Approximate Jensen-convexity and convexity with respect to a subfield

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In this talk we show that if a function satisfies the Jensen-inequality (or the inequality describing $\mathbb{Q}$-convexity) with an appropriate error term then the function is Jensen-convex (without error) as well.

First we consider a function f, which is defined on an open interval I of $\mathbb{R}$. Let ℓ_I be the length of the interval I and define J_I as $]0, \ell_I[$, furthermore let $\psi : J_I \to [0, +\infty[$ be such that $\lim\limits_{t\to 0+} \frac{\psi(t)}{t^2} = 0$. We prove that if $f : I \to \mathbb{R}$ satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} + \psi(|x-y|)$$

for every $x, y \in I$, then f is Jensen-convex.

We also prove that if a real function f, which is defined on a K-algebraically open K-convex subset D of a vector space X over K (where K is a subfield of $\mathbb{R}$), satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + c\left[\lambda(1-\lambda)\left|x-y\right|\right]^p$$

for every $x, y \in D$ and $\lambda \in [0,1] \cap K$, with a fixed non-negative real number c and a fixed exponent $p > 1$, then it has to be K-convex , i.e., f satisfies the above inequality with $c = 0$ as well. Considering $K = \mathbb{Q}$, we can see that the last statement can be applied for (approximately) Jensen-convex functions.

Moreover we show that the $(\alpha, \mathbb{F})$-convexity of a function f is equivalent to two other statements. Namely, a function f is $(\alpha, \mathbb{F})$-convex on a nonempty convex subset D of a real linear space X if and only if the inequality

$$\frac{f(u) - f(u-sh) - \alpha(-sh)}{s} \leq \frac{f(u+rh) - f(u) + \alpha(rh)}{r}$$

is satisfied for all $r, s \in \mathbb{F}_+,\ u \in D,\ h \in X$ (where $u - sh, u + rh \in D$), or there exists a function A (defined on $D \times X$) such that

$$f(u+rh) - f(u) \leq r\,A(u,h) - \alpha(rh)$$

for all $u \in D,\ r \in \mathbb{F}\ h \in X$ (where $u + rh \in D$).

We also prove that under certain conditions for the functions f and α, the mapping A described above can be written as

$$A(u,h) = \lim_{s\to,\ s\in\mathbb{F}_+} \frac{f(u+sh) - f(u)}{s}$$

for all $x \in D,\ h \in X$. Moreover, the mapping $h \mapsto A(u,h)$ is positive $\mathbb{F}$-homogeneous and subadditive for every $u \in D$.

This is a joint work with Zoltán Boros from the University of Debrecen.