Unique Periodic Solutions of a Delay Differential Equation with Piecewise Linear Feedback Function

Ábel Garab

We consider the delayed differential equation:

$$\dot{x}(t) = -a \cdot x(t) + b \cdot f(c \cdot x(t-1)), \tag{1}$$

where $a, b, c \in \mathbb{R}$ are positive parameters such that $a < bc$ and $f(\xi) = \frac{1}{2}(|\xi + 1| - |\xi - 1|$, for all $\xi \in \mathbb{R}$.

This equation is often applied in models of neural networks and time delay appears due to finite conduction velocities or synaptic transmission. In neural systems, periodic solutions are of great importance. Our purpose is to give sufficient conditions for nonexistence and uniqueness of periodic solutions of (1) with prescribed oscillation frequencies. Such conditions have been provided by Krisztin and Walther [2] in the case when f is an odd C^1 function satisfying some convexity properties.

To formulate our main result, we need some preparation. The natural phase space for (1) is $C := C([-1, 0], \mathbb{R})$. If $x(\cdot)$ is a real function on some interval I, then we let $x_t \in C$ be defined by

$$x_t(\theta) = x(t + \theta) \quad \text{for all} \quad \theta \in [-1, 0],$$

at least where it makes sense. Now, define the following functional: $V : C \setminus \{0\} \to \{0, 2, 4, \dots, \infty\}$,

$$V(\varphi) = \begin{cases} \mathrm{sc}(\varphi) & \text{if} \quad \mathrm{sc}(\varphi) \quad \text{is even or infinite,} \\ \mathrm{sc}(\varphi) + 1 & \text{if} \quad \mathrm{sc}(\varphi) \quad \text{is odd,} \end{cases}$$

where $\mathrm{sc}(\varphi)$ denotes the number of sign changes of φ on the interval $[-1, 0]$. According to [3], V is a discrete Lyapunov functional and consequently for any periodic solution $x : \mathbb{R} \to \mathbb{R}$ of (1), there exists an appropriate $k = k(x) \in \{0, 1, 2, \dots\}$ such that $V(x_t) = 2k$, for all $t \in \mathbb{R}$. Thus, we can write the last formula in a shorter way: $V(x) = 2k$. Now we are ready to formulate our theorem. The proof is based on the technique applied in [2].

Theorem. *Let $a, b, c > 0$ and $k \in \mathbb{N}$ be fixed such that $a < bc$ and let $\nu = \nu(a, b, c) = \sqrt{(bc)^2 - a^2} + \arccos \frac{a}{bc}$. Then the following statements hold.*

1. *If $\nu < 2k\pi$ then there exists no periodic solution of (1) in $V^{-1}(2l)$ where $l \geq k,\ l \in \mathbb{N}$.*

2. *If $2k\pi < \nu < (2k+2)\pi$, then equation (1) has at most one periodic solution in $V^{-1}(2l)$ for all $l \leq k,\ l \in \mathbb{N}$.*

3. *Otherwise, if $\nu = 2k\pi$ is the case, then equation (1) has infinitely many periodic solutions in $V^{-1}(2k)$ (namely the periodic solutions of the linearized equation of (1)), but all their ranges are contained in the interval $[-1/c, 1/c]$.*

Győri and Hartung [1] proved that in case of $a > b$ and $c = 1$, all solutions of equation (1) tends to an equilibrium point. After this, Vas [4] showed that if $b_0 = b_0(a)$ is defined by the equation $\sqrt{{b_0}^2 - a^2} + \arccos \frac{a}{b_0} = 2\pi$ and $b > b_0$, then there exists a periodic solution of (1). It ramained an open problem whether there exists a periodic function in the case of $a \geq b \geq b_0$. As a corollary of our theorem we obtain that there exists no periodic solution in this case. This is a joint work with Dr. Tibor Krisztin.

References

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