Numerikus Integrálás

Alapötletek:

1) $\int_a^b f[x]\, dx \approx \int_a^b L[x]\, dx$

2) Ekvidisztáns alappontrendszer $a \le x_0 < \ldots < x_{n-1} \le b$, $x_k - x_{k-1} = h$

Memo: $\int_a^b f[x]\, dx \approx \int_a^b L[x]\, dx = \int_a^b \sum_{i=0}^{n-1} f[x_i]\, l_i[x]\, dx = \sum_{i=0}^{n-1} f[x_i] \int_a^b l_i[x]\, dx = \sum_{i=0}^{n-1} f[x_i]\, w_i$

■ Lagrange előállítása: bázispolinomok lineáris kombinációja

In[1]:=
$$\text{LagrBase}[j\_, x\_List, var\_] := \left(\prod_{k=1}^{j} \frac{var - x[[k]]}{x[[j+1]] - x[[k]]}\right) \left(\prod_{k=j+2}^{\text{Length}[x]} \frac{var - x[[k]]}{x[[j+1]] - x[[k]]}\right)$$

In[2]:=
$$\text{LagrInterp}[x\_List, y\_List, var\_] := \sum_{j=0}^{\text{Length}[x]-1} y[[j+1]]\ \text{LagrBase}[j, x, var]$$

■ Simpson 1 (egyszerû kvadratúraformula)

In[3]:=
```
LagrInterp[{a, (a + b) / 2, b}, {fa, fm, fb}, x]
```

Out[3]=
$$\frac{fb\,(-a+x)\left(\frac{1}{2}(-a-b)+x\right)}{(-a+b)\left(\frac{1}{2}(-a-b)+b\right)} + \frac{fm\,(-a+x)\,(-b+x)}{\left(-a+\frac{a+b}{2}\right)\left(-b+\frac{a+b}{2}\right)} + \frac{fa\left(\frac{1}{2}(-a-b)+x\right)(-b+x)}{\left(a+\frac{1}{2}(-a-b)\right)(a-b)}$$

In[4]:=
$$\int_a^b \text{LagrInterp}[\{a, (a+b)/2, b\}, \{fa, fm, fb\}, x]\, dx$$

Out[4]=
$$-\frac{1}{6}(a-b)(fa+fb+4\,fm)$$

In[5]:=
```
LagrInterp[{-1, 0, 1}, {1, 0, 1}, x] // Expand
```

Out[5]=
$$x^2$$

■ Feladat

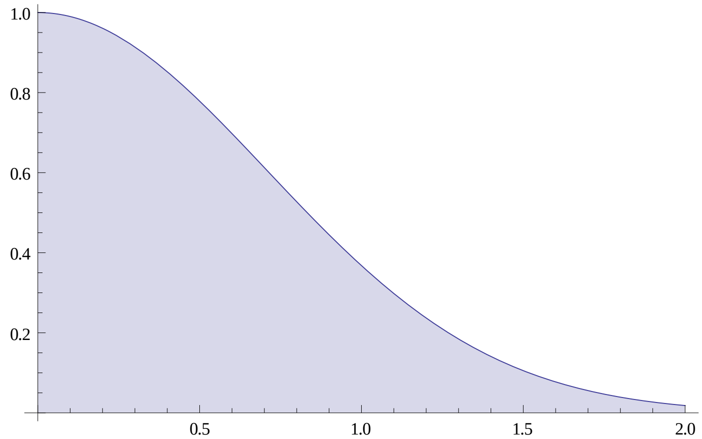
Végezzük el az alábbi lépésket, ha a $\int_0^2 e^{-x^2}\, dx$ határozott integrált közelítjük!

In[6]:=
```
f[x_] := E^(-x^2);
{a1, b1} = {0, 2};
```

$\int_0^2 e^{-x^2}\, dx$ határozott integrált közelítése a Simpson formulával

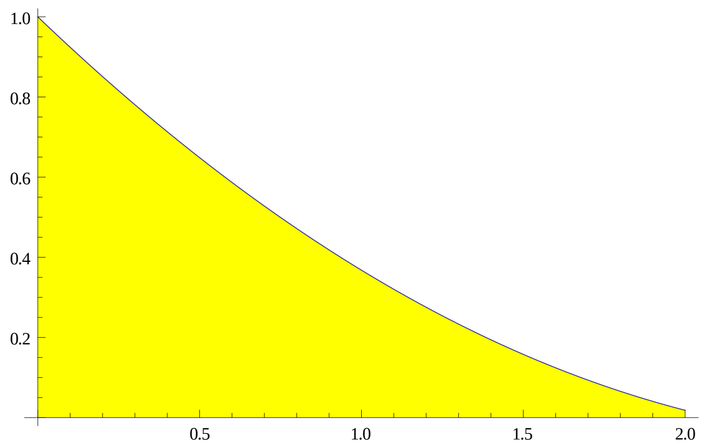
In[8]:=
```
g1a = Plot[f[x], {x, a1, b1}, Filling → Axis]
```

Out[8]=



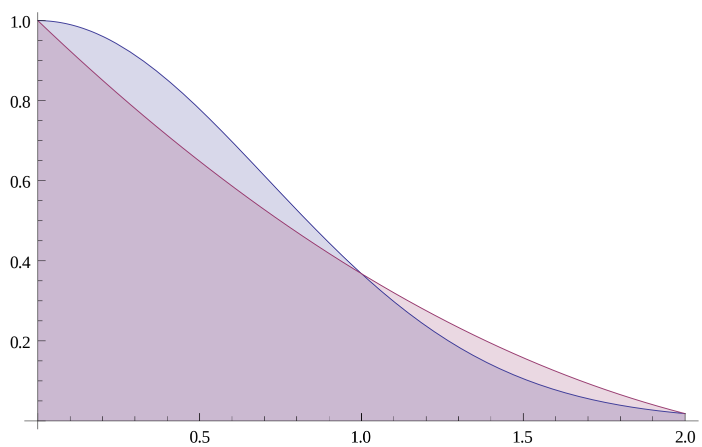
In[9]:=
```
g1b = Plot[LagrInterp[{a1, (a1 + b1) / 2, b1}, {f[a1], f[(a1 + b1)/2], f[b1]}, x],
  {x, a1, b1}, Filling → Axis, FillingStyle → Yellow]
```

Out[9]=



In[12]:=
```
g1c = Plot[{f[x], LagrInterp[{a1, (a1 + b1) / 2, b1}, {f[a1], f[(a1 + b1)/2], f[b1]}, x]},
  {x, a1, b1}, Filling → Axis, FillingStyle → Automatic]
```

Out[12]=



Newton-Leibniz

In[13]:= $\int_{a1}^{b1} f[x]\, dx$

Out[13]= $\frac{1}{2}\sqrt{\pi}\ \text{Erf}[2]$

In[14]:= `N[%]`

Out[14]= 0.882081

Beépített numerikus integrálás

In[15]:= `NIntegrate[f[x], {x, a1, b1}]`

Out[15]= 0.882081

Interpolációs polinom kiintegrálása

In[16]:= $\int_{a1}^{b1} \text{LagrInterp}\left[\{a1, (a1+b1)/2, b1\}, \left\{f[a1], f\left[\frac{a1+b1}{2}\right], f[b1]\right\}, x\right] dx$

Out[16]= $\frac{1}{3} + \frac{1}{3\, e^4} + \frac{4}{3\, e}$

In[17]:= `N[%]`

Out[17]= 0.829944

Simpson formula direkt alkalmazása

In[18]:= $\frac{b1 - a1}{6}\left(f[a1] + 4\, f\left[\frac{a1+b1}{2}\right] + f[b1]\right)$

Out[18]= $\frac{1}{3}\left(1 + \frac{1}{e^4} + \frac{4}{e}\right)$

In[19]:= `N[%]`

Out[19]= 0.829944

- **Feladat**

Végezzük el a fenti lépésket, ha a $\int_1^2 \frac{1}{x}\, dx$ határozott integrált közelítjük!

In[20]:=
```
g[x_] := 1 / x;
{a2, b2} = {1, 2};
```

Newton-Leibniz

In[25]:= $\int_{a2}^{b2} g[x]\, dx$

Out[25]= Log[2]

In[26]:= `N[%]`

Out[26]= 0.693147

Interpolációs polinom kiintegrálása

In[23]:= $\int_{a2}^{b2} \text{LagrInterp}\left[\{a2, (a2+b2)/2, b2\}, \left\{g[a2], g\left[\frac{a2+b2}{2}\right], g[b2]\right\}, x\right] dx$

Out[23]= $\frac{25}{36}$

Simpson formula direkt alkalmazása

In[27]:= $\frac{b2 - a2}{6}\left(g[a2] + 4\ g\left[\frac{a2 + b2}{2}\right] + g[b2]\right)$

Out[27]= $\frac{25}{36}$

In[28]:= `N[%]`

Out[28]= 0.694444

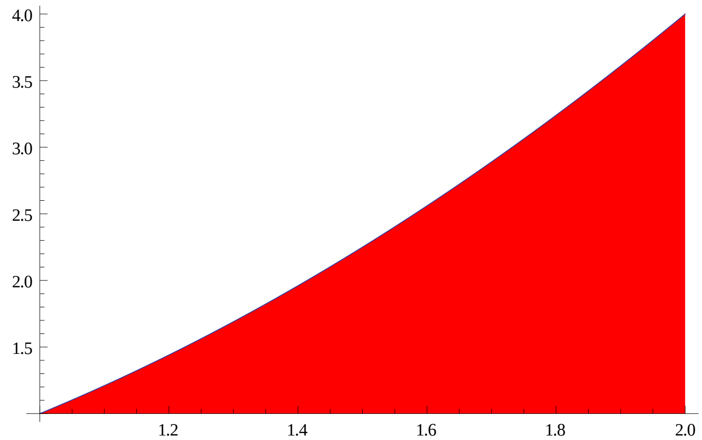
- **Trapéz 1 (egyszerû kvadratúraformula)**

In[29]:= `Integrate[x^2, {x, 1, 2}]`

Out[29]= $\frac{7}{3}$

In[30]:= `g1 = Plot[x^2, {x , 1, 2}, FillingStyle → Red, Filling → Axis]`

Out[30]=

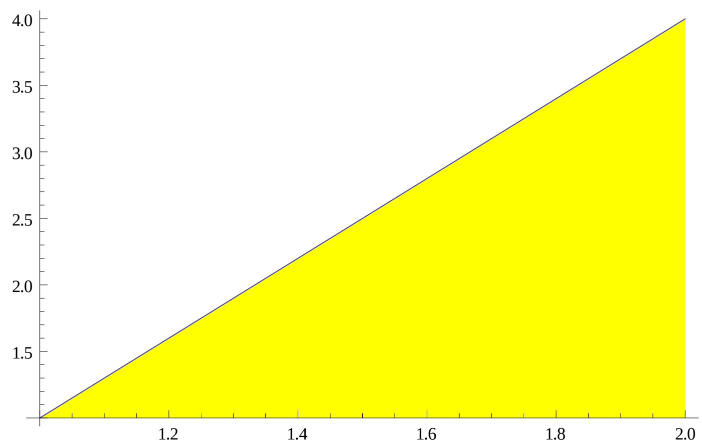


In[31]:= `N[{7 / 3, 5 / 2}]`

Out[31]= {2.33333, 2.5}

In[33]:= `g2 = Plot[3 x - 2, {x , 1, 2}, FillingStyle → Yellow, Filling → Axis]`

Out[33]=



Lineáris interpolációs polinom — trapézszabály

In[34]:= $\int_a^b$ LagrInterp[{a, b}, {fa, fb}, x] ⅆx

Out[34]= $-\frac{1}{2}(a-b)(fa+fb)$

In[35]:= `Integrate[x^2, {x, 1, 2}]`

Out[35]= $\frac{7}{3}$

In[36]:= `LagrInterp[{1, 2}, {1, 4}, x] // Expand`

Out[36]= $-2+3x$

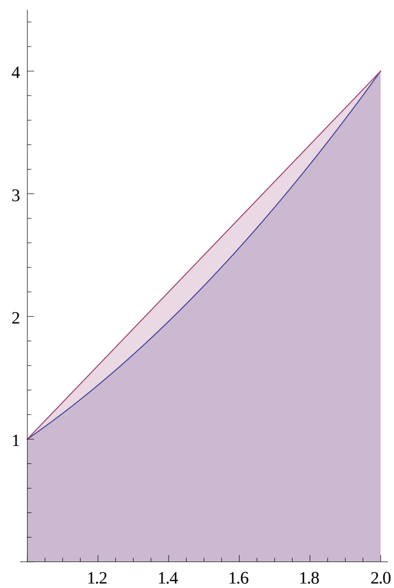
In[37]:= `Integrate[%, {x, 1, 2}] // N`

Out[37]= 2.5

In[40]:= $\frac{1}{2}$ (b - a) (fa + fb) /. {a → 1, b → 2, fa → 1, fb → 4}

Out[40]= $\frac{5}{2}$

In[38]:=
```
Plot[{x^2, 3 x - 2}, {x, 1, 2}, AxesOrigin → {1, 0}, PlotRange → {0, 9/2},
 FillingStyle → Automatic, Filling → Axis, AspectRatio → 1.5, ImageSize → {300, 300}]
```

Out[38]=


Zárt NC formulák

```
Clear[h]
```

$x_0 = a,\ x_{n-1} = b,\ h = \frac{b-a}{n}$

In[41]:=
```
EDPoints[sp_, n_, h_] := Table[sp + (i) h, {i, 0, n - 1}]
```

In[42]:=
```
FPoints[f_, n_] := Table[ToExpression[StringJoin[ToString[f], ToString[i]]], {i, 0, n - 1}]
```

In[43]:=
```
EDPoints[a, 5, h]
```

Out[43]= {a, a + h, a + 2 h, a + 3 h, a + 4 h}

In[44]:=
```
FPoints[f, 5]
```

Out[44]= {f0, f1, f2, f3, f4}

In[45]:=
```
LagrInterp[EDPoints[a, 2, h], FPoints[f, 2], x]
```

Out[45]= $\frac{f1\,(-a+x)}{h} - \frac{f0\,(-a-h+x)}{h}$

In[46]:=
```
Factor[Integrate[LagrInterp[EDPoints[a, 2, h], FPoints[f, 2], x], {x, a, a + h}]]
```

Out[46]= $\frac{1}{2}\,(f0+f1)\,h$

Figyelem: h minden sorban más

In[47]:=
```
TableForm[
 Table[Factor[Integrate[LagrInterp[EDPoints[a, n, h], FPoints[f, n], x], {x, a, a + (n - 1) h}]],
  {n, 2, 10}]]
```

Out[47]//TableForm=

$$\frac{1}{2}\,(f0+f1)\,h$$
$$\frac{1}{3}\,(f0+4\,f1+f2)\,h$$
$$\frac{3}{8}\,(f0+3\,f1+3\,f2+f3)\,h$$
$$\frac{2}{45}\,(7\,f0+32\,f1+12\,f2+32\,f3+7\,f4)\,h$$
$$\frac{5}{288}\,(19\,f0+75\,f1+50\,f2+50\,f3+75\,f4+19\,f5)\,h$$
$$\frac{1}{140}\,(41\,f0+216\,f1+27\,f2+272\,f3+27\,f4+216\,f5+41\,f6)\,h$$
$$\frac{7\,(751\,f0+3577\,f1+1323\,f2+2989\,f3+2989\,f4+1323\,f5+3577\,f6+751\,f7)\,h}{17\,280}$$
$$\frac{4\,(989\,f0+5888\,f1-928\,f2+10\,496\,f3-4540\,f4+10\,496\,f5-928\,f6+5888\,f7+989\,f8)\,h}{14\,175}$$
$$\frac{9\,(2857\,f0+15\,741\,f1+1080\,f2+19\,344\,f3+5778\,f4+5778\,f5+19\,344\,f6+1080\,f7+15\,741\,f8+2857\,f9)\,h}{89\,600}$$

Itt a Simpson formula (második a sorban):

In[48]:=
```
SF = Factor[Integrate[LagrInterp[EDPoints[a, 3, h], FPoints[f, 3], x], {x, a, a + h, a + 2 h}]] /.
  {h → (b - a) / 2}
```

Out[48]= $\frac{1}{6}\,(-a+b)\,(f0+4\,f1+f2)$

Súlyok (weights):

In[49]:=
```
Map[Coefficient[SF, #] &, {f0, f1, f2}]
```

Out[49]= $\left\{\frac{1}{6}\,(-a+b),\ \frac{2}{3}\,(-a+b),\ \frac{1}{6}\,(-a+b)\right\}$

- **Nyílt NC formulák**

Itt a végpontok nem tartoznak az alappontokhoz.

In[50]:=
```
TableForm[
 Table[Factor[Integrate[LagrInterp[Drop[EDPoints[a, n - 1, h], 1], Drop[FPoints[f, n - 1], 1], x],
   {x, a, a + (n - 1) h}]], {n, 3, 10}]]
```

Out[50]//TableForm=

$$2\, f1\, h$$
$$\frac{3}{2}\,(f1 + f2)\, h$$
$$\frac{4}{3}\,(2\, f1 - f2 + 2\, f3)\, h$$
$$\frac{5}{24}\,(11\, f1 + f2 + f3 + 11\, f4)\, h$$
$$\frac{3}{10}\,(11\, f1 - 14\, f2 + 26\, f3 - 14\, f4 + 11\, f5)\, h$$
$$\frac{7\,(611\, f1 - 453\, f2 + 562\, f3 + 562\, f4 - 453\, f5 + 611\, f6)\, h}{1440}$$
$$\frac{8}{945}\,(460\, f1 - 954\, f2 + 2196\, f3 - 2459\, f4 + 2196\, f5 - 954\, f6 + 460\, f7)\, h$$
$$\frac{9\,(1787\, f1 - 2803\, f2 + 4967\, f3 - 1711\, f4 - 1711\, f5 + 4967\, f6 - 2803\, f7 + 1787\, f8)\, h}{4480}$$

Az elsô a téglalap szabály
Már a harmadik sorban negatív súlyok!

■ Hibabecslés

Pontossági rend: Ellenõrizzük, hogy a Simpson formula esetén r=3!

In[51]:=
```
polylist = {1, x, x^2, x^3, x^4};
```

In[52]:= $\int_a^b \text{polylist}\, dx$

Out[52]= $\left\{-a + b, -\frac{a^2}{2} + \frac{b^2}{2}, -\frac{a^3}{3} + \frac{b^3}{3}, -\frac{a^4}{4} + \frac{b^4}{4}, -\frac{a^5}{5} + \frac{b^5}{5}\right\}$

In[53]:= $\text{Table}\left[\frac{b-a}{6}\left(a^n + 4\left(\frac{a+b}{2}\right)^n + b^n\right), \{n, 0, 4\}\right] \text{ // Expand}$

Out[53]= $\left\{-a + b, -\frac{a^2}{2} + \frac{b^2}{2}, -\frac{a^3}{3} + \frac{b^3}{3}, -\frac{a^4}{4} + \frac{b^4}{4}, -\frac{5\,a^5}{24} + \frac{a^4\, b}{24} - \frac{a^3\, b^2}{12} + \frac{a^2\, b^3}{12} - \frac{a\, b^4}{24} + \frac{5\, b^5}{24}\right\}$

Pontossági rend: A trapéz formula esetén csak r=1!

In[56]:= $\text{Table}\left[\frac{b-a}{2}\,(a^n + b^n), \{n, 0, 2\}\right]$

Out[56]= $\left\{-a + b, \frac{1}{2}\,(-a + b)\,(a + b), \frac{1}{2}\,(-a + b)\,\left(a^2 + b^2\right)\right\}$

In[57]:= $\text{Simplify}\left[\text{Take}\left[\int_a^b \text{polylist}\, dx, 3\right] - \%\right]$

Out[57]= $\left\{0, 0, \frac{1}{6}\,(a - b)^3\right\}$

A Simpson formula alkalmazásával közelítjük a $\int_0^1 e^{-x^2}\, dx$ integrált. Becsüljük meg a hibát!

$-(b-a)^5\, f^{(IV)}[\xi]/2880$

32 × 90

In[73]:=
```
t = Table[Factor[D[E^(-x^2), {x, n}]], {n, 0, 4}]
```

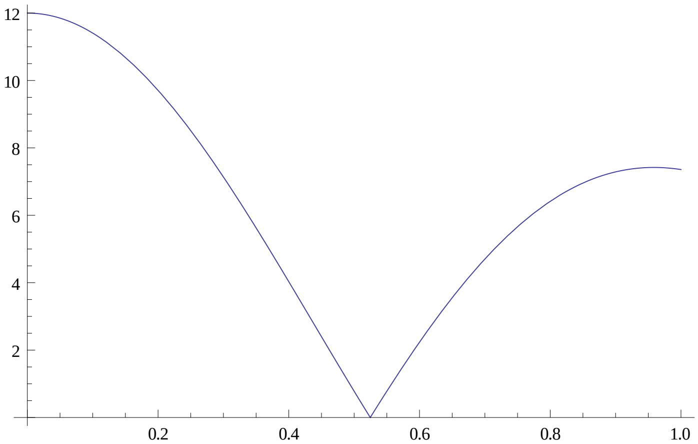
Out[73]= $\{e^{-x^2}, -2 e^{-x^2} x, 2 e^{-x^2} (-1+2 x^2), -4 e^{-x^2} x (-3+2 x^2), 4 e^{-x^2} (3-12 x^2+4 x^4)\}$

Nagyon durva becslése a negyedik deriváltnak (76), intervallum hossza 1.

In[74]:=
```
76 / (90 × 32) // N
```

Out[74]= 0.0263889

In[75]:=
```
Plot[Abs[Last[t]], {x, 0, 1}]
```

Out[75]=


In[76]:=
```
NIntegrate[E^(-x^2), {x, 0, 1}]
```

Out[76]= 0.746824

In[77]:=
```
1 / 6 (1 + 4 E^(-1 / 4) + 1 / E) // N
```

Out[77]= 0.74718

- **képlethibák levezetése**

TRAPÉZ

In[79]:=
```
Factor[∫_a^b (x - a) (x - b) dx]
```

Out[79]= $\frac{1}{6} (a-b)^3$

-1/ 12 f''(ξ) $(b-a)^3$

SIMPSON

In[80]:=
```
Factor[∫_a^b (x - a) (x - (a + b) / 2)^2 (x - b) dx]
```

Out[80]= $\frac{1}{120} (a-b)^5$

ÉRINTÔ (TÉGLA)

In[81]:=
```
Factor[∫_a^b (x - (a + b) / 2)^2 ⅆx]
```

Out[81]= $-\frac{1}{12}(a-b)^3$

Simpson 2 (szabály, összetett kvadratúraformula)

Simpson súlyfgv. (**W**eights for **S**impson method)

In[58]:=
```
Clear[WS];
WS[0, _] = 1;
WS[m_, m_] = 1;
WS[n_?EvenQ, _] := 2;
WS[n_?OddQ, _] := 4;
```

In[63]:=
```
Sum[WS[i, 4] F[i], {i, 0, 4}]
```

Out[63]= F[0] + 4 F[1] + 2 F[2] + 4 F[3] + F[4]

In[64]:=
```
Sum[WS[i, 10] F[i], {i, 0, 10}]
```

Out[64]= F[0] + 4 F[1] + 2 F[2] + 4 F[3] + 2 F[4] + 4 F[5] + 2 F[6] + 4 F[7] + 2 F[8] + 4 F[9] + F[10]

$\int_1^2 \frac{1}{x}\, dx$ határozott integrált közelítése (f primitiválható)

In[65]:=
```
f[x_] = 1 / x;
```

m=10, h=1/ 10 alappontok: $x_k = 1 + k / 10$; $(k = 0, 1, 2, \ldots, 10)$

In[66]:=
```
m = 10; h = 1/10;
```

In[67]:=
```
∑_{k=0}^m F[1 + k / 10] WS[k, m]
```

Out[67]= $F[1] + 4 F[\frac{11}{10}] + 2 F[\frac{6}{5}] + 4 F[\frac{13}{10}] + 2 F[\frac{7}{5}] + 4 F[\frac{3}{2}] + 2 F[\frac{8}{5}] + 4 F[\frac{17}{10}] + 2 F[\frac{9}{5}] + 4 F[\frac{19}{10}] + F[2]$

In[68]:=
```
h/3 ∑_{k=0}^m f[1 + k / 10] WS[k, m]
```

Out[68]= $\frac{48\,408\,065}{69\,837\,768}$

In[69]:=
```
N[%]
```

Out[69]= 0.69315

In[70]:=
```
N[Log[2]]
```

Out[70]= 0.693147

In[71]:=
```
∫_1^2 f[x] ⅆx
```

Out[71]= Log[2]

In[72]:=
```
N[%]
```

Out[72]= 0.693147

Megjegyzés. m növelésével hiba csökkenthetõ! $1/180\ \frac{-(b-a)^5}{m^4}\ f^{(IV)}[\xi]$