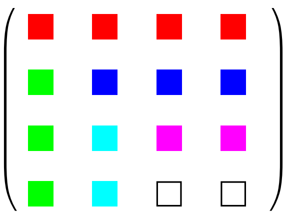
$R^H\ R$ felbontás parkettázással (Cholesky) [LL^T]



■ Feladat

Adjuk meg a Cholesky felbontást.

In[5]:=
```
B = (1 2
     2 9);
```

■ Megoldás

□ White-Box: a parketta algoritmus lépésrôl lépésre.

$$B = R^T\ R = \begin{pmatrix} r_{11} & 0 \\ r_{12} & r_{22} \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$$

$$r_{11}{}^2 = 1 \Longrightarrow r_{11} = 1\ r_{12} = 2\ ,\ \ r_{22}{}^2 = 5 \Longrightarrow r_{22} = \sqrt{5}$$

■ Black-Box: A Mathematica built–in függvény a felbontás jobboldali komponensét adja.

In[6]:= `R = CholeskyDecomposition[B];`

In[7]:= `MatrixForm[R]`

Out[7]//MatrixForm= $\begin{pmatrix} 1 & 2 \\ 0 & \sqrt{5} \end{pmatrix}$

In[8]:= `{MatrixForm[Transpose[R]], MatrixForm[R]}`

Out[8]= $\left\{ \begin{pmatrix} 1 & 0 \\ 2 & \sqrt{5} \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & \sqrt{5} \end{pmatrix} \right\}$

$B = R^H\ R$

In[9]:= `MatrixForm[Transpose[R].R]`

Out[9]//MatrixForm= $\begin{pmatrix} 1 & 2 \\ 2 & 9 \end{pmatrix}$

■ Feladat

$$B = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 5 \\ -2 & 5 & 21 \end{pmatrix};$$

Adjuk meg a Cholesky felbontást és a módosított felbontást.

In[10]:=
$$B = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 5 \\ -2 & 5 & 21 \end{pmatrix};$$

In[12]:=
```
(R = CholeskyDecomposition[B]) // MatrixForm
```

Out[12]//MatrixForm=
$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

In[13]:=
```
Map[MatrixForm, {ConjugateTranspose[R], R, ConjugateTranspose[R].R}]
```

Out[13]=
$$\left\{ \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 5 \\ -2 & 5 & 21 \end{pmatrix} \right\}$$

In[17]:=
```
L = ConjugateTranspose[R].DiagonalMatrix[1 / {R[[1, 1]], R[[2, 2]], R[[3, 3]]}];
DD = DiagonalMatrix[{4, 9, 16}];
```

Módosított Cholesky, ha az alapfelbontás már ismert

In[19]:=
```
L.DD.ConjugateTranspose[L] // MatrixForm
```

Out[19]//MatrixForm=
$$\begin{pmatrix} 4 & 2 & -2 \\ 2 & 10 & 5 \\ -2 & 5 & 21 \end{pmatrix}$$

In[20]:=
```
Clear[L]
```

Módosított Cholesky (parkettázás, rek. összefüggések megfelelô sorrendben)

In[21]:=
```
(LL = Table[If[j == k, 1, If[j > k, L_{j,k}, 0]], {j, 3}, {k, 3}]).
  DiagonalMatrix[{d_1, d_2, d_3}].Transpose[LL] // MatrixForm
```

Out[21]//MatrixForm=
$$\begin{pmatrix} d_1 & d_1 L_{2,1} & d_1 L_{3,1} \\ d_1 L_{2,1} & d_2 + d_1 L_{2,1}^2 & d_1 L_{2,1} L_{3,1} + d_2 L_{3,2} \\ d_1 L_{3,1} & d_1 L_{2,1} L_{3,1} + d_2 L_{3,2} & d_3 + d_1 L_{3,1}^2 + d_2 L_{3,2}^2 \end{pmatrix}$$

■ Feladat

In[28]:=
$$B = \begin{pmatrix} 1 & -2 + i & i \\ -2 - i & 19 & -3 \\ -i & -3 & 3 \end{pmatrix};$$

■ Megoldás

In[29]:=
```
R = CholeskyDecomposition[B];
```

In[30]:= `MatrixForm[R]`

Out[30]//MatrixForm=

$$\begin{pmatrix} 1 & -2+i & i \\ 0 & \sqrt{14} & (-2+i)\sqrt{\frac{2}{7}} \\ 0 & 0 & \frac{2}{\sqrt{7}} \end{pmatrix}$$

In[31]:= `ConjugateTranspose[R].R // MatrixForm`

Out[31]//MatrixForm=

$$\begin{pmatrix} 1 & -2+i & i \\ -2-i & 19 & -3 \\ -i & -3 & 3 \end{pmatrix}$$

In[32]:= `MatrixForm[B]`

Out[32]//MatrixForm=

$$\begin{pmatrix} 1 & -2+i & i \\ -2-i & 19 & -3 \\ -i & -3 & 3 \end{pmatrix}$$

- **Feladat**

Adjuk meg a Cholesky felbontást.

$$B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix};$$

```
Det[B]
```

```
CholeskyDecomposition[B]
```

B nem pozitív defintit, a felbontást nem adja meg a built–in fgv.
Nézzük az összes 2x2-es és 1x1-esfôminort

```
MapIndexed[Take[#, #2][[1]] &, Minors[B, 2]]
```

```
MapIndexed[Take[#, #2][[1]] &, Minors[B, 1]]
```

- **Megoldás**

B nem PD, de van neki Cholesky felbontása

In[25]:= `R = {{1, 0, -1}, {0, 1, -1}, {0, 0, 0}}; MatrixForm[R]`

Out[25]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

In[26]:= `Transpose[R].R // MatrixForm`

Out[26]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

In[27]:=
```
B // MatrixForm
```

Out[27]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

- **Feladat**

In[23]:=
$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

- **Megoldás**

A mátrix szimmetrikus, de nincs neki Cholesky felbontása.
A mátrixnak van negatív fôminora!

In[24]:=
```
MapIndexed[Take[#, #2][[1]] &, Minors[B, 2]]
```

Out[24]= {0, -1, -1}

Sajátérték+Sajátvektor (Memo)

In[33]:=
$$A1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix};$$

In[34]:=
$$\text{Eigenvalues}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right]$$

Out[34]=
$$\left\{\frac{1}{2}\left(5+\sqrt{33}\right), \frac{1}{2}\left(5-\sqrt{33}\right)\right\}$$

In[35]:=
$$\lambda \text{ /. Solve}\left[\text{Det}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \lambda \text{ IdentityMatrix}[2]\right] == 0\right]$$

Out[35]=
$$\left\{\frac{1}{2}\left(5-\sqrt{33}\right), \frac{1}{2}\left(5+\sqrt{33}\right)\right\}$$

In[36]:=
```
CharacteristicPolynomial[A1, λ]
```

Out[36]= $-2 - 5\lambda + \lambda^2$

In[37]:=
```
Solve[% == 0, λ]
```

Out[37]=
$$\left\{\left\{\lambda \to \frac{1}{2}\left(5-\sqrt{33}\right)\right\}, \left\{\lambda \to \frac{1}{2}\left(5+\sqrt{33}\right)\right\}\right\}$$

In[38]:=
$$\text{Eigenvectors}\left[\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right]$$

Out[38]=
$$\left\{\left\{-\frac{4}{3}+\frac{1}{6}\left(5+\sqrt{33}\right), 1\right\}, \left\{-\frac{4}{3}+\frac{1}{6}\left(5-\sqrt{33}\right), 1\right\}\right\}$$

In[39]:=
```
Eigensystem[{{1, 2}, {3, 4}}]
```

Out[39]= $\left\{\left\{\frac{1}{2}\left(5+\sqrt{33}\right), \frac{1}{2}\left(5-\sqrt{33}\right)\right\}, \left\{\left\{-\frac{4}{3}+\frac{1}{6}\left(5+\sqrt{33}\right), 1\right\}, \left\{-\frac{4}{3}+\frac{1}{6}\left(5-\sqrt{33}\right), 1\right\}\right\}\right\}$

In[40]:=
```
{Tr[A1], Plus @@ Eigenvalues[A1], Simplify[Plus @@ Eigenvalues[A1]]}
```

Out[40]= $\left\{5, \frac{1}{2}\left(5-\sqrt{33}\right)+\frac{1}{2}\left(5+\sqrt{33}\right), 5\right\}$

In[41]:=
```
{Det[A1], Times @@ Eigenvalues[A1], Simplify[Times @@ Eigenvalues[A1]]}
```

Out[41]= $\left\{-2, \frac{1}{4}\left(5-\sqrt{33}\right)\left(5+\sqrt{33}\right), -2\right\}$

In[42]:=
```
?Det
```

Det[*m*] gives the determinant of the square matrix *m*. ≫

In[43]:=
```
?Tr
```

Tr[*list*] finds the trace of the matrix or tensor *list*.
Tr[*list*, *f*] finds a generalized trace, combining terms with *f* instead of Plus.
Tr[*list*, *f*, *n*] goes down to level *n* in *list*. ≫

■ Sajátértékek kapcsolatai

p(A) sajátértékei

In[44]:=
```
A = {{-1, 1}, {4, 2}};
```

In[45]:=
```
Eigenvalues[A]
```

Out[45]= {3, −2}

In[46]:=
```
B = A.A.A; MatrixForm[B]
```

Out[46]//MatrixForm=
$$\begin{pmatrix} -1 & 7 \\ 28 & 20 \end{pmatrix}$$

In[47]:=
```
MatrixPower[A, 3] // MatrixForm
```

Out[47]//MatrixForm=
$$\begin{pmatrix} -1 & 7 \\ 28 & 20 \end{pmatrix}$$

In[48]:=
```
A^3 // MatrixForm
```

Out[48]//MatrixForm=
$$\begin{pmatrix} -1 & 1 \\ 64 & 8 \end{pmatrix}$$

In[49]:=
```
Eigenvalues[A]
```

Out[49]= {3, −2}

In[50]:=
```
Eigenvalues[B]
```

Out[50]= {27, −8}

Sejtés. B-nek a sajátértékei A sajátértékeinek köbei

In[51]:=
```
Eigenvalues[A]
```

Out[51]= {3, −2}

In[52]:=
```
Eigenvalues[Inverse[A]]
```

Out[52]= $\left\{-\frac{1}{2}, \frac{1}{3}\right\}$

Gersgorin körök

Cél: s.é absz. értékeinek becslése

$u_j = \mathrm{Re}[a_{j\,j}]$, $v_j = \mathrm{Im}[a_{j\,j}]$, $r_j = \sum_{k, k \neq j} |a_{j\,k}|$

In[53]:=
```
GerschgorinDisks[(a_List)?MatrixQ] := Module[{abs = Abs[a], disks},
  disks = Transpose[{Through /@ {Re, Im} /@ Tr[a, List], (Plus @@ # &) /@ abs - Tr[abs, List]}];
  {
   {Hue[.5], Disk @@@ disks},
   Circle @@@ disks
  }
 ]
```

In[54]:=
$$B1 = \begin{pmatrix} 4 & -1 & i \\ 1 & -i & 0 \\ 2 & 0 & -1+i \end{pmatrix};$$

In[55]:=
```
EVL = N[Eigenvalues[B1]]
```

Out[55]= {3.72637 + 0.512201 i, −0.987708 + 0.620314 i, 0.261334 − 1.13252 i}

In[56]:=
```
EVCoord[el_] := Map[{Re[#], Im[#]} &, el]
```

In[57]:=
```
EVCoord[EVL]
```

Out[57]= {{3.72637, 0.512201}, {−0.987708, 0.620314}, {0.261334, −1.13252}}

In[58]:=
```
Re[1 + 2 I]
```

Out[58]= 1

In[59]:=
```
Im[1 + 2 I]
```

Out[59]= 2

In[60]:=
```
Map[#^3 + 1 &, {-1, 2, 5, -9}]
```

Out[60]= {0, 9, 126, −728}

In[61]:=
```
Show[Graphics[GerschgorinDisks[B1]], Sequence @@
  Map[Graphics[{RGBColor[1, 0, 1], PointSize[.02], Point[#]}] &, EVCoord[EVL]], Sequence @@
  Map[Graphics[{Black, PointSize[.02], Point[#]}] &, EVCoord[Table[B1[[j, j]], {j, Length[B1]}]]],
 Axes → True, ImageSize → {400, 300}, PlotRange → {{-4, 8}, {-6, 6}}, AspectRatio → 1]
```

Out[61]=