Num. Math.—7

gyak fel. (elôzô óra)

Példa $\| A \|_2$ definíció alapján közelítve, ill mint felt. optimalizálási probléma megoldása

```
A = ({{12, -2, 1}, {-2, 15, 3}, {1, -2, 10}});
```

Táblázatot készítünk sok véletlen vektorokból és erre képezzük a maximumot! Ez jól közelíti a def-ban szereplô sup értéket.

```
T = Table[Table[Random[Real, {-1, 1}], {3}], {10^5}];
T[[1]]
{-0.0514173, -0.844396, 0.792783}
Max[Map[Norm[A.#] / Norm[#] &, T]]
16.1931
N[Norm[A]]
16.1931
```

Feltételes szélsôéertékprobléma megoldásaként a norma.

```
NMaximize[{Norm[A.{x1, x2, x3}], x1^2 + x2^2 + x3^2 == 1}, {x1, x2, x3}]
{16.1931, {x1 → -0.43126, x2 → 0.897697, x3 → 0.0903124}}
Norm[A]
```

$$\sqrt{\text{Root}\left[-3294225 + 72816\, \#1 - 492\, \#1^2 + \#1^3 \&, 3\right]}$$

```
Maximize[{Norm[A.{x1, x2, x3}], x1^2 + x2^2 + x3^2 == 1}, {x1, x2, x3}]
```

$$\left\{\sqrt{-\text{Root}\left[3294225 + 72816\, \#1 + 492\, \#1^2 + \#1^3 \&, 1\right]},\right.$$
$$\{x1 \to \text{Root}\left[-3459321856 + 37626759296\, \#1^2 - 125676752925\, \#1^4 + 125676752925\, \#1^6 \&, 3\right],$$
$$x2 \to \text{Root}\left[-939299904 + 20827845521\, \#1^2 - 125676752925\, \#1^4 + 125676752925\, \#1^6 \&, 6\right],$$
$$\left. x3 \to \text{Root}\left[-9486400 + 1203778745\, \#1^2 - 5027070117\, \#1^4 + 5027070117\, \#1^6 \&, 4\right]\}\right\}$$

GS explicit (n=3) $\left[x_j = x_j^{(m-1)},\ y_j = x_j^{(m)}\right.$ átnevezéssel]

```
-DiagonalMatrix[1 / {a11, a22, a33}].({{0, a12, a13}, {0, 0, a23}, {0, 0, 0}}.{y1, y2, y3} +
    {{0, 0, 0}, {a21, 0, 0}, {a31, a32, 0}}.{x1, x2, x3} - {a14, a24, a34})
```

$$\left\{-\frac{-a14 + a12\ y2 + a13\ y3}{a11}, -\frac{-a24 + a21\ x1 + a23\ y3}{a22}, -\frac{-a34 + a31\ x1 + a32\ x2}{a33}\right\}$$

Jacobi explicit

```
% /. {y1 → x1, y2 -> x2, y3 -> x3}
```

$$\left\{-\frac{-a14 + a12\ x2 + a13\ x3}{a11}, -\frac{-a24 + a21\ x1 + a23\ x3}{a22}, -\frac{-a34 + a31\ x1 + a32\ x2}{a33}\right\}$$

$$A = \begin{pmatrix} 1 & -2+i & i \\ -2-i & 19 & -3 \\ -i & -3 & 3 \end{pmatrix};$$

Oljuk meg Ax=e1 GS iterációval (nem diag dom!)

```
PartJG[AV_] := {UpperTriangularize[AV, 1],
  LowerTriangularize[AV, -1], DiagonalMatrix[Table[AV[[j, j]], {j, Length[AV]}]]}
Map[MatrixForm, PartJG[A]]
```

$$\left\{\begin{pmatrix} 0 & -2+i & i \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ -2-i & 0 & 0 \\ -i & -3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right\}$$

```
PA = PartJG[A]
```

{{{0, -2 + i, i}, {0, 0, -3}, {0, 0, 0}},
 {{0, 0, 0}, {-2 - i, 0, 0}, {-i, -3, 0}}, {{1, 0, 0}, {0, 19, 0}, {0, 0, 3}}}

```
b = Inverse[PA[[3]] + PA[[2]]].{1, 0, 0}
```

$$\left\{1, \frac{2}{19} + \frac{i}{19}, \frac{2}{19} + \frac{22\, i}{57}\right\}$$

```
B = -Inverse[PA[[3]] + PA[[2]]].(PA[[1]])
```

$$\left\{\{0, 2-i, -i\}, \left\{0, \frac{5}{19}, \frac{4}{19} - \frac{2\, i}{19}\right\}, \left\{0, \frac{34}{57} + \frac{2\, i}{3}, \frac{31}{57} - \frac{2\, i}{19}\right\}\right\}$$

```
Eigenvalues[B] // Abs // N
```

{0.8697, 0.13532, 0.}

```
b
```

$$\left\{1, \frac{2}{19} + \frac{i}{19}, \frac{2}{19} + \frac{22\, i}{57}\right\}$$

```
NestList[B.# + b &, b, 50] // N
```

{{1., 0.105263 + 0.0526316 ⅈ, 0.105263 + 0.385965 ⅈ},
{1.64912 - 0.105263 ⅈ, 0.195753 + 0.136657 ⅈ, 0.23084 + 0.686365 ⅈ},
{2.21453 - 0.153278 ⅈ, 0.277624 + 0.208793 ⅈ, 0.328717 + 0.946969 ⅈ},
{2.71101 - 0.188754 ⅈ, 0.347206 + 0.272337 ⅈ, 0.410125 + 1.17601 ⅈ},
{3.14276 - 0.212657 ⅈ, 0.406766 + 0.328709 ⅈ, 0.477651 + 1.37629 ⅈ},
{3.51853 - 0.226999 ⅈ, 0.457738 + 0.378601 ⅈ, 0.533404 + 1.55145 ⅈ},
{3.84552 - 0.23394 ⅈ, 0.501326 + 0.422736 ⅈ, 0.579306 + 1.70458 ⅈ},
{4.12996 - 0.235161 ⅈ, 0.538579 + 0.461756 ⅈ, 0.616966 + 1.83841 ⅈ},
{4.37733 - 0.232033 ⅈ, 0.570399 + 0.496237 ⅈ, 0.647744 + 1.95535 ⅈ},
{4.59238 - 0.22567 ⅈ, 0.597561 + 0.526688 ⅈ, 0.672784 + 2.05748 ⅈ},
{4.77929 - 0.216969 ⅈ, 0.620732 + 0.553568 ⅈ, 0.693055 + 2.14667 ⅈ},
{4.9417 - 0.20665 ⅈ, 0.640485 + 0.577284 ⅈ, 0.709368 + 2.22452 ⅈ},
{5.08277 - 0.195285 ⅈ, 0.657312 + 0.598197 ⅈ, 0.722407 + 2.29245 ⅈ},
{5.20528 - 0.183324 ⅈ, 0.671637 + 0.616631 ⅈ, 0.732745 + 2.35172 ⅈ},
{5.31163 - 0.171119 ⅈ, 0.683822 + 0.632871 ⅈ, 0.740861 + 2.40341 ⅈ},
{5.40393 - 0.15894 ⅈ, 0.694178 + 0.647173 ⅈ, 0.747158 + 2.44848 ⅈ},
{5.48401 - 0.14699 ⅈ, 0.702973 + 0.659762 ⅈ, 0.751969 + 2.48777 ⅈ},
{5.55347 - 0.135418 ⅈ, 0.710436 + 0.670839 ⅈ, 0.755575 + 2.522 ⅈ},
{5.61371 - 0.124333 ⅈ, 0.716762 + 0.680581 ⅈ, 0.758206 + 2.55182 ⅈ},
{5.66592 - 0.113806 ⅈ, 0.722119 + 0.689145 ⅈ, 0.760055 + 2.57778 ⅈ},
{5.71117 - 0.103884 ⅈ, 0.726652 + 0.696671 ⅈ, 0.76128 + 2.60039 ⅈ},
{5.75037 - 0.0945888 ⅈ, 0.730482 + 0.703283 ⅈ, 0.762012 + 2.62007 ⅈ},
{5.78432 - 0.0859288 ⅈ, 0.733716 + 0.709088 ⅈ, 0.762359 + 2.6372 ⅈ},
{5.81372 - 0.0778982 ⅈ, 0.736442 + 0.714184 ⅈ, 0.762409 + 2.65209 ⅈ},
{5.83916 - 0.0704821 ⅈ, 0.738738 + 0.718656 ⅈ, 0.762232 + 2.66504 ⅈ},
{5.86117 - 0.0636584 ⅈ, 0.740669 + 0.722578 ⅈ, 0.761888 + 2.6763 ⅈ},
{5.88022 - 0.0574005 ⅈ, 0.74229 + 0.726017 ⅈ, 0.761423 + 2.68609 ⅈ},
{5.89669 - 0.0516783 ⅈ, 0.743648 + 0.729032 ⅈ, 0.760875 + 2.69459 ⅈ},
{5.91092 - 0.0464598 ⅈ, 0.744786 + 0.731673 ⅈ, 0.760272 + 2.70198 ⅈ},
{5.92322 - 0.0417123 ⅈ, 0.745736 + 0.733986 ⅈ, 0.75964 + 2.70839 ⅈ},
{5.93385 - 0.0374028 ⅈ, 0.746528 + 0.736012 ⅈ, 0.758995 + 2.71396 ⅈ},
{5.94303 - 0.0334988 ⅈ, 0.747187 + 0.737785 ⅈ, 0.758353 + 2.7188 ⅈ},
{5.95095 - 0.0299688 ⅈ, 0.747733 + 0.739337 ⅈ, 0.757723 + 2.72299 ⅈ},
{5.95779 - 0.0267825 ⅈ, 0.748186 + 0.740694 ⅈ, 0.757114 + 2.72663 ⅈ},
{5.96369 - 0.0239111 ⅈ, 0.74856 + 0.741881 ⅈ, 0.75653 + 2.72978 ⅈ},
{5.96878 - 0.0213272 ⅈ, 0.748867 + 0.742919 ⅈ, 0.755976 + 2.73251 ⅈ},
{5.97317 - 0.0190055 ⅈ, 0.749119 + 0.743826 ⅈ, 0.755454 + 2.73488 ⅈ},
{5.97695 - 0.0169219 ⅈ, 0.749325 + 0.744618 ⅈ, 0.754966 + 2.73693 ⅈ},
{5.9802 - 0.0150545 ⅈ, 0.749492 + 0.74531 ⅈ, 0.75451 + 2.73871 ⅈ},
{5.98301 - 0.0133826 ⅈ, 0.749628 + 0.745914 ⅈ, 0.754088 + 2.74025 ⅈ},
{5.98542 - 0.0118875 ⅈ, 0.749736 + 0.746442 ⅈ, 0.753699 + 2.74158 ⅈ},
{5.9875 - 0.0105518 ⅈ, 0.749823 + 0.746902 ⅈ, 0.75334 + 2.74273 ⅈ},
{5.98928 - 0.00935978 ⅈ, 0.749892 + 0.747303 ⅈ, 0.753012 + 2.74373 ⅈ},
{5.99082 - 0.00829689 ⅈ, 0.749946 + 0.747653 ⅈ, 0.752711 + 2.74459 ⅈ},
{5.99214 - 0.00735001 ⅈ, 0.749987 + 0.747959 ⅈ, 0.752437 + 2.74534 ⅈ},
{5.99327 - 0.00650719 ⅈ, 0.750019 + 0.748225 ⅈ, 0.752188 + 2.74598 ⅈ},
{5.99424 - 0.0057576 ⅈ, 0.750043 + 0.748457 ⅈ, 0.751962 + 2.74654 ⅈ},
{5.99508 - 0.00509144 ⅈ, 0.75006 + 0.748659 ⅈ, 0.751757 + 2.74702 ⅈ},
{5.9958 - 0.00449985 ⅈ, 0.750072 + 0.748834 ⅈ, 0.751572 + 2.74743 ⅈ},
{5.99641 - 0.00397487 ⅈ, 0.75008 + 0.748987 ⅈ, 0.751405 + 2.74779 ⅈ},
{5.99694 - 0.0035093 ⅈ, 0.750084 + 0.749121 ⅈ, 0.751254 + 2.7481 ⅈ}}

```
LinearSolve[A, {1, 0, 0}]
```

$$\left\{6, \frac{3}{4} + \frac{3\,i}{4}, \frac{3}{4} + \frac{11\,i}{4}\right\}$$

Valós gyökök közelítése (Newton, Newton-Raphson)

- **1**

Legyen f kétszer diff. ∃x∈[a,b]: f[x]=0? pl folytonossági megfontolások
Alapötlet: legyen $x_0$egy jó közelítése a gyöknek és használjuk a deriváltat egy jobb közelítés megadására

$$f\,'[x_n] = (0 - f[x_n]) / (x_{n+1} - x_n) \Longrightarrow x_{n+1} = x_n + \frac{-f[x_n]}{f\,'[x_n]}$$

Példa. f[x]=Cos[x]-x^3

```
f[x_] := Cos[x] - x^3;
NSolve[f[x] == 0, x]
```

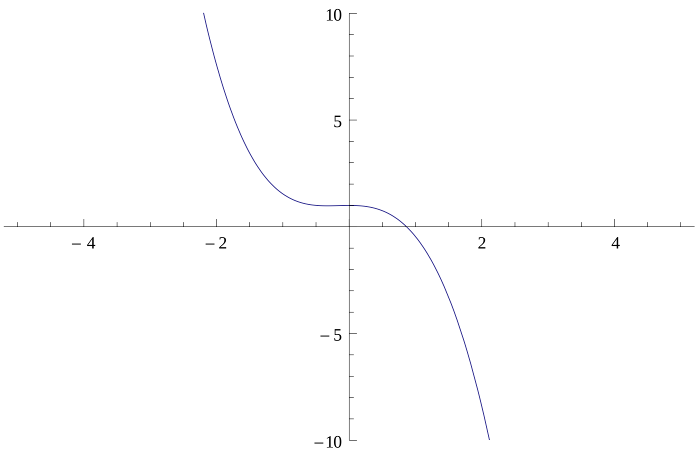
NSolve::nsmet : This system cannot be solved with the methods available to NSolve. ≫

```
NSolve[-x^3 + Cos[x] == 0, x]
```

```
NSolve[x^2 - 5 x + 6 == 0, x]
```

```
{{x → 2.}, {x → 3.}}
```

```
Plot[f[x], {x, -5, 5}, PlotRange → {-10, 10}]
```




```
FindRoot[f[x], {x, 1}]
```

```
{x → 0.865474}
```

```
Options[FindRoot]
```

```
{AccuracyGoal → Automatic, Compiled → Automatic, DampingFactor → 1, Evaluated → True,
 EvaluationMonitor → None, Jacobian → Automatic, MaxIterations → 100, Method → Automatic,
 PrecisionGoal → Automatic, StepMonitor → None, WorkingPrecision → MachinePrecision}
```

Most mi is megoldjuk iterációval! ⟶ Nest, NestList, NestWhile

```
Nest[F, x, 3]
```

```
F[F[F[x]]]
```

```
NestList[F, x, 3]
```

```
{x, F[x], F[F[x]], F[F[F[x]]]}
```

A Newton-Raphson iteráció magfüggvénye

```
NewtonRaphsonStep[x_] := N[x - f[x] / f'[x]];
```

```
Nest[NewtonRaphsonStep, 1, 3]
```

0.865474

```
NestList[NewtonRaphsonStep, 1, 3]
```

{1, 0.880333, 0.865684, 0.865474}

```
h[x_] := 1 / (2 x + 1);
```

```
Solve[h[x] == x, x]
```

$\left\{\{x \to -1\}, \left\{x \to \frac{1}{2}\right\}\right\}$

```
N[NestList[h, -101 / 100, 20]]
```

{-1.01, -0.980392, -1.04082, -0.924528, -1.17778, -0.737705,
-2.10345, -0.311828, 2.65714, 0.158371, 0.75945, 0.396999, 0.557414, 0.472851,
0.513953, 0.493119, 0.503464, 0.498274, 0.500865, 0.499568, 0.500216}

```
NestList[h, 1, 14]
```

{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}

Ezt egy kicsit nehezebb megérteni, mindenki próbálkozzon azért!

```
NestWhile[NewtonRaphsonStep, 1, Unequal, 2]
```

0.865474

```
InputForm[%]
```

```
?MachinePrecision
```

MachinePrecision is a symbol used to indicate machine-number precision. ≫

Konvergencia gyorsaság?

```
f[x_] := x^2 - 5
```

```
NewtonRaphsonStepb[x_] := N[x - f[x] / f'[x], 40];
```

```
NestList[NewtonRaphsonStepb, 3, 7] // TableForm
```

3
2.333333333333333333333333333333333333333
2.238095238095238095238095238095238095238
2.236068895643363728470111448834853090172
2.23606797749997819409459355858145010648
2.23606797749978969640917367667633375744
2.23606797749978969640917366873127623544
2.2360679774997896964091736687312762354

```
N[Sqrt[5], 40]
```

2.236067977499789696409173668731276235441

Asz. hibakonstans ebben az esetben

```
T = NestList[NewtonRaphsonStepb, 3, 7];
```

```
Table[(T[[m]] - Sqrt[5]) / (T[[m - 1]] - Sqrt[5])^2, {m, 2, 6}]
```

```
{0.16666666666666666666666666666666666667, 0.21428571428571428571428571428571428571428 57,
 0.22340425531914893617021276595744 7, 0.2236067059356592659719075 7, 0.223606797750}
```

```
f''[x] / (2 f'[x]) /. x → Sqrt[5] // N
```

```
0.223607
```

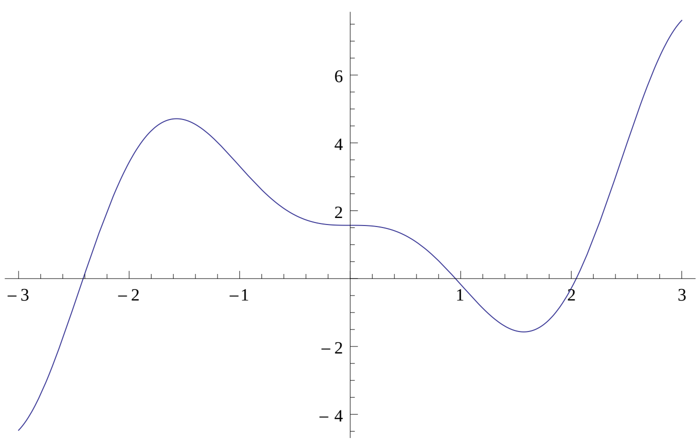
■ Feladat

Adjuk meg 3 tizedesjegy pontossággal, felhasználva a NewtonRaphsonStep fgv-t, az alábbi egyenletek összes zéróhelyét, melyek benne vanak a 0 3 sugarú környezetében.

```
eq1 = x^2 == 5;
eq2 = x^3 == 2;
eq3 = 4 x Cos[x]^2 + π / 2 == 2 x + Sin[2 x];
```

A harmadik egyenlet esete (3 valós gyök [-3,3]-ban)

```
f[x_] := 4 x Cos[x]^2 + π / 2 - 2 x - Sin[2 x];
Plot[f[x], {x, -3, 3}]
```




```
{Nest[NewtonRaphsonStep, -2, 4], Nest[NewtonRaphsonStep, 1, 4], Nest[NewtonRaphsonStep, 2, 4]}
```

```
{-2.41614, 0.952848, 2.04516}
```

Az elsô egyenlet
(f függvényt újra definiáljuk mert ez globális a NewtnRaphsonStep-re!)

```
f[x_] := x^2 - 5
```

```
NewtonRaphsonStep[x] // Simplify
```

$$\frac{2.5}{x} + 0.5\,x$$

Négyzetgyökvonás

```
Nest[NewtonRaphsonStep, 2, 4]
```

```
2.23607
```

$$N\left[\sqrt{5}\right]$$

```
2.23607
```

Most osztályozzuk a valós egyenes pontjait:
A osztály) Konvergencia van és lim=Sqrt[5],
B osztály) Konvergencia van és lim=-Sqrt[5],
C osztály) Nincs konvergencia vagy az interációssorozat nem értelmezhetô

```
x0
```

```
x0
```

```
TableForm[Table[Nest[NewtonRaphsonStep, x0, 10], {x0, -5, 5, 1/4}]]
```

Power::infy : Infinite expression $\frac{1}{0}$ encountered. ≫

Infinity::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

```
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
-2.23607
Indeterminate
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
2.23607
```

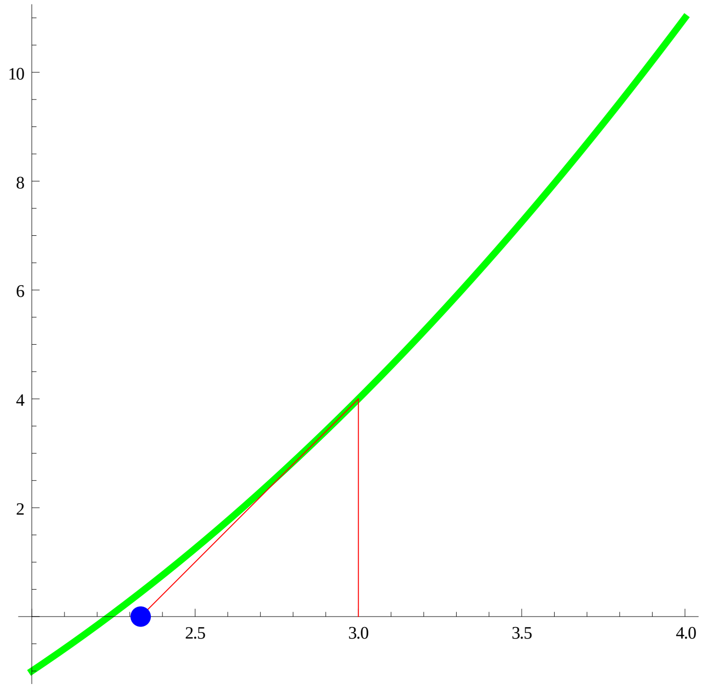
Projektmunka: NR Vizualizáció

Egy lépés viz.

```
Clear[f, g]
f[x_] := x^2 - 5;
g[x_] := x^3 - 2;
```

```
NewtonRaphsonVis[f_, x0_] := Module[{x},
  Plot[f[x], {x, x0 - 1, x0 + 1}, PlotStyle → {Thickness[.01], RGBColor[0, 1, 0]}, AspectRatio → 1,
   Epilog → {RGBColor[1, 0, 0], Line[{{x0, f[x0]}, {-f[x0] / (D[f[x], x] /. x → x0) + x0, 0}}],
     Line[{{x0, 0}, {x0, f[x0]}}], RGBColor[0, 0, 1], PointSize[.03],
     Point[{-f[x0] / (D[f[x], x] /. x → x0) + x0, 0}]}]]
```

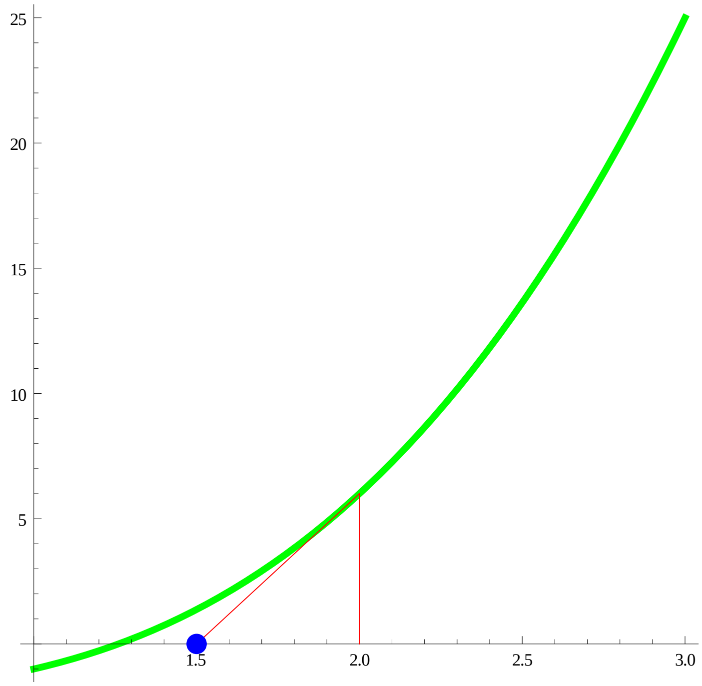
```
NewtonRaphsonVis[f, 3]
```




```
Nest[NewtonRaphsonStep, 3, 1]
```

2.33333

```
NewtonRaphsonVis[g, 2]
```



Memo: f globális NR-re nézve!

```
f[x_] = g[x];
```

```
Nest[NewtonRaphsonStep, 2, 1]
```

2.25

■ Kitekintés: Többdimenziós NR

f[x,y]=0
g[x,y]=0

Vektoriteráció

```
F[{x_, y_}] := {x + 1, 2 y}
```

```
NestList[F, {0, 1}, 5] // TableForm
```

0	1
1	2
2	4
3	8
4	16
5	32

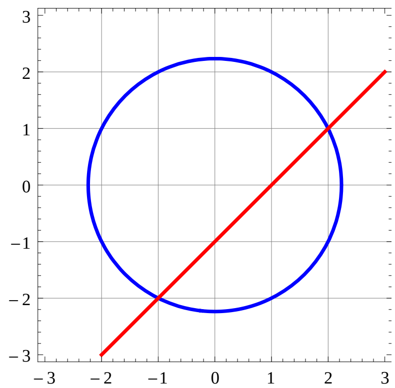
```
f1[x_, y_] = x^2 + y^2 - 5;
f2[x_, y_] = x - y - 1;
```

```
Solve[{f1[x, y] == 0, f2[x, y] == 0}, {x, y}]
```

{{x → -1, y → -2}, {x → 2, y → 1}}

Meg tudjuk-e közelíteni hasonlóan a megoldásokat?

```
ContourPlot[{f1[x, y] == 0, f2[x, y] == 0}, {x, -3, 3}, {y, -3, 3}, ImageSize → {200, 200},
 GridLines → Automatic, ContourStyle → {{Blue, Thickness[.01]}, {Red, Thickness[.01]}}]
```



Általánosítás 2 dimenzióra a Jacobi mátrix segítségével:

```
J[x_, y_] = ( D[f1[x, y], x]  D[f1[x, y], y] )
            ( D[f2[x, y], x]  D[f2[x, y], y] )
```

```
{{2 x, 2 y}, {1, -1}}
```

```
J[1 / 2, 1 / 2]
```

```
{{1, 1}, {1, -1}}
```

```
NewtonRaphsonStep2D[{x_, y_}] := N[{x, y} - Inverse[J[x, y]].{f1[x, y], f2[x, y]}];
```

```
TableForm[NestList[NewtonRaphsonStep2D, {1 / 2, 1 / 2}, 5]]
```

$\frac{1}{2}$	$\frac{1}{2}$
3.25	2.25
2.28409	1.28409
2.02262	1.02262
2.00017	1.00017
2.	1.

```
TableForm[NestList[NewtonRaphsonStep2D, {-3, -1 / 2}, 5]]
```

−3	$-\frac{1}{2}$
−1.89286	−2.89286
−1.16658	−2.16658
−1.00832	−2.00832
−1.00002	−2.00002
−1.	−2.