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# Periodic Solutions for an Equation with Delay

We study a scalar delay differential equation of the form

$$\dot{x}(t) = -\mu x(t) + f\left(x\left(t-1\right)\right), \tag{1}$$

where $\mu > 0$ and $f$ is a so called feedback function. Usually $f$ is chosen to be continuous or it is a step function. Such equations model the voltage of a single neuron or the synchronized behavior of neurons in a system. Many results are given. For strictly increasing and smooth feedback functions, the dynamics is described by Krisztin, Walter and Wu.

Because of the presence of the delay, the initial data of any solution is not a real number but a continuous function with domain $[-1, 0]$; one has to give $\varphi \in C = C\left(\left[-1, 0\right], \mathbb{R}\right)$ to obtain a solution $x^{\varphi}$ on $[-1, \infty)$. (A continuous function $x^{\varphi} : [-1, \infty) \to \mathbb{R}$ is a solution of Eq.(1) with initial function $\varphi$ if it is differentiable a.e. on $(0, \infty)$, satisfies the equation a.e. on $(0, \infty)$ and $x^{\varphi}|_{[-1,0]} = \varphi$.) The solution is unique and it is easily given by the so called variation-of-constant formula: if $x^{\varphi}$ is already known for $t \in [-1, n]$, then for $t \in (n, n+1]$,

$$x^{\varphi}\left(t\right) = x^{\varphi}\left(n\right) e^{-\mu(t-n)} + \int_{n}^{t} e^{-\mu(t-s)} f\left(x^{\varphi}\left(s-1\right)\right) \mathrm{d}s.$$

As the phase space is $C = C\left(\left[-1, 0\right], \mathbb{R}\right)$, most of the problems related to Eq.(1) are of infinite dimension and therefore need deep technical tools to solve.

To understand the asymptotic behavior of the solutions of Eq.(1), it is essential to describe the periodic solutions of Eq.(1). The aim of the talk is to show that if $f$ is a step function or a "smoothed" step function, then the periodic solutions are given by the fixed points of finite dimensional maps. Then, if the constructed periodic solutions are hyperbolic, one can prove the existence of periodic solutions for feedback functions "close" to the previous ones applying the theory of Poincaré maps.

A further goal is to describe the connecting orbits among periodic solutions and equilibrium points. Tools of the proof are: the Floquet theory, a discrete Lyapunov functional introduced by Mallet-Paret and Sell and the Poincaré - Bendixson Theorem proven by the same authors.