Positive operators arising from contractions

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Let $\mathcal{H}$ be a complex Hilbert space. If $T \in \mathcal{B}(\mathcal{H})$ is a contraction i.e..: $\|T\| \leq 1$, then the sequence $\{T^{*n}T^n\}_{n=1}^{\infty}$ of positive operators is decreasing, so it has a positive limit in the strong operator topology (SOT):

$$A_T = \lim_{n\to\infty} T^{*n}T^n.$$

We say that A_T *is induced by* T, or A_T is the *asymptotic limit* of T. The subspace

$$\mathcal{N}(A_T) = \mathcal{H}_0(T) := \{x \in \mathcal{H} \colon \lim_{n\to\infty} \|T^n x\| = 0\}$$

is hyperinvariant for T and it is called the *stable subspace* of T. The subspace

$$\mathcal{N}(A_T - I) = \mathcal{H}_1(T) := \{x \in \mathcal{H} \colon \lim_{n\to\infty} \|T^n x\| = \|x\|\}$$

is the largest invariant subspace where T is an isometry.
We will describe those positive operators that arises from a contraction in such a way. Moreover we can ensure uniform convergence, and expect from the case when $0 < \dim \mathcal{H}_1 < \aleph_0$ we can choose a co-stable contraction i.e.: $\mathcal{H}_0(T^*) = 0$. After that we give some sufficient condition for two contraction, having the same asymptotic limit.

Keywords: Hilbert space contraction, asymptotic limit, positive operators

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