Doubly biased Connectivity Game

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Positional game is a pair $(V, \mathcal{F})$, where V is a finite set and $\mathcal{F} \subseteq 2^V$. We refer to V as the *board* and $\mathcal{F}$ as the *winning sets*. We look at the so-called *Maker-Breaker games*, played by Maker and Breaker who take turns in occupying the previously unoccupied elements of the board. Maker's goal is to occupy all the elements of a winning set by the end of game, and Breaker's goal is to prevent him from doing that, i.e., his goal is to occupy at least one element in every winning set. In many games one of the players can win quite easily. To even out the players' chances to win *biased games* are introduced. For two positive integers a and b, in biased $(a : b)$ game Maker claims a elements in each move, and Breaker claims b elements in each move. Now, if Maker can win a $(1 : 1)$ game easily, a standard approach is to look at the same game with $(1 : b)$ bias, increasing b until the game becomes more balanced.

Chvátal and Erdős [1] were first to consider biased Maker-Breaker positional games played on the edge set of the complete graph $E(K_n)$, $n \to \infty$. In *Connectivity Game* winning sets are the edge sets of spanning trees of K_n, and Maker wins if he claims all the edges of one spanning tree. In [1] it is shown that Breaker wins the $(1 : b)$ Connectivity Game, if $b > \frac{n(1+\epsilon)}{\log n}$, $\epsilon > 0$. Gebauer and Szabó proved in [2] that Maker can win the same game, if $b \le (\ln n - \ln\ln n - 6)\frac{n}{\ln^2 n}$.

We study doubly biased $(a : b)$ Connectivity Game on $E(K_n)$, where both a and b can be greater than one. For each $a = a(n)$, our hope is to determine $b_0(a, n) = b_0(a)$, so that for $b < b_0(a)$ the game is Maker's win, and for $b > b_0(a)$ the game is Breaker's win. We refer to $b_0(a)$ as the *threshold bias for a*.

For $a = o(\ln n)$ we prove that

$$\frac{an}{\ln n} - \frac{an(\ln\ln n + a - 1)}{\ln^2 n} < b_0(a) < \frac{an}{\ln n} - (1 - o(1))\frac{an\ln a}{\ln^2 n}.$$

When $a = c\ln n$, $0 < c \le 1$, we obtain $\frac{cn}{c+1} - o(n) < b_0(a) < cn(1 - o(1))$. When $a = c\ln n, c > 1$, we get $\frac{cn}{c+1} - o(n) < b_0(a) < \frac{2cn}{2c+1} + o(n)$. When $a = \omega(\ln n)$ and $a = o\left(\frac{n}{\ln n}\right)$, then $n - \frac{n(\ln^2 n + a)}{a\ln n} < b_0(a) < n - (1 - o(1))\frac{n\ln n}{2a}$. When $a = \frac{cn}{\ln n}, c > 0$, we have $n - \frac{2\ln n\ln\ln n}{c}(1 + o(1)) < b_0(a) < n - (1 - o(1))\frac{\ln n\ln\ln n}{2c}$. For $a = \omega\left(\frac{n}{\ln n}\right)$, we obtain

$$n - \frac{2n(\ln n - \ln a + 2)}{a} < b_0(a) < n - (1 - o(1))\frac{(n - 2a)(\ln n - \ln 2a) + 4a}{2a}.$$

References

[1] V. Chvátal and P. Erdős, Biased Positional Games, *Annals of Discrete Math.* **2**(1978), 221-228.

[2] H. Gebauer and T. Szabó, Asymptotic random graph intuition for the biased connectivity game, *Random Structures & Algorithms* **35**(2010), 431-443.

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