Slicing the Sierpiński gasket

Balázs Bárány[1]

Denote by $\Lambda \subset \mathbb{R}^2$ the *right-angle Sierpiński gasket.* Precisely, Λ is the attractor of the Iterated Function System (IFS)

$$\left\{ S_0(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right),\ S_1(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right),\ S_2(x,y) = \left(\frac{x}{2} + \frac{1}{4}, \frac{y}{2} + \frac{\sqrt{3}}{4}\right) \right\}.$$

It is well known since S_0, S_1, S_2 are contractions that the Λ set is the unique non-empty compact set satisfying

$$\Lambda = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\dots,i_n=0}^{2} S_{i_1} \circ \cdots \circ S_{i_n}([0,1]^2) \text{ and } \Lambda = S_0(\Lambda) \cup S_1(\Lambda) \cup S_2(\Lambda).$$

We investigate the dimension of intersections of the Sierpiński gasket with lines. Let us denote the Hausdorff dimension of a bounded set V by $\dim_H V$ and respectively the box dimension by $\dim_B V$. Denote the s-dimensional Hausdorff measure by $\mathcal{H}^s$. It is well known that $s := \dim_H \Lambda = \dim_B \Lambda = \frac{\log 3}{\log 2}$ and $0 < \mathcal{H}^s(\Lambda) < \infty$.

Define proj_θ as the projection onto the line through the origin making angle θ with the x-axis. Our goal is to analyze the dimension theory of the slices $E_{\theta,a} = L_{\theta,a} \cap \Lambda$, where $L_{\theta,a} = \{(x,y) : \mathrm{proj}_\theta(x,y) = a\} = \{(x, a + x\tan\theta) : x \in \mathbb{R}\}$ for $a \in \mathrm{proj}_\theta(\Lambda)$. Denote by ν the natural self-similar measure of Λ. That is, $\nu = \frac{\mathcal{H}^s|_\Lambda}{\mathcal{H}^s(\Lambda)}$. In this case, ν satisfies that

$$\nu = \sum_{i=0}^{2} \frac{1}{3} \nu \circ S_i^{-1}.$$

Denote by ν_θ the projection of ν by angle θ. That is, $\nu_\theta = \nu \circ \mathrm{proj}_\theta^{-1}$. Similarly, let Λ_θ be the projection of Λ.

Theorem 1. *Let us suppose that* $\tan\theta \in \mathbb{Q}$ *and* $\theta \in (0, \frac{\pi}{2})$. *Then there exist constants* $\alpha(\theta), \beta(\theta)$ *depending only on* θ *such that*

(1) $\alpha(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} < s - 1$ *for Lebesgue almost all* $a \in \Lambda_\theta$,
(2) $\beta(\theta) := \dim_B E_{\theta,a} = \dim_H E_{\theta,a} > s - 1$ *for* ν_θ*-almost all* $a \in \Lambda_\theta$.

If $\tan\theta \in \mathbb{Q}$ then proj_θ satisfies a dimension conservation formula, that is,

$$\beta(\theta) + \dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} = \beta(\theta)\} = s.$$

Moreover, we provide a multifractal analysis for the set of points in the projection for which the associated slice has a prescribed dimension. We describe the function

$$\chi(\delta) = \dim_H \{a \in \Lambda_\theta : \dim_H E_{\theta,a} = \delta\}$$

for every $\alpha(\theta) \le \delta \le\ b_{\max}$, where $b_{\max}$ is a constant depending on θ and $\tan\theta \in \mathbb{Q}$. We will prove that the function χ is decreasing, continuous and concave.

The talk is based on [1] which is a joint work with Károly Simon and Andrew Ferguson.

The results descussed above are supported by the grant TÁMOP - 4.2.2.B-10/1–2010-0009.

References

[1] B. Bárány, A. Ferguson, K. Simon: Slicing the Sierpiński gasket, to appear in Nonlinearity , (2012).

[1]Department of Stochastics, Institute of Mathematics, Technical University of Budapest, 1521 Budapest, P.O.Box 91, Hungary, balubsheep@gmail.com, www.math.bme.hu/∼balubs