 470400 t^{11} - 234144 t^{12} + 65792 t^{13} - 2784 t^{14} - 3840 t^{15} + 768 t^{16});$$

$$p22 = -t^{\wedge} 3 (65 - 774 t + 4344 t^2 - 15694 t^3 + 41224 t^4 - 82096 t^5 + 124784 t^6 - 144768 t^7 + 127840 t^8 - 83808 t^9 + 37824 t^{10} - 10080 t^{11} + 1152 t^{12});$$

$$p01 = (1 - t) (1 - 19 t + 161 t^2 - 924 t^3 + 4066 t^4 - 13822 t^5 + 35840 t^6 - 70584 t^7 + 105400 t^8 - 118168 t^9 + 96704 t^{10} - 54208 t^{11} + 17760 t^{12} - 1440 t^{13} - 1024 t^{14} + 256 t^{15});$$

$$p02 = t^3 (25 - 290 t + 1636 t^2 - 6010 t^3 + 15992 t^4 - 32064 t^5 + 48928 t^6 - 56632 t^7 + 49024 t^8 - 30752 t^9 + 13120 t^{10} - 3360 t^{11} + 384 t^{12});$$

$$p11 = 8 t^{\wedge} 3 (t - 1) (-125 + 4005 t - 60169 t^2 + 570158 t^3 - 3862166 t^4 + 20045945 t^5 - 83223823 t^6 + 284217439 t^7 - 813613881 t^8 + 1978202656 t^9 - 4124892528 t^{10} + 7432364016 t^{11} - 11645707296 t^{12} + 15960414880 t^{13} - 19242695952 t^{14} + 20530254512 t^{15} - 19487643408 t^{16} + 16504985216 t^{17} - 12441863808 t^{18} + 8263236224 t^{19} - 4749495424 t^{20} + 2309017088 t^{21} - 926839040 t^{22} + 300215040 t^{23} - 76503296 t^{24} + 14585856 t^{25} - 1794048 t^{26} + 73728 t^{27} + 8192 t^{28});$$

$$p12 = 8 t^{\wedge} 3 (1 - t) (-25 + 765 t - 11171 t^2 + 105634 t^3 - 736638 t^4 + 4055851 t^5 - 18297419 t^6 + 69029409 t^7 - 220386161 t^8 + 599953820 t^9 - 1399432680 t^{10} + 2804818368 t^{11} - 4835265904 t^{12} + 7165985152 t^{13} - 9113381872 t^{14} + 9916520688 t^{15} - 9197207856 t^{16} + 7238388416 t^{17} - 4811313152 t^{18} + 2688147584 t^{19} - 1256283520 t^{20} + 488377856 t^{21} - 156990208 t^{22} + 41689856 t^{23} - 9309952 t^{24} + 1799168 t^{25} - 280576 t^{26} + 24576 t^{27});$$

$$p31 = 8 t^{\wedge} 3 (t - 1) (-125 + 4005 t - 59369 t^2 + 553998 t^3 - 3730470 t^4 + 19565545 t^5 - 83537119 t^6 + 297885767 t^7 - 901975729 t^8 + 2350136384 t^9 - 5333296784 t^{10} + 10652922784 t^{11} - 18880321184 t^{12} +$$

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29 852 879 392 t13 - 42 255 474 560 t14 + 53 630 116 864 t15 -
60 971 031 168 t16 + 61 759 492 352 t17 - 55 128 623 872 t18 + 42 657 332 224 t19 -
28 037 451 008 t20 + 15 314 942 464 t21 - 6 803 466 496 t22 + 2 405 327 616 t23 -
657 957 120 t24 + 131 469 312 t25 - 16 441 344 t26 + 663 552 t27 + 73 728 t28);
p32 = 8 t^3 (1 - t) (-25 + 765 t - 11 171 t^2 + 107 634 t^3 - 787 518 t^4 + 4 665 067 t^5 -
22 973 819 t^6 + 95 297 577 t^7 - 336 259 225 t^8 + 1 016 942 748 t^9 - 2 647 457 192 t10 +
5 940 364 352 t11 - 11 479 425 312 t12 + 19 070 848 736 t13 - 27 171 006 848 t14 +
33 097 766 976 t15 - 34 341 234 112 t16 + 30 212 895 232 t17 - 22 415 990 528 t18 +
13 930 544 128 t19 - 7 190 272 256 t20 + 3 053 491 200 t21 - 1 060 255 488 t22 +
303 176 448 t23 - 73 578 240 t24 + 15 455 232 t25 - 2 525 184 t26 + 221 184 t27);
p51 = 128 t^8 (t - 1) (-1500 + 28 960 t - 257 939 t^2 + 1 404 604 t^3 - 5 211 955 t^4 +
13 795 890 t^5 - 25 855 137 t^6 + 29 614 248 t^7 + 3 063 112 t^8 - 113 012 293 t^9 +
333 031 055 t10 - 652 968 101 t11 + 993 241 608 t12 - 1 221 269 400 t13 +
1 224 164 696 t14 - 994 694 744 t15 + 647 101 024 t16 - 331 790 224 t17 +
131 533 872 t18 - 39 147 024 t19 + 8 229 120 t20 - 1 046 016 t21 + 41 472 t22 + 4608 t23);
p52 = 128 t^8 (1 - t) (-100 + 80 t + 20 039 t^2 - 260 728 t^3 + 1 786 957 t^4 - 8 269 540 t^5 + 28 531 279 t^6 -
77 414 974 t^7 + 170 435 054 t^8 - 309 998 835 t^9 + 470 441 223 t10 - 598 385 907 t11 +
638 090 092 t12 - 567 729 488 t13 + 417 123 064 t14 - 249 429 768 t15 + 119 783 328 t16 -
46 124 976 t17 + 14 587 824 t18 - 3 960 432 t19 + 919 872 t20 - 157 824 t21 + 13 824 t22);
p71 = 2048 t^13 (t - 1) (-25 + 56 t + 2520 t^2 - 25 565 t^3 + 120 032 t^4 - 330 197 t^5 +
533 239 t^6 - 339 915 t^7 - 575 358 t^8 + 1 899 107 t^9 - 2 748 394 t10 + 2 566 194 t11 -
1 675 233 t12 + 781 067 t13 - 257 689 t14 + 57 232 t15 - 7392 t16 + 288 t17 + 32 t18);
p72 = 2048 t^13 (1 - t) (-1 - 50 t + 626 t^2 - 2319 t^3 - 1948 t^4 + 47 711 t^5 -
201 331 t^6 + 485 563 t^7 - 779 658 t^8 + 878 711 t^9 - 712 736 t10 + 425 072 t11 -
193 639 t12 + 72 003 t13 - 23 071 t14 + 6068 t15 - 1096 t16 + 96 t17);

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Two particular checks and a plot

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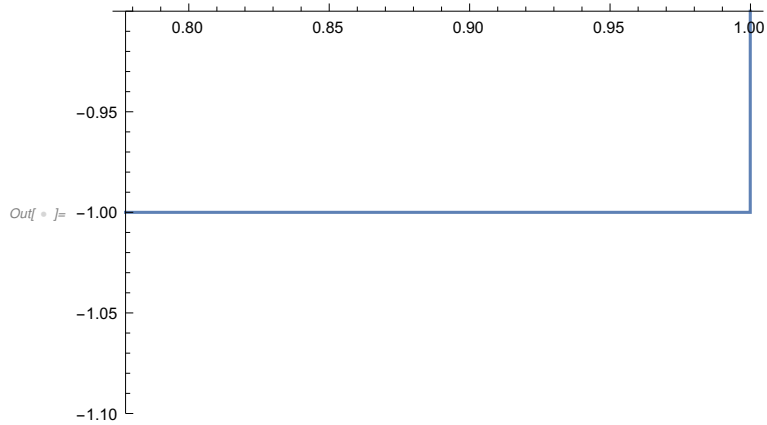
In[ ]:= RootReduce [
  1 / q {Sqrt[p71 + p72 ωω], p61 + p62 ωω, -Sqrt[p51 + p52 ωω], p41 + p42 ωω, Sqrt[p31 + p32 ωω],
  p21 + p22 ωω, -Sqrt[p11 + p12 ωω], p01 + p02 ωω}.Table[Sqrt[A]^j, {j, 7, 0, -1}] /.
  A → αα /. ωω → ω /. {{t → 4/5}, {t → 1/(11 - 4 Sqrt[6])}}]
Out[ ]:= {-1, -1}

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In[ ]:= Plot[
  1 / q {Sqrt[p71 + p72 ωω], p61 + p62 ωω, -Sqrt[p51 + p52 ωω], p41 + p42 ωω, Sqrt[p31 + p32 ωω],
    p21 + p22 ωω, -Sqrt[p11 + p12 ωω], p01 + p02 ωω}.Table[Sqrt[A]^j, {j, 7, 0, -1}] /.
  A → αα /. ωω → ω, {t, 0.777 ..., 1}, PlotRange → {-1.1, -.9}, WorkingPrecision → 30]

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Note that we will need 4 lemmas:

$$\alpha^*b_1=u_1/q, \alpha^3*b_3=u_3/q, \alpha^5*b_5=u_5/q, \alpha^7*b_7=u_7/q$$

u_1, u_3, u_5, u_7 has the form $(P_1+P_2)\omega/Q$, where P_1, P_2, Q are suitable polynomials of t .

$$\text{In[]:= } u1 = (-10 + 198 t - 1742 t^2 + 9029 t^3 - 30986 t^4 + 74600 t^5 - 128398 t^6 + 153928 t^7 - 113584 t^8 + 22480 t^9 + 49912 t^{10} - 57728 t^{11} + 28480 t^{12} - 7808 t^{13} + 2656 t^{14} - 1280 t^{15} + 256 t^{16}) - \omega \omega t (10 - 174 t + 1409 t^2 - 7126 t^3 + 25020 t^4 - 63866 t^5 + 121016 t^6 - 171312 t^7 + 180480 t^8 - 140328 t^9 + 80640 t^{10} - 35392 t^{11} + 12352 t^{12} - 3104 t^{13} + 384 t^{14});$$

$$u3 = \frac{1}{4(-1+t)^3 t^3} (85 t - 1530 t^2 + 11801 t^3 - 45890 t^4 + 41000 t^5 + 573384 t^6 - 3922012 t^7 + 14743612 t^8 - 38993060 t^9 + 77564192 t^{10} - 118920088 t^{11} + 141854664 t^{12} - 132268192 t^{13} + 97062688 t^{14} - 56854208 t^{15} + 27058816 t^{16} - 10397056 t^{17} + 2955904 t^{18} - 459904 t^{19} - 25728 t^{20} + 24576 t^{21} - 3072 t^{22} + t(5 + 26 t - 1583 t^2 + 18062 t^3 - 114412 t^4 + 486044 t^5 - 1488148 t^6 + 3355172 t^7 - 5454140 t^8 + 5774672 t^9 - 2173480 t^{10} - 4792760 t^{11} + 11086592 t^{12} - 12846496 t^{13} + 10198080 t^{14} - 6137984 t^{15} + 2960896 t^{16} - 1140864 t^{17} + 322944 t^{18} - 57216 t^{19} + 4608 t^{20}) \omega \omega);$$

$$u5 = \frac{1}{4(-1+t)^6 t^3} (-250 t + 2987 t^2 + 4045 t^3 - 313967 t^4 + 2996100 t^5 - 16116136 t^6 + 58167348 t^7 - 149013636 t^8 + 272559440 t^9 - 335591112 t^{10} + 197736736 t^{11} + 199225392 t^{12} - 723484336 t^{13} + 1121333232 t^{14} - 1215613648 t^{15} + 1022423712 t^{16} - 691412576 t^{17} + 377059456 t^{18} - 161992864 t^{19} + 52100096 t^{20} - 11373440 t^{21} + 1286912 t^{22} + 47232 t^{23} - 33792 t^{24} + 3072 t^{25} - t(10 + 391 t - 9797 t^2 + 89447 t^3 - 426220 t^4 + 970280 t^5 + 778040 t^6 - 14382620 t^7 + 57858672 t^8 - 144875368 t^9 + 260147248 t^{10} - 352440432 t^{11} + 368727984 t^{12} - 300424464 t^{13} + 188428624 t^{14} - 84962240 t^{15} + 18995168 t^{16} + 8535104 t^{17} - 11947808 t^{18} + 6943232 t^{19} - 2508672 t^{20} + 576000 t^{21} - 77184 t^{22} + 4608 t^{23}) \omega \omega);$$

$$u7 = \frac{1}{4(-1+t)^6 t^3} (-33 t - 16 t^2 + 9292 t^3 - 122404 t^4 + 793766 t^5 - 2977444 t^6 + 5622684 t^7 + 4021212 t^8 - 63726144 t^9 + 227073744 t^{10} - 503694112 t^{11} + 787840912 t^{12} - 888389920 t^{13} + 694326576 t^{14} - 308427104 t^{15} - 34342848 t^{16} + 181724576 t^{17} - 160749120 t^{18} + 84638784 t^{19} - 29409792 t^{20} + 6561152 t^{21} - 788736 t^{22} + 4736 t^{23} + 11264 t^{24} - 1024 t^{25} + t(-1 - 90 t + 1532 t^2 - 7096 t^3 - 28190 t^4 + 520588 t^5 - 3157956 t^6 + 11559500 t^7 - 28309248 t^8 + 46157648 t^9 - 42374448 t^{10} - 6235696 t^{11} + 91494176 t^{12} - 165214832 t^{13} + 179249632 t^{14} - 132624128 t^{15} + 66918944 t^{16} - 20646976 t^{17} + 1733440 t^{18} + 1652608 t^{19} - 877696 t^{20} + 214528 t^{21} - 27776 t^{22} + 1536 t^{23}) \omega \omega);$$

$$\text{In[]:= Together} [\alpha \wedge 1 (p11 + p12 \omega \omega) - u1 \wedge 2 /. \omega \omega \rightarrow \omega]$$

$$\text{Out[]:= } 0$$

$$\text{In[]:= Together} [\alpha \wedge 3 (p31 + p32 \omega \omega) - u3 \wedge 2 /. \omega \omega \rightarrow \omega]$$

$$\text{Out[]:= } 0$$

$$\text{In[]:= Together} [\alpha \wedge 5 (p51 + p52 \omega \omega) - u5 \wedge 2 /. \omega \omega \rightarrow \omega]$$

$$\text{Out[]:= } 0$$

In[*]:=

Together[$\alpha^7 (p71 + p72 \omega\omega) - u7^2 / . \omega\omega \rightarrow \omega$]

Out[*]:= 0

In[*]:= Together[$1/q (u7 + (p61 + p62 \omega\omega) \alpha^3 + u5 +$

$(p41 + p42 \omega\omega) \alpha^2 + u3 + (p21 + p22 \omega\omega) \alpha + u1 + p01 + p02 \omega\omega) / . \omega\omega \rightarrow \omega$]

Out[*]:= -1

In[*]:= Resolve[ForAll[t, $\text{0.777...} < t < 1, \alpha^1 (p11 + p12 \omega\omega) == u1^2 / . \omega\omega \rightarrow \omega$], Reals]

Out[*]:= True

In[*]:= Resolve[ForAll[t, $\text{0.777...} < t < 1, \alpha^3 (p31 + p32 \omega\omega) == u3^2 / . \omega\omega \rightarrow \omega$], Reals]

Out[*]:= True

In[*]:= Resolve[ForAll[t, $\text{0.777...} < t < 1, \alpha^5 (p51 + p52 \omega\omega) == u5^2 / . \omega\omega \rightarrow \omega$], Reals]

Out[*]:= True

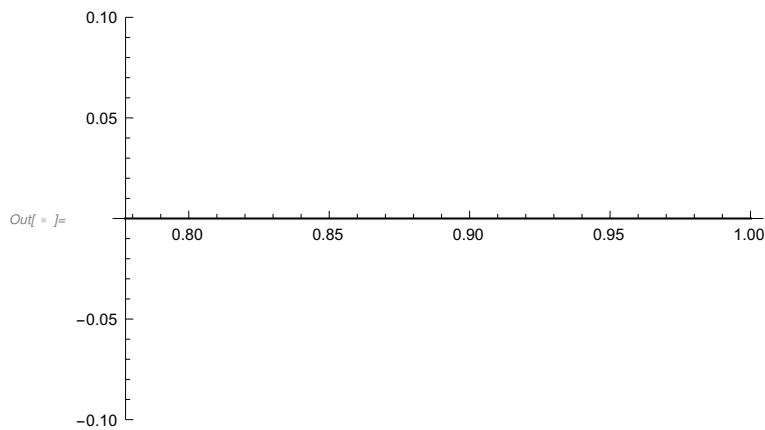
In[*]:= Resolve[ForAll[t, $\text{0.777...} < t < 1, \alpha^7 (p71 + p72 \omega\omega) == u7^2 / . \omega\omega \rightarrow \omega$], Reals]

Out[*]:= True

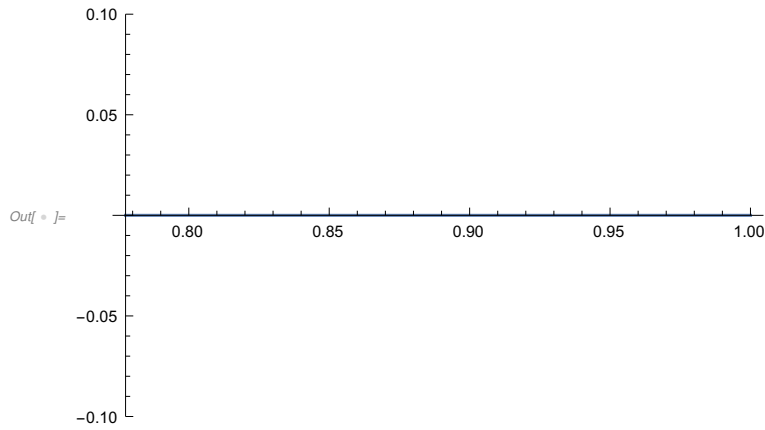
Plots

In[*]:= Plot[-Sqrt[$\alpha^1 (p11 + p12 \omega\omega) - u1 / . \omega\omega \rightarrow \omega$,

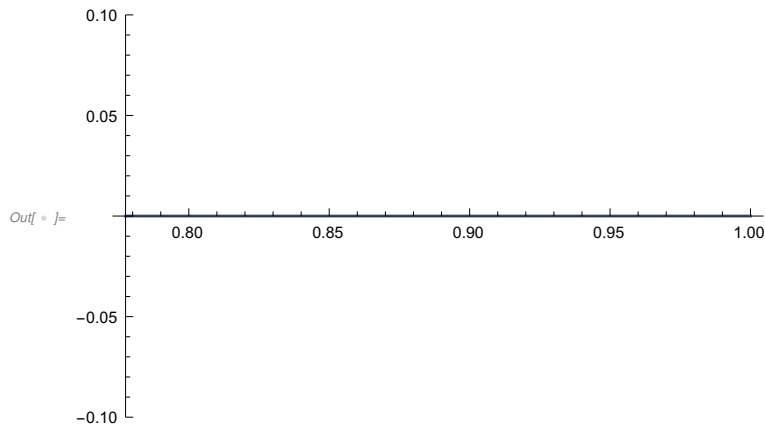
{t, 0.777... , 1}, PlotRange $\rightarrow \{-.1, .1\}$, WorkingPrecision $\rightarrow 30$]



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In[ ]:= Plot[Sqrt[ $\alpha^3 (p31 + p32 \omega\omega)$ ] - u3 /.  $\omega\omega \rightarrow \omega$ ,
  {t, 0.777..., 1}, PlotRange -> {- .1, .1}, WorkingPrecision -> 30]
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```
In[ ]:= Plot[-Sqrt[ $\alpha^5 (p51 + p52 \omega\omega)$ ] - u5 /.  $\omega\omega \rightarrow \omega$ ,
  {t, 0.777..., 1}, PlotRange -> {- .1, .1}, WorkingPrecision -> 30]
```



```
In[ ]:= Plot[Sqrt[ $\alpha^7 (p71 + p72 \omega\omega)$ ] - u7 /.  $\omega\omega \rightarrow \omega$ ,
  {t, 0.777..., 1}, PlotRange -> {- .1, .1}, WorkingPrecision -> 30]
```

