On the dimension theory of Iterated Function Systems with fixed point correspondence

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Let us denote the Hausdorff dimension of a compact subset Λ of $\mathbb{R}$ by $\dim_H \Lambda$ and respectively the Box dimension by $\dim_B \Lambda$.

Let $\{f_0, \dots, f_{m-1}\}$ be a family of contracting similarity map such that $|f_i(x) - f_i(y)| = |\lambda_i||x-y|$ for all x, y and for some $-1 < \lambda_i < 1$. Then there exists a unique, nonempty compact subset Λ of $\mathbb{R}$ which satisfies $\Lambda = \bigcup_{i=0}^{m-1} f_i(\Lambda)$. We call this set Λ the attractor of the iterated function system (IFS) $\{f_0(x), \dots, f_{m-1}(x)\}$. In this case we say that the attractor Λ (or the IFS itself) is self-similar.

It is well known that the Hausdorff dimension and the Box dimension of the attractor is the unique solution of

$$\sum_{i=0}^{m-1} |\lambda_i|^s = 1, \tag{0.1}$$

if the open set condition (OSC) holds, i.e. there exists a nonempty open set U such that $f_i(U) \subseteq U$ for every $i = 0, \dots, m-1$ and $f_i(U) \cap f_j(U) = \emptyset$ for every $i \neq j$, see [2].

Even if the OSC does not hold, the solution of equation (0.1) is called similarity dimension of the IFS. The similarity dimension is always an upper bound for the Hausdorff dimension of the attractor. In the case when the IFS has overlapping structure, i.e. the open set condition does not hold, the Hausdorff dimension of the attractor Λ of IFS $\{f_i(x) = \lambda_i x + d_i\}_{i=0}^{m-1}$ is

$$\dim_B \Lambda = \dim_H \Lambda = \min\{s, 1\} \text{ for a.e. } (d_0, \dots, d_{m-1}) \in \mathbb{R}^m \tag{0.2}$$

where s is the unique solution of (0.1), see [3].

In [1] we considered the IFS $\{\gamma x, \lambda x, \lambda x + 1\}$, $\gamma < \lambda$ on the real line. The problem of the computation of the dimension of the attractor of this IFS was raised by Pablo Shmerkin at the conference in Greifswald in 2008. The novelty of the result obtained in [1] about the dimension of Λ was to tackle the difficulty which comes from the fact that the first two maps have the same fixed point. It is clear that the IFS does not satisfy the OSC, and trivially we are not able to apply the formula (0.2). However, we are able to calculate the Hausdorff dimension for almost every contraction ratios. The talk is based on the paper [1].

Theorem 1. *Let Λ be the attractor of $\{\gamma x, \lambda x, \lambda x + 1\}$ such that $0 < \gamma < \lambda$ then $\dim_H \Lambda = \min\{1, s\}$ for Lebesgue-a.e. γ, where s is the unique solution of*

$$2\lambda^s + \gamma^s - \lambda^s \gamma^s = 1.$$

References

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