Two-dimensional moduli spaces of irregular Higgs bundles

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We give a complete description of the two-dimensional moduli spaces of stable Higgs bundles of rank 2 over $\mathbb{C}P^1$ with unique pole of order 4 as singularity, having regular leading-order term, and endowed with a generic compatible parabolic structure such that the parabolic degree of the Higgs bundle is 0.

The motivation of this study is that the moduli spaces of irregular Higgs bundles are linked to the generalization of Hodge structures, Riemann–Hilbert correspondence and integrable systems.

In our study the Higgs bundles are $(\mathcal{E}, \Theta)$ pairs, where $\mathcal{E}$ is a rank 2 vector bundle over $\mathbb{C}P^1$. If K denotes the canonical holomorphic line bundle on $\mathbb{C}P^1$, then Θ is the meromorphic section of $\mathsf{End}(\mathcal{E} \otimes K)$ and Θ has a single pole of order 4. One needs to distinguish two cases, depending on whether the leading-order term (at singularity) is a regular semi-simple endomorphism (untwisted case), or has non-vanishing nilpotent part (twisted case). The polar part of irregular Higgs bundles depends on some complex parameters.

It is known [1] that if one fixes finitely many points on $\mathbb{C}P^1$ and suitable polar parts for a Higgs bundle near those points, then one gets a holomorphic symplectic moduli space of Higgs bundles over $\mathbb{C}P^1$ with the given asymptotic behavior at the singularities. We denote this moduli space by $\mathcal{M}$. In our case, $\mathcal{M}$ turns out to be of complex dimension 2, and this imply that $\mathcal{M}$ is an elliptic fibration over a curve [3]. Our results will confirm this expectation, with one singular fiber of type $\widetilde{E}_7$ (untwisted case) or $\widetilde{E}_8$ (twisted case). On the other hand, there are several possibilities for the other singular fibers.

Theorem 1. *Assume that the polar part of the Higgs bundles is untwisted. Then $\mathcal{M}$ is biregular to the complement of the fiber at infinity ($\widetilde{E}_7$) in an elliptic fibration of the rational elliptic surface such that the set of other singular fibers of the fibration is:*

(1) a type III fiber if $\Delta = 0$ and $\lambda_+ = 0$;

(2) a type II and an I_1 fiber if $\Delta = 0$ and $\lambda_+ \neq 0$;

(3) an I_2 and an I_1 fiber if $\Delta \neq 0$ and $\lambda_+ = 0$;

(4) and three I_1 fibers otherwise,

where Δ and λ_+ depend on the complex parameters of the polar part of irregular Higgs bundles. (For the definition of various types of singular fibers see [2].)

Similarly to Theorem 1, there is a theorem provides a complete description of the singular fibers of the fibration in the twisted case.

[1] O. Biquard, Ph. Boalch, Wild non-abelian Hodge theory on curves, *Compos. Math.*, **140 (1)**, (2004), 179–204.

[2] K. Kodaira, On compact analytic surfaces: II, *Ann. Math.*, **77**, (1963), 563–626.

[3] Sz. Szabó, The birational geometry of irregular Higgs bundles, (2015), `arXiv:1502.02003`.