On the Hall product of preinjective Kronecker modules

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Let K be the **Kronecker quiver**, k a finite field (or the field of complex numbers), kK the path algebra of K over k (called the **Kronecker algebra**) and mod-kK the category of finitely generated right modules over the Kronecker algebra. We identify the category mod-kK with the category rep-kK of the finite dimensional k-representations of the Kronecker quiver. The indecomposables in mod-kK are divided into three families: the preprojectives, the regulars and the preinjectives. The preprojective (respectively) preinjective indecomposable modules are up to isomorphism uniquely determined by their dimension vectors. For $n \in \mathbb{N}$ we will denote by P_n (respectively by I_n) the indecomposable preprojective module of dimension $(n+1, n)$ (respectively the indecomposable preinjective module of dimension $(n, n+1)$. So P_0, P_1 are the projective indecomposable modules and I_0, I_1 are the injective indecomposable modules. P_n and I_n are isomorphic as representations with

$$P_n : k^{n+1} \underset{\binom{E_n}{0}}{\overset{\binom{0}{E_n}}{\leftleftarrows}} k^n \quad \text{and} \quad I_n : k^n \underset{(E_n\,0)}{\overset{(0\,E_n)}{\leftleftarrows}} k^{n+1} \ ,$$

where E_n is the identity matrix.

By the Krull-Schmidt theorem, every module $M \in$ mod-kK has the following decomposition up to isomorphism (for a partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ we are using the notation $R_p(\lambda) = R_p(\lambda_1) \oplus \cdots \oplus R_p(\lambda_s)$):

$$M \cong (P_{c_1} \oplus \cdots \oplus P_{c_n}) \oplus (\oplus_{p \in \mathbb{P}^1_k} R_p(\lambda^p)) \oplus (I_{d_1} \oplus \cdots \oplus I_{d_m}),$$

where

(1) $(c_1, \ldots, c_n)$ is a finite increasing sequence of non-negative integers

(2) λ^p is a partition for every $p \in \mathbb{P}^1_k$ (the projective line over k)

(3) $(d_1, \ldots, d_m)$ is a finite decreasing sequence of non-negative integers

The sequences from (1), (2) and (3) are called the **Kronecker invariants** of the module M. They determine M up to isomorphism and correspond to the classical Kronecker invariants in the following manner: the minimal indices for rows correspond to the integers from sequence (1) parameterizing the preprojective part, the minimal indices for columns correspond to integers from (3) parameterizing the preinjective part and the finite (and infinite) elementary divisors correspond to the partitions in (2) – a partition λ^p describes the dimensions of the Jordan blocks corresponding to p.

When $k = \mathbb{C}$, we identify a (linear) matrix pencil $A + \lambda B \in \mathcal{M}_{m,n}(k[\lambda])$ with the pair of matrices (A, B) and further with the representation of the **Kronecker module** $M_{A,B} : k^m \underset{B}{\overset{A}{\leftleftarrows}} k^n$. The strict equivalence $A + \lambda B \sim A' + \lambda B'$ means isomorphism of modules $M_{A,B} \cong M_{A',B'}$ and a pencil $A' + \lambda B'$ is a subpencil of $A + \lambda B$ iff the module $M_{A',B'}$ is a subfactor of $M_{A,B}$ (i.e. there is a module N such that $M_{A',B'} \twoheadleftarrow N \hookrightarrow M_{A,B}$ or equivalently there is a module L such that $M_{A,B} \twoheadrightarrow L \hookleftarrow M_{A',B'}$. We recall that a pencil $A' + \lambda B'$ is called a **subpencil** of $A + \lambda B$ if there are pencils $A_{12} + \lambda B_{12}$, $A_{21} + \lambda B_{21}$ and $A_{22} + \lambda B_{22}$ such that

$$A + \lambda B \sim \begin{pmatrix} A' + \lambda B' & A_{12} + \lambda B_{12} \\ A_{21} + \lambda B_{21} & A_{22} + \lambda B_{22} \end{pmatrix}.$$

The **matrix subpencil problem** (i.e. if $A + \lambda B$ and $A' + \lambda B'$ are linear matrix pencils, find necessary and sufficient conditions in terms of their classical Kronecker invariants for $A' + \lambda B'$ to be subpencil of $A + \lambda B$) is an important open problem with numerous interdisciplinary applications. Our work represents some steps towards solving this challenge, using the Hall algebra approach.

We denote by $[M]$ the isomorphism class of $M \in$ mod-kK. The **Hall algebra** $\mathcal{H}(kK, \mathbb{Q})$ associated to the Kronecker algebra kK is the free $\mathbb{Q}$-space having as basis the isomorphism classes in mod-kK together with a multiplication defined by $[N_1][N_2] = \sum_{[M]} F^M_{N_1 N_2}[M]$, where the structure constants $F^M_{N_1 N_2} = |\{M \supseteq U | U \cong N_2, M/U \cong N_1\}|$ are called the **Hall numbers**. Using associativity we have $[N_1] \ldots [N_t] = \sum_{[M]} F^M_{N_1 \ldots N_t}[M]$, where $F^M_{N_1 \ldots N_t} = |\{M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0 | M_{i-1}/M_i \cong N_i, i = \overline{1,t}\}|$ (the **Hall product**).

The ground and motivation for our work is the following result: for $M, M' \in$ mod-kK, M' is a subfactor of $M \iff F^M_{XM'Y} \neq 0$ for some $X, Y \in$ mod-$kK \iff [M] \in \{[X][M'][Y]\}$ for some $X, Y \in$ mod-$kK \iff$ $[M] \in \{[X_1] \ldots [X_n][M'][Y_1] \ldots [Y_m]\}$ for some $m, n \in \mathbb{N}^*$ and $X_1, \ldots X_n, Y_1, \ldots Y_m \in$ mod-kK.

We study combinatorial aspects of the Hall products (having multiple terms) of preprojective and preinjective modules in mod-kK. We describe the Hall product of these classes of modules and some methods, algorithms to compute them.

References

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