Play with LEGO!

The Kalmár-Steinhaus function and selective compounds of games

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Colloquium on Combinatorial Games
Luxembourg 10 May 2010

Outline

1. The Kalmár-Steinhaus function and selective compounds of games

2. Play with LEGO!

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2. Play with LEGO!

How to win a game?

Game: $\mathcal{G} = (P, M, T)$

- P is the set of positions
- $M \subseteq P \times P$ is the set of moves
- $T = \{p \in P \mid \nexists q \in P : (p, q) \in M\}$ is the set of terminal positions

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Normal play: the last player to move is the winner.

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Partition P into good and bad positions such that

$$\text{good} \xrightarrow{\forall} \text{bad} \quad \text{and} \quad \text{bad} \xrightarrow{\exists} \text{good}.$$

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Winning strategy: always move to a good position!

The Sprague-Grundy function

There is a unique function $\gamma\colon P \longrightarrow \mathbb{N}_0$ such that for all $p \in P$

$$\gamma(p) = \operatorname{mex}\{\gamma(q) \mid p \to q\} = \min\left(\mathbb{N}_0 \setminus \{\gamma(q) \mid p \to q\}\right).$$

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The good positions are exactly the zeros of this function:

$$p \text{ is good} \iff \gamma(p) = 0.$$

Playing several games simultaneously

There are many ways to play games $\mathcal{G}_1, \ldots, \mathcal{G}_n$ simultaneously:

- $\mathcal{G}_1 + \cdots + \mathcal{G}_n$ (sum, disjunctive compound): move in one of them

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The game of nim

- position: k heaps of beans
- move: take away some beans from one heap

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The game of nim

- position: k heaps of beans
- move: take away some beans from one heap

This is a sum of k one-heap nim games:

$$\mathcal{N}_k = \mathcal{N}_1 + \cdots + \mathcal{N}_1.$$

The SG function of the sum

Theorem

The exists a binary operation $\oplus$ *on* $\mathbb{N}_0$ *such that*

$$\gamma_{\mathcal{G}_1+\mathcal{G}_2}\left(p_1, p_2\right) = \gamma_{\mathcal{G}_1}\left(p_1\right) \oplus \gamma_{\mathcal{G}_2}\left(p_2\right).$$

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Corollary

$$\gamma_{\mathcal{N}_k}\left(n_1, \ldots, n_k\right) = \gamma_{\mathcal{N}_1}\left(n_1\right) \oplus \cdots \oplus \gamma_{\mathcal{N}_1}\left(n_k\right) = n_1 \oplus \cdots \oplus n_k$$

Nim addition

$\oplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Nim addition

⊕	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

binary addition without carrying
(bitwise XOR)

The Kalmár-Steinhaus function

There is a unique function $\varkappa \colon P \to \mathbb{N}_0$ such that for all $p \in P$

- if there is an even number in $\{\varkappa(q) \mid p \to q\}$, then

$$\varkappa(p) = 1 + \min\left(\{\varkappa(q) \mid p \to q\} \cap 2\mathbb{N}_0\right), \qquad \text{(\textbf{l}east \textbf{e}ven)}$$

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- if all numbers in $\{\varkappa(q) \mid p \longrightarrow q\}$ are odd, then

$$\varkappa(p) = 1 + \max\{\varkappa(q) \mid p \longrightarrow q\}. \qquad \text{(**g**reatest **o**dd)}$$

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$$\varkappa(p) = 1 + \min\left(\{\varkappa(q) \mid p \to q\} \cap 2\mathbb{N}_0\right), \qquad \textbf{(least even)}$$

- if all numbers in $\{\varkappa(q) \mid p \to q\}$ are odd, then

$$\varkappa(p) = 1 + \max\{\varkappa(q) \mid p \to q\}. \qquad \textbf{(greatest odd)}$$

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In short,

$$\varkappa(p) = 1 + \operatorname{lego}\{\varkappa(q) \mid p \to q\}.$$

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In short,

$$\varkappa(p) = 1 + \operatorname{lego}\{\varkappa(q) \mid p \to q\}.$$

The good positions are easily determined from this function:

$$p \text{ is good} \iff \varkappa(p) \text{ is even}.$$

The KS function of the conjunctive/selective compound

Theorem

$$\varkappa_{\mathcal{G}_1 \wedge \mathcal{G}_2}(p_1, p_2) = \min\left(\varkappa_{\mathcal{G}_1}(p_1), \varkappa_{\mathcal{G}_2}(p_2)\right) = \varkappa_{\mathcal{G}_1}(p_1) \wedge \varkappa_{\mathcal{G}_2}(p_2).$$

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Theorem

There exists a binary operation $\boxplus$ *on* $\mathbb{N}_0$ *such that*

$$\varkappa_{\mathcal{G}_1 \vee \mathcal{G}_2}(p_1, p_2) = \varkappa_{\mathcal{G}_1}(p_1) \boxplus \varkappa_{\mathcal{G}_2}(p_2).$$

The operation ⊞

⊞	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	5	5	7	7
2	2	3	4	5	6	7	8	9
3	3	3	5	5	7	7	9	9
4	4	5	6	7	8	9	10	11
5	5	5	7	7	9	9	11	11
6	6	7	8	9	10	11	12	13
7	7	7	9	9	11	11	13	13

The operation $\boxplus$

$\boxplus$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	5	5	7	7
2	2	3	4	5	6	7	8	9
3	3	3	5	5	7	7	9	9
4	4	5	6	7	8	9	10	11
5	5	5	7	7	9	9	11	11
6	6	7	8	9	10	11	12	13
7	7	7	9	9	11	11	13	13

$a \boxplus b =$

- $a + b$, if a or b is even
- $a + b - 1$, if a and b are odd

Grundy nim, disjunctive version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\gamma(n)$	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4

Grundy nim, disjunctive version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\gamma(n)$	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4

$$\begin{aligned}\gamma(10) &= \text{mex}\left(\gamma(1) \oplus \gamma(9), \gamma(2) \oplus \gamma(8), \gamma(3) \oplus \gamma(7), \gamma(4) \oplus \gamma(6)\right) \\ &= \text{mex}\left(0 \oplus 1, 0 \oplus 2, 1 \oplus 0, 0 \oplus 1\right) \\ &= \text{mex}\left(1, 2, 1, 1\right) = 0\end{aligned}$$

Grundy nim, disjunctive version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\gamma(n)$	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4

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$$n \text{ is good} \iff ???$$

Grundy nim, conjunctive version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\mathscr{K}(n)$	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Grundy nim, conjunctive version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\varkappa(n)$	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$$\begin{aligned}\varkappa(10) &= 1 + \text{lego}\left(\varkappa(1) \wedge \varkappa(9), \varkappa(2) \wedge \varkappa(8), \varkappa(3) \wedge \varkappa(7), \varkappa(4) \wedge \varkappa(6)\right) \\ &= 1 + \text{lego}\left(0 \wedge 1, 0 \wedge 1, 1 \wedge 1, 1 \wedge 1\right) \\ &= 1 + \text{lego}\left(0, 0, 1, 1\right) = 1 + 0 = 1\end{aligned}$$

Grundy nim, conjunctive version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\varkappa(n)$	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$$\begin{aligned}\varkappa(10) &= 1 + \text{lego}\left(\varkappa(1) \wedge \varkappa(9)\,,\varkappa(2) \wedge \varkappa(8)\,,\varkappa(3) \wedge \varkappa(7)\,,\varkappa(4) \wedge \varkappa(6)\right)\\ &= 1 + \text{lego}\left(0 \wedge 1, 0 \wedge 1, 1 \wedge 1, 1 \wedge 1\right)\\ &= 1 + \text{lego}\left(0, 0, 1, 1\right) = 1 + 0 = 1\end{aligned}$$

$$n \text{ is good} \iff n = 1, 2$$

Grundy nim, selective version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\varkappa(n)$	0	0	1	2	3	3	4	5	5	6	7	7	8	9	9	10	11	11

Grundy nim, selective version

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\varkappa(n)$	0	0	1	2	3	3	4	5	5	6	7	7	8	9	9	10	11	11

$$
\begin{aligned}
\varkappa(10) &= 1 + \text{lego}\left(\varkappa(1) \boxplus \varkappa(9), \varkappa(2) \boxplus \varkappa(8), \varkappa(3) \boxplus \varkappa(7), \varkappa(4) \boxplus \varkappa(6)\right) \\
&= 1 + \text{lego}\left(0 \boxplus 5, 0 \boxplus 5, 1 \boxplus 4, 2 \boxplus 3\right) \\
&= 1 + \text{lego}\left(5, 5, 5, 5\right) = 1 + 5 = 6
\end{aligned}
$$

Grundy nim, selective version

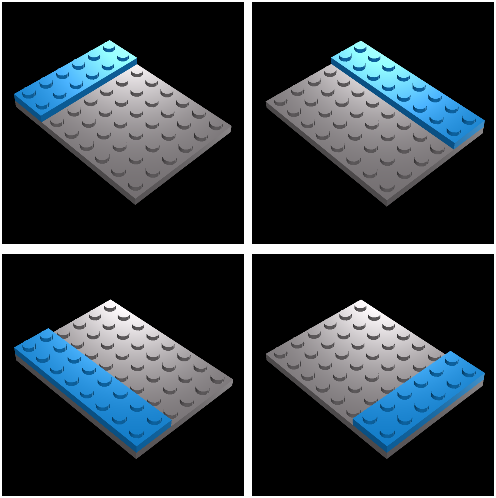
n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\varkappa(n)$	0	0	1	2	3	3	4	5	5	6	7	7	8	9	9	10	11	11

$$\begin{aligned}\varkappa(10) &= 1 + \text{lego}\left(\varkappa(1) \boxplus \varkappa(9), \varkappa(2) \boxplus \varkappa(8), \varkappa(3) \boxplus \varkappa(7), \varkappa(4) \boxplus \varkappa(6)\right) \\ &= 1 + \text{lego}\left(0 \boxplus 5, 0 \boxplus 5, 1 \boxplus 4, 2 \boxplus 3\right) \\ &= 1 + \text{lego}\left(5, 5, 5, 5\right) = 1 + 5 = 6\end{aligned}$$

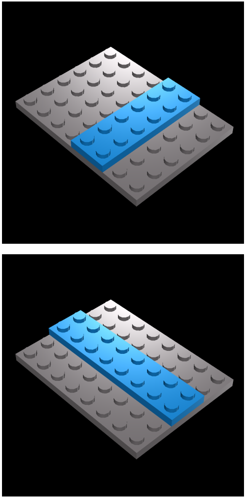
$$n \text{ is good} \iff n = 1, 2 \text{ or } n \equiv 1 \pmod 3$$

Outline

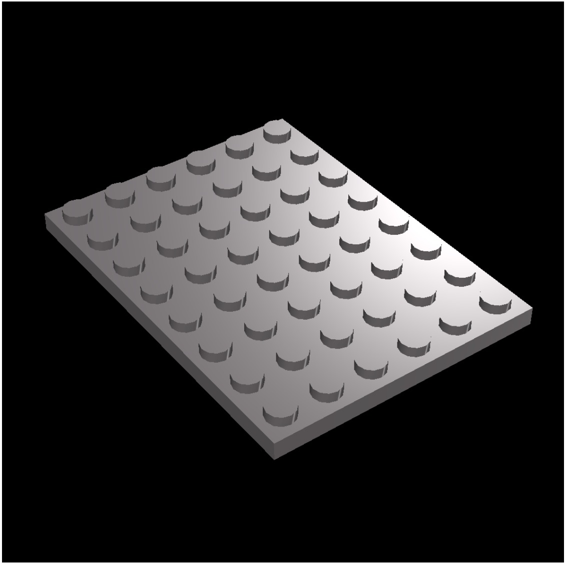
Side moves



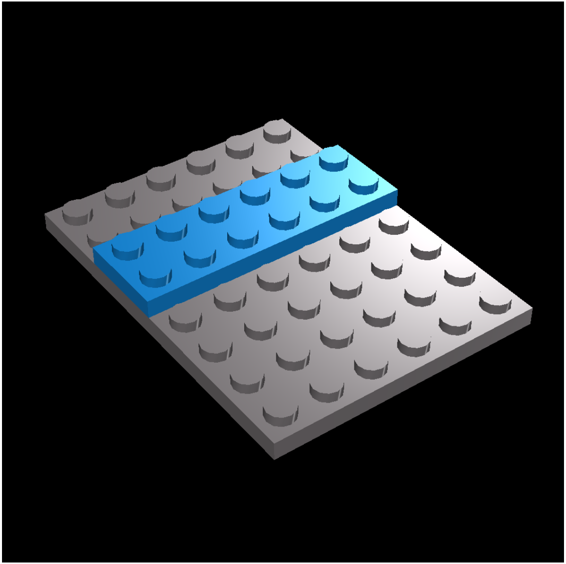
Middle moves



Playing LEGO(s/m,$h \leq \infty$) selectively



Playing LEGO(s/m, $h \leq \infty$) selectively



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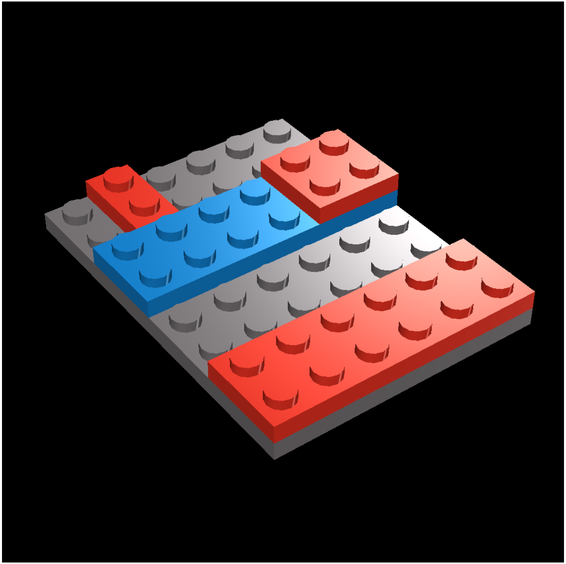

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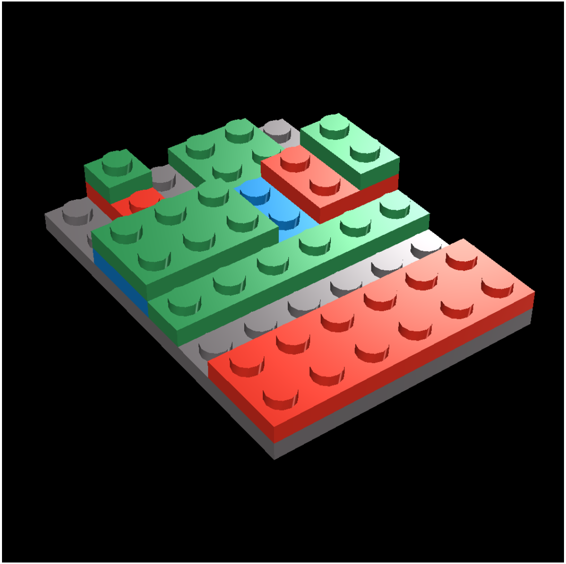

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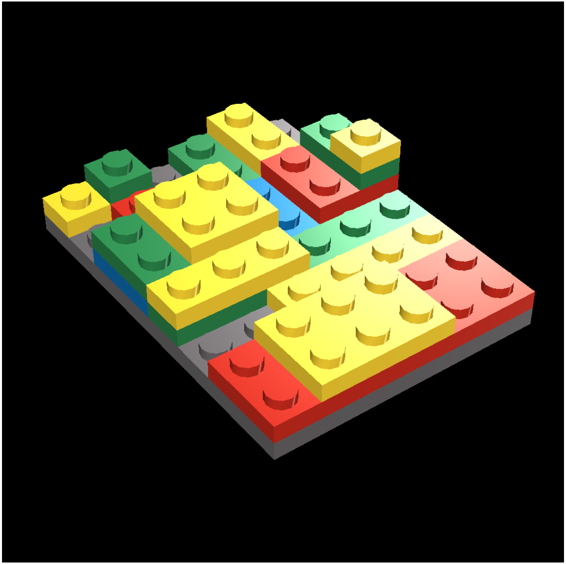

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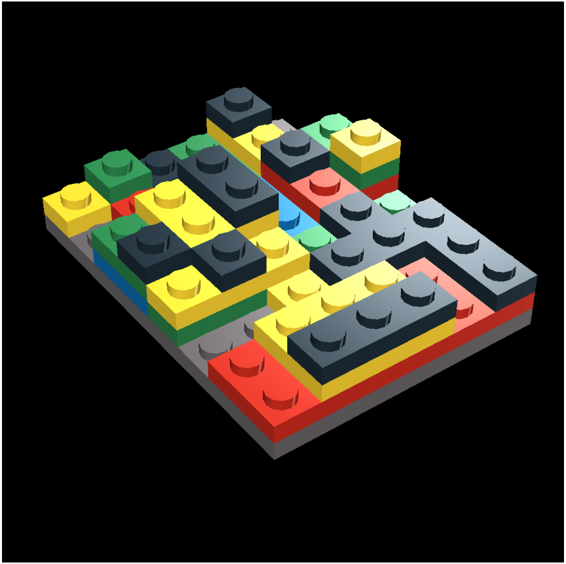

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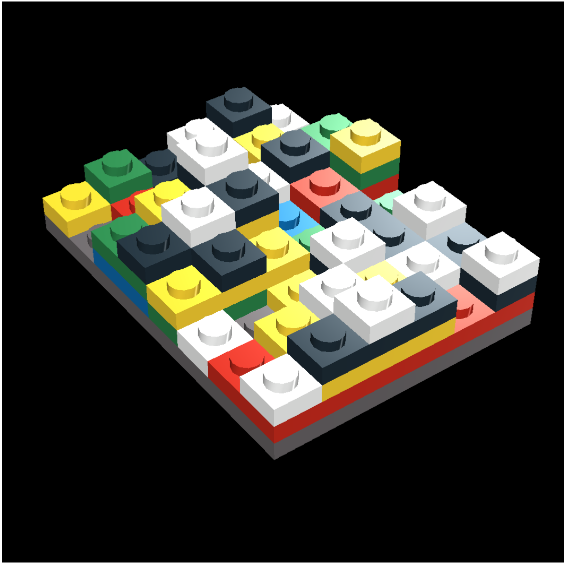


LEGO(s, $h \leq \infty$), disjunctive version

8	1	1	1	1	1	1	1	1
7	0	1	0	1	0	1	0	1
6	1	1	1	1	1	1	1	1
5	0	1	0	1	0	1	0	1
4	1	1	1	1	1	1	1	1
3	0	1	0	1	0	1	0	1
2	1	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0	1
γ	1	2	3	4	5	6	7	8

LEGO(s,$h \leq \infty$), disjunctive version

8	1	1	1	1	1	1	1	1
7	0	1	0	1	0	1	0	1
6	1	1	1	1	1	1	1	1
5	0	1	0	1	0	1	0	1
4	1	1	1	1	1	1	1	1
3	0	1	0	1	0	1	0	1
2	1	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0	1
γ	1	2	3	4	5	6	7	8

(a, b) is good

$$\Updownarrow$$

ab is odd

LEGO(s,$h \leq \infty$), conjunctive version

8	1	3	3	5	5	5	5	6
7	1	3	3	4	4	4	4	5
6	1	3	3	4	4	4	4	5
5	1	3	3	4	4	4	4	5
4	1	3	3	4	4	4	4	5
3	1	2	2	3	3	3	3	3
2	1	2	2	3	3	3	3	3
1	0	1	1	1	1	1	1	1
$\varkappa$	1	2	3	4	5	6	7	8

LEGO(s,$h \leq \infty$), conjunctive version

8	1	3	3	5	5	5	5	6
7	1	3	3	4	4	4	4	5
6	1	3	3	4	4	4	4	5
5	1	3	3	4	4	4	4	5
4	1	3	3	4	4	4	4	5
3	1	2	2	3	3	3	3	3
2	1	2	2	3	3	3	3	3
1	0	1	1	1	1	1	1	1
$\varkappa$	1	2	3	4	5	6	7	8

$$(a, b) \text{ is good} \Updownarrow \lfloor \log_2 a \rfloor = \lfloor \log_2 b \rfloor$$

LEGO(s, $h \leq \infty$), selective version

8	7	13	23	29	39	45	55	62
7	6	13	20	27	34	41	48	55
6	5	10	17	21	29	34	41	45
5	4	9	14	19	24	29	34	39
4	3	5	11	14	19	21	27	29
3	2	5	8	11	14	17	20	23
2	1	2	5	5	9	10	13	13
1	0	1	2	3	4	5	6	7
$\varkappa$	1	2	3	4	5	6	7	8

LEGO(s, $h \leq \infty$), selective version

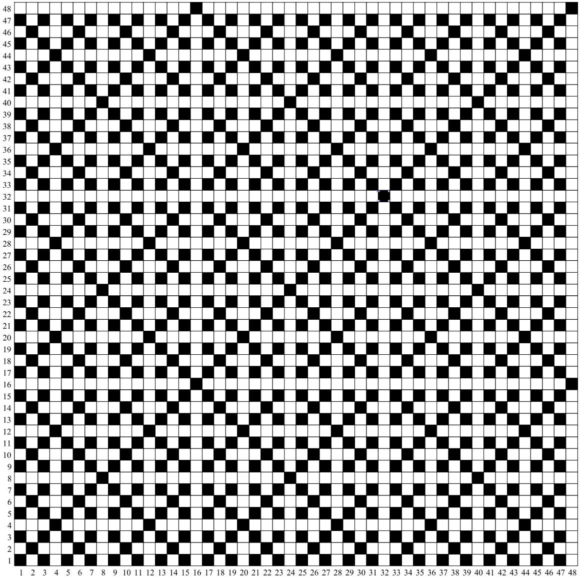
8	7	13	23	29	39	45	55	62
7	6	13	20	27	34	41	48	55
6	5	10	17	21	29	34	41	45
5	4	9	14	19	24	29	34	39
4	3	5	11	14	19	21	27	29
3	2	5	8	11	14	17	20	23
2	1	2	5	5	9	10	13	13
1	0	1	2	3	4	5	6	7
$\varkappa$	1	2	3	4	5	6	7	8

(a, b) is good

$\Updownarrow$

$?\,(a) = ?\,(b)$

The good positions $\text{LEGO}(s, h \leq \infty)$



$\text{good} \xrightarrow{\forall} \text{bad} \qquad \text{bad} \xrightarrow{\exists} \text{good}$

For $a \in \mathbb{N}$ let $\|a\| = n$ if the binary representation of a has the following form:

$$a = \cdots\ \mathbf{1}\ \overbrace{0\cdots0}^{n}.$$

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Key properties of the norm

For all $a \in \mathbb{N}$ we have

1. $\nexists a_1, a_2 : \|a\| = \|a_1\| = \|a_2\|$ and $a = a_1 + a_2$,
2. $\forall n < \|a\| \ \exists a_1, a_2 : n = \|a_1\| = \|a_2\|$ and $a = a_1 + a_2$.

Theorem

An $a \times b$ rectangle is good iff $\|a\| = \|b\|$; a position is good iff all its pieces are good.

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bad$\xrightarrow{\exists}$good: if $\|b\| < \|a\|$, then

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□

LEGO(s, $h \leq \infty$), selective version

8	7	13	23	29	39	45	55	62
7	6	13	20	27	34	41	48	55
6	5	10	17	21	29	34	41	45
5	4	9	14	19	24	29	34	39
4	3	5	11	14	19	21	27	29
3	2	5	8	11	14	17	20	23
2	1	2	5	5	9	10	13	13
1	0	1	2	3	4	5	6	7
$\varkappa$	1	2	3	4	5	6	7	8

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2	1	2	5	5	9	10	13	13
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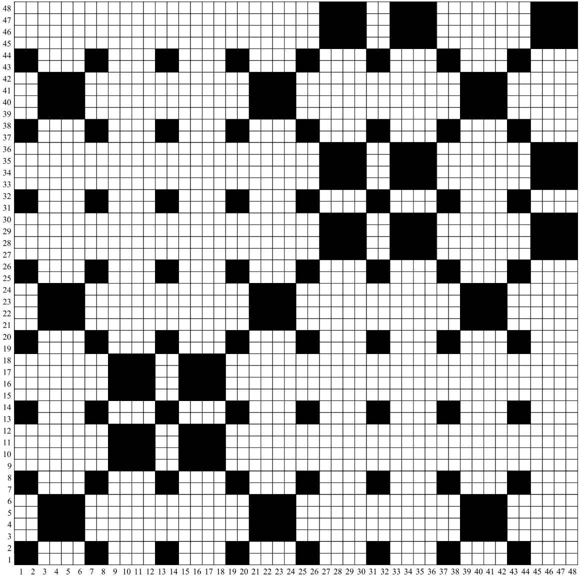
$\varkappa(a, b) =$

- $ab - 1$, if $\|a\| = 0$ or $\|b\| = 0$
- $ab - 2$, if $\|a\| = \|b\| \geq 1$
- $ab - 3$, otherwise

The good positions $\text{LEGO}(m, h \leq \infty)$



If a is even, then let $\|a\| = n$ if the ternary representation of a has one of the following two forms:

$$a = \cdots\ \mathbf{2}\ \overbrace{0\cdots 0}^{n},$$

$$a = \cdots\ \mathbf{1}\ \overbrace{{}^{0}\!/_{2}\cdots {}^{0}\!/_{2}\ \mathbf{1}\ 0\cdots 0}^{n}.$$

If a is odd, then let $\|a\| = n$ if the ternary representation of a has the following form:

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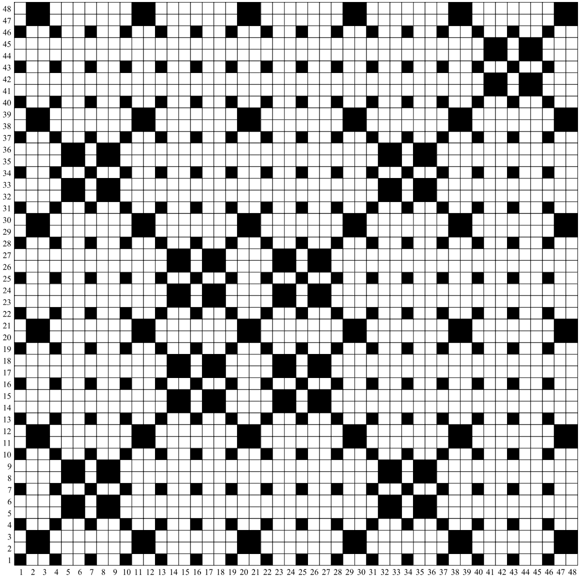
2. $\forall n < \|a\|\ \ \exists a_1, a_2, a_3 : n = \|a_1\| = \|a_2\| = \|a_3\|$ and $a = a_1 + a_2 + a_3$.

The good positions $LEGO(s/m, h \leq \infty)$



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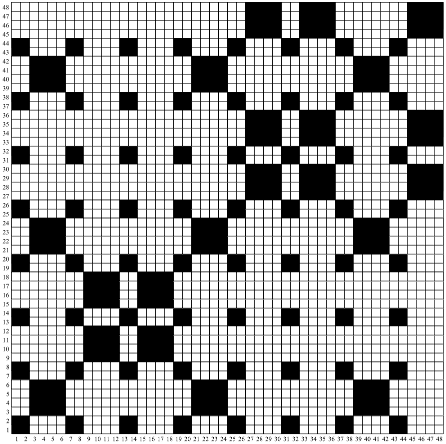
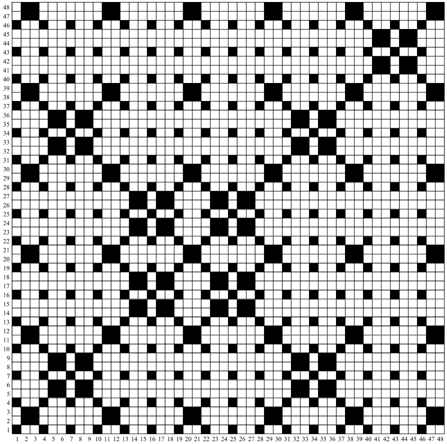
LEGO(m, $h \leq \infty$) vs. LEGO(s/m, $h \leq \infty$)

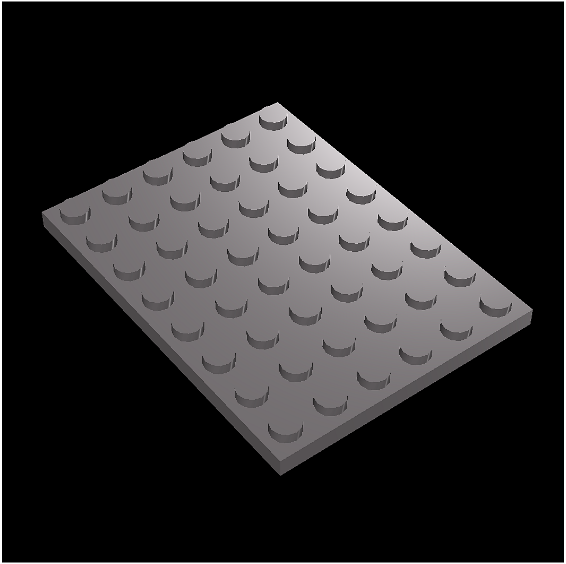




good $\xrightarrow{\forall}$ bad   bad $\xrightarrow{\exists}$ good

$(2a-1, 2b)$, $(2a, 2b-1)$, $(2a-1, 2b-1)$, $(2a, 2b)$ are good in LEGO(m, $h \leq \infty$) $\Longleftrightarrow$ (a, b) is good in LEGO(s/m, $h \leq \infty$)

Playing LEGO(s, $h \leq 2$) selectively



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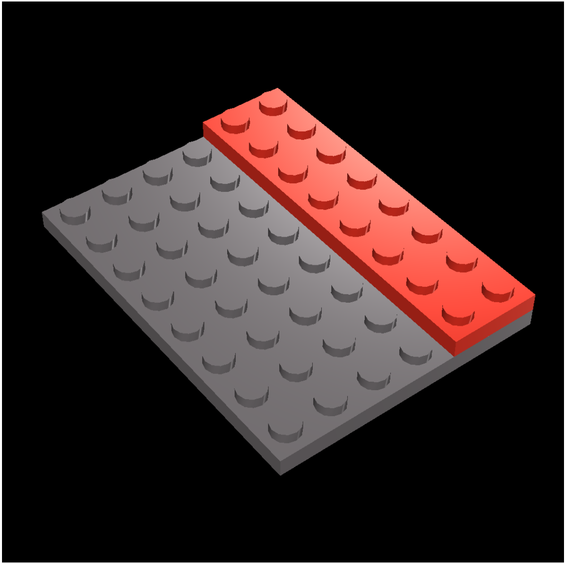

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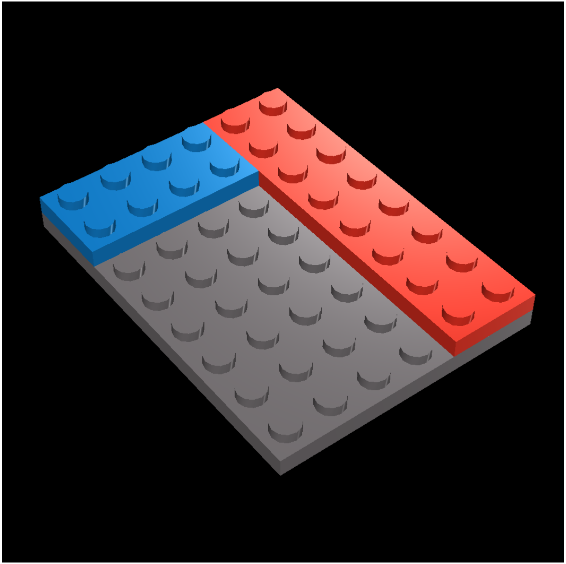

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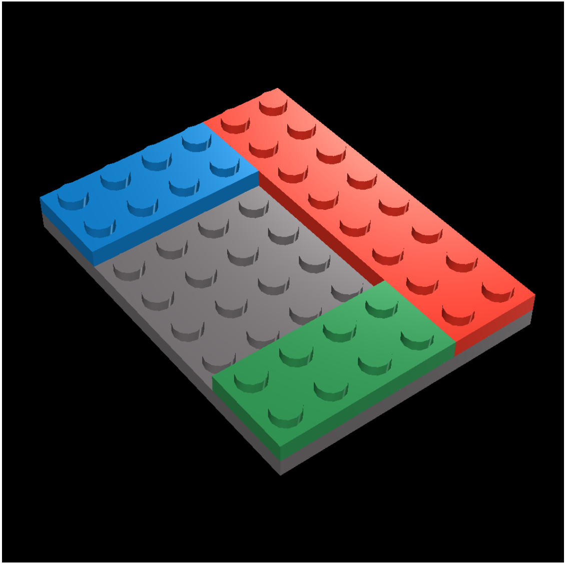

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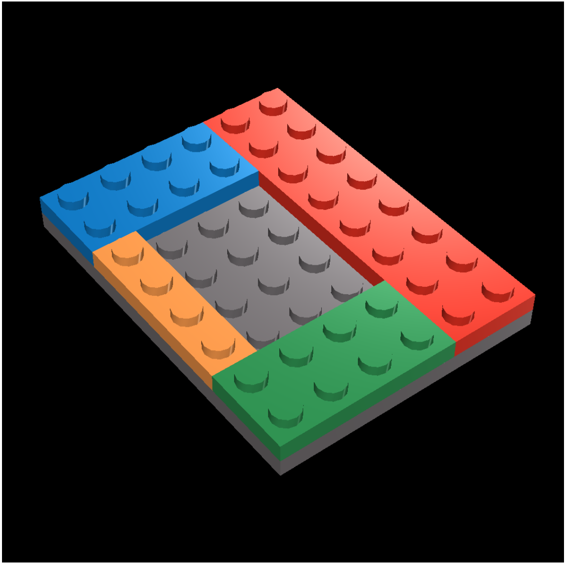

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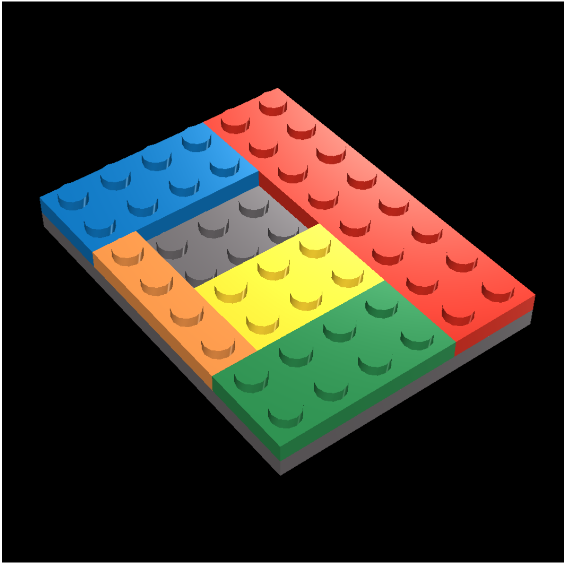

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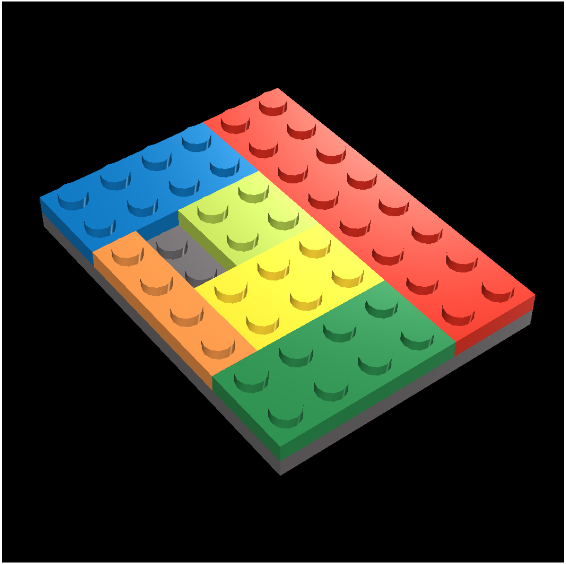

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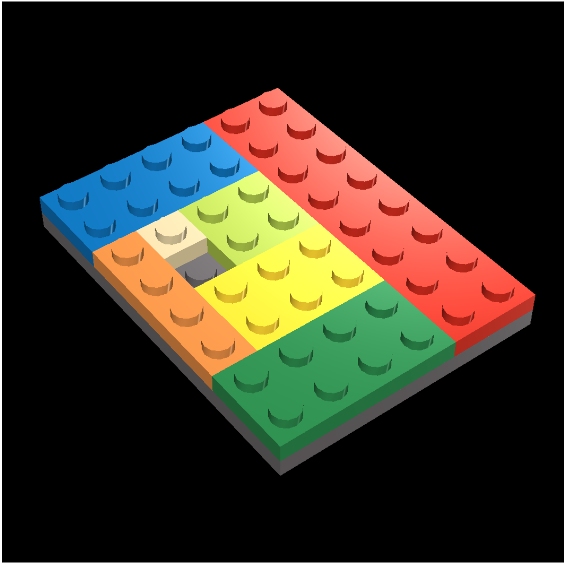

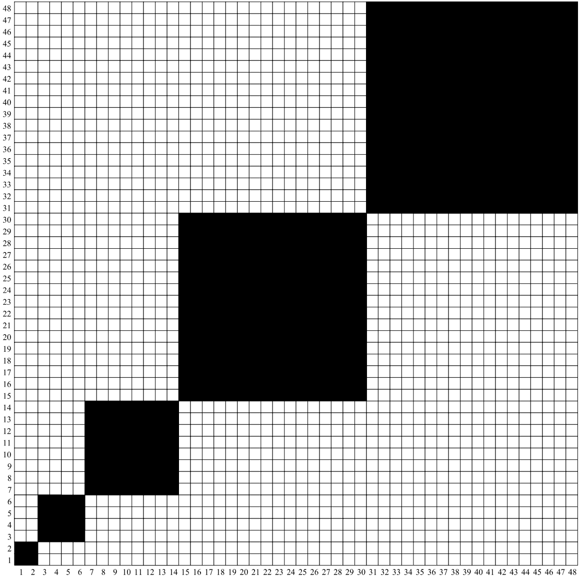

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48

For $a \in \mathbb{N}$ let $\|a\| = \lfloor \log_2 (a+1) \rfloor$.

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Hyper-LEGO

Theorem

In the d-dimensional version of any of the previously mentioned games

$$(a_1, \ldots, a_d) \text{ is good} \iff \|a_1\| \oplus \cdots \oplus \|a_d\| = 0.$$

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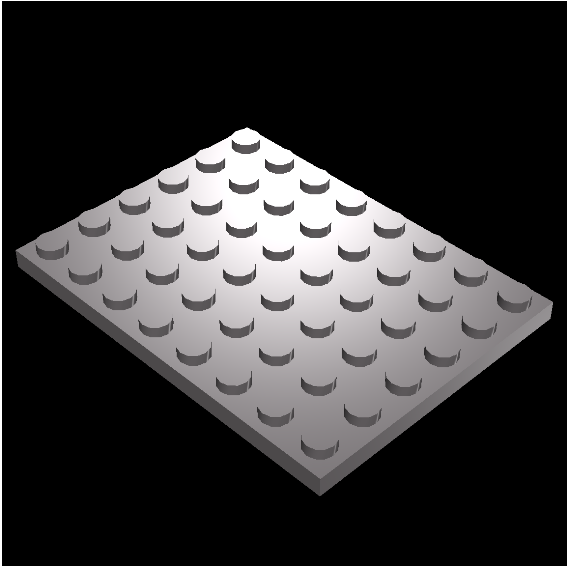
Proof.

Play "poker nim" with the norms. □

Playing LEGO(s, $h \leq 3$) selectively



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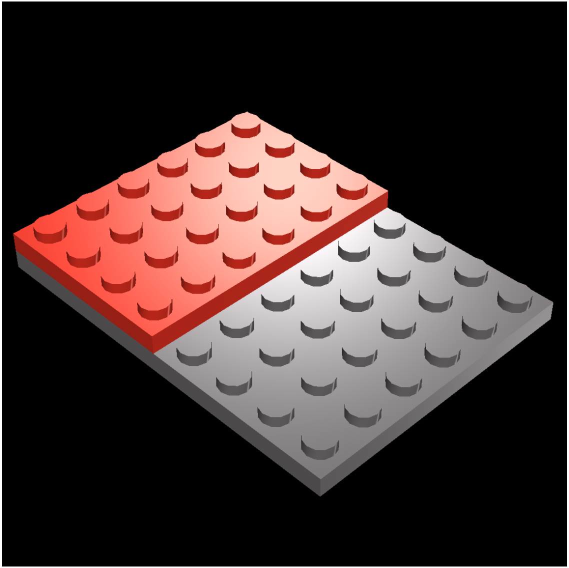

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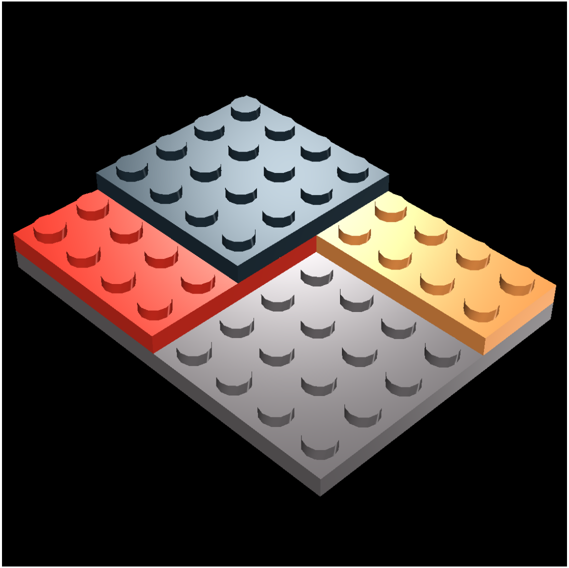

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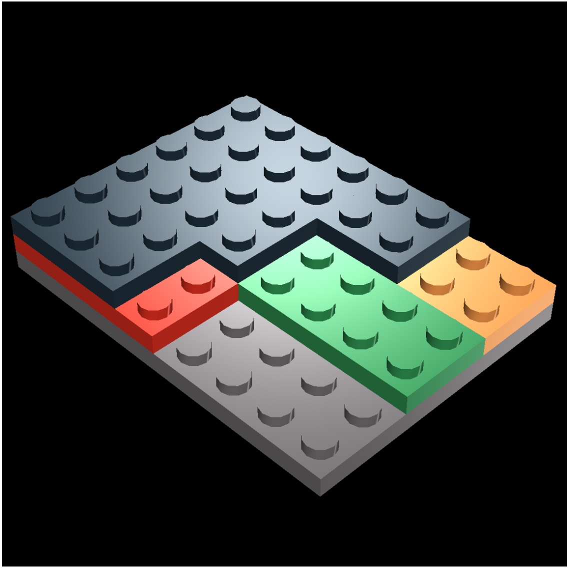

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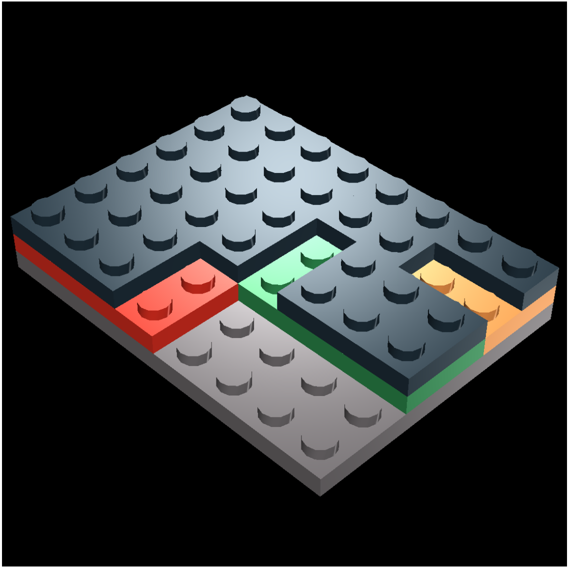

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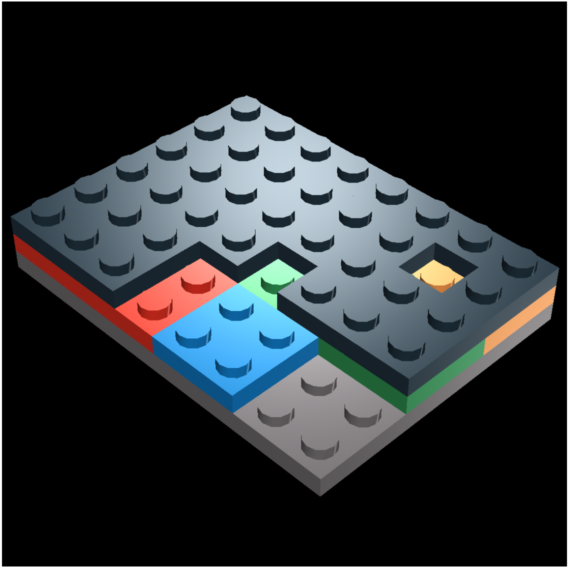

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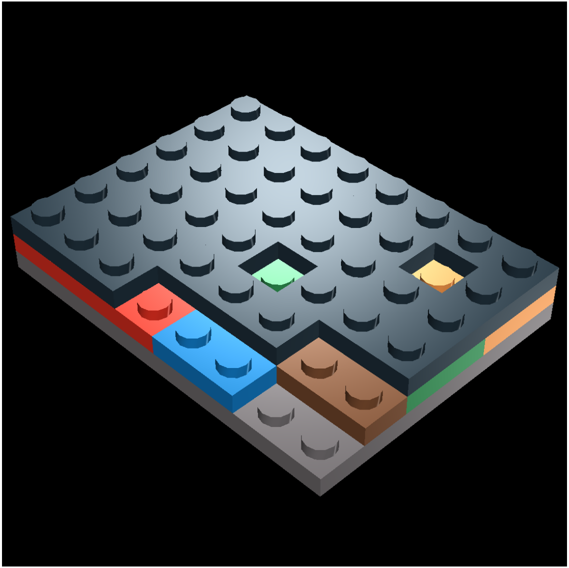
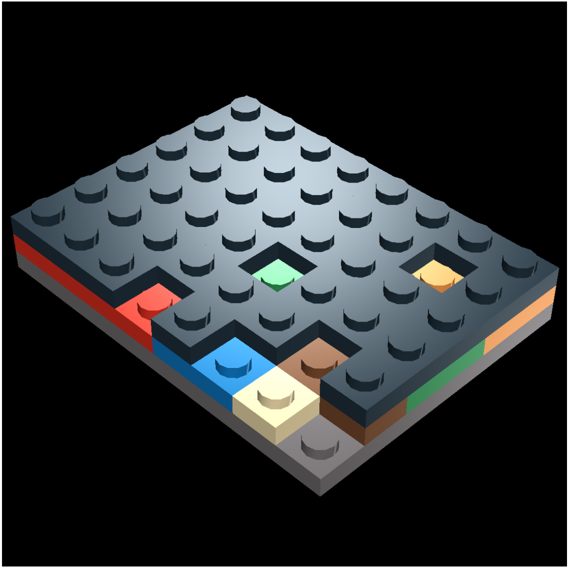
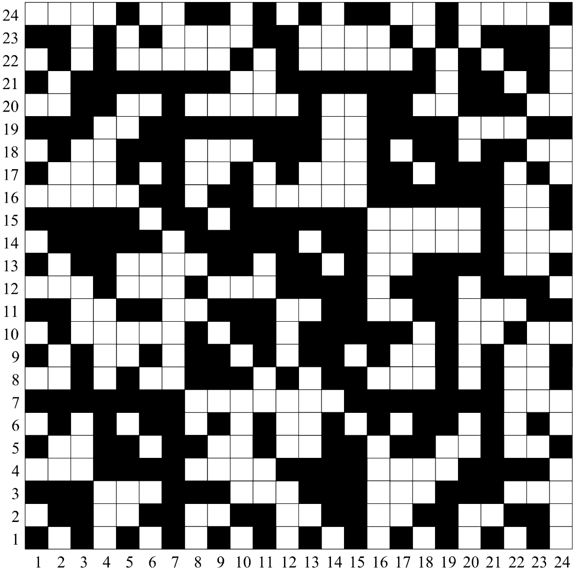


Playing LEGO(s, $h \leq 3$) selectively



The good positions LEGO(s, $h \leq 3$)



Theorem

An $a \times b$ rectangle in the second layer is good iff $a = b$.

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An $a \times b$ rectangle in the second layer is good iff $a = b$.
An $a \times b$ rectangle in the first layer is good iff the sum of the partial quotients of the continued fraction representation of $\frac{a}{b}$ is odd.

Acknowledgement

The present project is supported by the National Research Fund, Luxembourg, and cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).