FEJEZETEK A SZÁMELMÉLETBŐL ELŐADÁS

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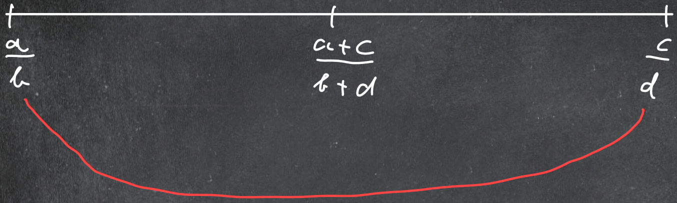
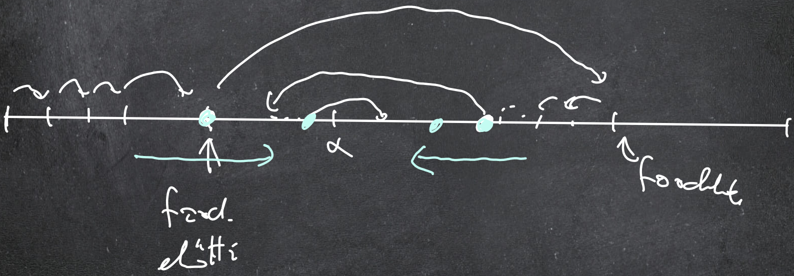
SZTE Bolyai Intézet

2021. május 4.

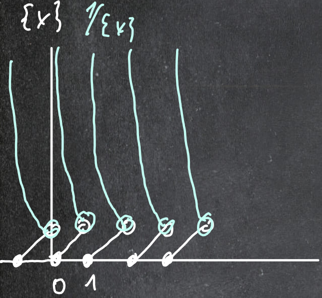


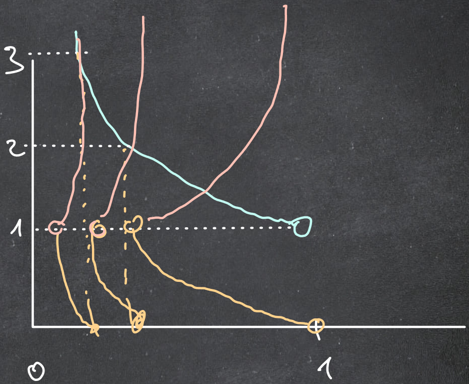
$\frac{1}{bd} \to 0$



$$t(x) = \frac{1}{\{x\}}$$

$$t^2(x) = \frac{1}{\{t(x)\}}$$





$[-3{,}2] = -4$

$\{-3{,}2\} = 0{,}8$

$-3{,}2$

$\{t(x)\}$

$$\frac{1}{\{t(x)\}} = t^2(x)$$

$$\pi = [3, 7, 15, 1, 292, 1, \dots]$$

$$-\pi = [-4, 1, 6, 15, 1, 292, 1, \dots]$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \dots]$$

$$-e = [-3, 3, 1, 1, 4, 1, 1, 6, \dots]$$

$\boxed{HF\,30}$

$$\{x\} \geq \frac{1}{2} \Rightarrow t^3(x) = t^2(-x)$$

$$\frac{1}{\left\{\frac{1}{\left\{\frac{1}{\{x\}}\right\}}\right\}} = \frac{1}{\left\{\frac{1}{\{-x\}}\right\}}$$

$$\{x\} < \frac{1}{2} \Rightarrow \{-x\} \geq \frac{1}{2}$$

$$\Downarrow HF30$$

$$t^3(-x) = t^2(x)$$

$$\sqrt{2} = [1; 2, 2, \dots]$$

$$t(\sqrt{2}) = \frac{1}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}+1$$

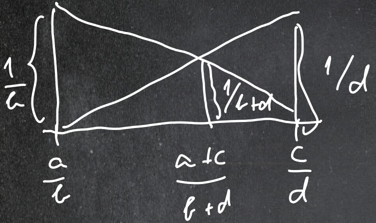
$$t(\sqrt{2}+1) = \frac{1}{\sqrt{2}+1-2} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$$

$$\frac{\sqrt{5}+1}{2} = [1; 1, 1, \dots]$$

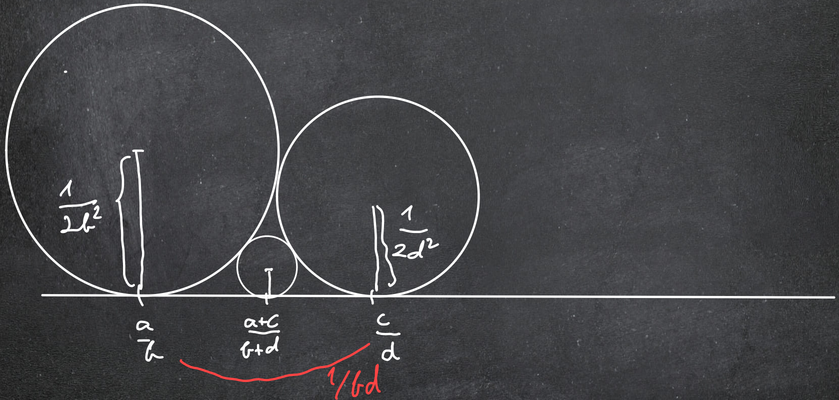
$$t\left(\frac{\sqrt{5}+1}{2}\right) = \frac{1}{\frac{\sqrt{5}+1}{2}-1} = \frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{2(\sqrt{5}+1)}{4} = \frac{\sqrt{5}+1}{2}$$

$$\alpha \sim \beta \iff \exists A \in \underset{\displaystyle [L, R]}{\underset{\shortparallel}{SL_2(\mathbb{Z})}} : \beta = A \bullet \alpha$$

$\alpha \sim \beta \iff \alpha$ és β SB-útja valahonnan kezdve megegyezik $\iff \alpha$ és β lánctörtje valahonnan kezdve megegy



FORD-körök



$$x = [1; \overline{1,2}, \overline{1,2}, \overline{1,2}, \dots] = 1 + \cfrac{1}{\underbrace{1+\cfrac{1}{2+\underbrace{1+\cfrac{1}{2+\frac{1}{\ddots}}}_{=\alpha}}}_{=\alpha}}$$

$$\alpha = [1; 2, 1, 2, 1, \dots]$$

$$\alpha = 1 + \cfrac{1}{2+\cfrac{1}{\alpha}} = 1 + \frac{\alpha}{2\alpha+1} = \frac{3\alpha+1}{2\alpha+1} \quad / \cdot 2\alpha+1$$

$$2\alpha^2 + \alpha = 3\alpha + 1$$

$$2\alpha^2 - 2\alpha - 1 = 0$$

$$\alpha = \frac{2 \pm \sqrt{4+8}}{4} = \frac{2 \pm 2\sqrt{3}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

$$x = 1 + \frac{1}{\alpha} = 1 + \frac{2}{\sqrt{3}+1} = \frac{\sqrt{3}+3}{\sqrt{3}+1} \cdot \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{3+3\sqrt{3}-\sqrt{3}-3}{2} = \underline{\underline{\sqrt{3}}}$$

HF31 $\sqrt{6} = [\ \ \ldots\ \ ]$

HF32 $? = [1; 2, 3, 2, 3, \ldots] = [1; \overline{2, 3}]$

<u>DEF</u> $\alpha \in \mathbb{R}$ <u>kvadratikus irracionalitás</u>, ha

$$\alpha = a + b\sqrt{d} \quad (a, b \in \mathbb{Q},\ d \in \mathbb{N}\ \ \underset{1}{\overset{d}{\vee}}\ \ \square\text{-mentes})$$

$$\sqrt{\frac{12}{7}} = \sqrt{\frac{2^2 \cdot 3 \cdot 7}{7^2}} = \frac{2}{7} \cdot \sqrt{3 \cdot 7}$$

$$\Longleftrightarrow \ \deg m_\alpha = 2$$

TÉTEL α lánctörtje valahonnan kezdve periodikus

$$\Updownarrow$$

α kvadr. irrac.

Biz. $\Uparrow$ Nehéz. LAGRANGE

$\Downarrow$ Könnyű EULER

$$L=\begin{pmatrix}1&0\\1&1\end{pmatrix} \quad L^a=\begin{pmatrix}1&0\\a&1\end{pmatrix}$$

$$R=\begin{pmatrix}1&1\\0&1\end{pmatrix} \quad R^a=\begin{pmatrix}1&a\\0&1\end{pmatrix}$$

$$\begin{pmatrix}a&1\\1&0\end{pmatrix}\notin SL_2(\mathbb{Z}) \qquad \begin{pmatrix}a&1\\1&0\end{pmatrix}\circ x=\frac{ax+1}{1x+0}=a+\frac{1}{x}$$

$$\begin{pmatrix}a_0&1\\1&0\end{pmatrix}\begin{pmatrix}a_1&1\\1&0\end{pmatrix}\circ x=a_0+\frac{1}{a_1+\frac{1}{x}}=[a_0;a_1,x]$$

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \circ x = [a_0, a_1, \ldots, a_n, x] = a_0 + \cfrac{1}{a_1 + \cdots \cfrac{}{a_n + \frac{1}{x}}}$$

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \circ \infty = [a_0, a_1, \ldots, a_n]$$

$$\alpha = [a_0, \ldots, a_k, b_1, \ldots, b_n, b_1, \ldots, b_n, \ldots]$$

$$\alpha = \underbrace{\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}}_{B} \cdot \underbrace{\begin{pmatrix} b_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_n & 1 \\ 1 & 0 \end{pmatrix}}_{B} \cdots \circ \infty$$

$$\left.\begin{aligned} \alpha &= ABB \cdots \circ \infty \qquad | A^{-1} \cdot \\ A^{-1} \circ \alpha &= B\underbrace{B \cdots \circ \infty}_{A^{-1} \circ \alpha} \end{aligned}\right\} \Rightarrow A^{-1} \circ \alpha = BA^{-1} \circ \alpha$$

$$\alpha = \underbrace{ABA^{-1}}_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \circ \alpha$$

$$\alpha = \frac{a\alpha + b}{c\alpha + d} \Rightarrow \deg_{\mathbb{Q}} \alpha = 2 \qquad \blacksquare$$

TÉTEL (GALOIS) α lánctörtje tisztán periodikus

$$\Updownarrow$$

$$\deg m_\alpha = 2 \text{ és } \alpha > 1,\ -1 < \bar{\alpha} < 0$$

m_α másik gyöke

KÖV. $\sqrt{d} = [a_0; b_1 \ldots b_n, b_1 \ldots b_n, \ldots]$

PELL-EGYENLETEK

$x^2 - dy^2 = 1$ pozitív Pell-egyenlet $\quad d \in \mathbb{N} \setminus \{1\}$ n-mentes

$x^2 - dy^2 = -1$ negatív Pell-egyenlet

$X^2 - 12y^2 = 1 \qquad x^2 - y^2 = 1 \qquad x^2 - 4y^2 = 1$

$X^2 - 3\cdot(\underbrace{2y}_{z})^2 = 1 \qquad (x+y)(x-y) = 1 \qquad (x-2y)(x+2y) = 1$

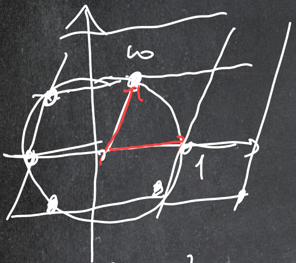
$x^2 - 2y^2 = \pm 1$

$(x + \sqrt{2}y)(x - \sqrt{2}y) = \pm 1$

$\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \qquad \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$

Gauss-egészek

Euler-egészek: $\mathbb{Z}\Big[\underbrace{\frac{1+\sqrt{3}i}{2}}_{\omega}\Big] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$



$x^3 + y^3 = z^3$

$$\boxed{\begin{aligned} &\mathbb{Z}^* = \{1, -1\} \\ &\mathbb{Z}[i]^* = \{1, -1, i, -i\} \\ &\mathbb{Z}[\omega]^* = \{\pm 1, \pm \tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} i\} \end{aligned}}$$

R tetsz. int. tart.

$\alpha \mid \beta \iff \exists \gamma \in R : \beta = \alpha \cdot \gamma$

$\alpha \sim \beta \iff \alpha \mid \beta$ és $\beta \mid \alpha \iff \exists \varepsilon \in R^* : \beta = \varepsilon \cdot \alpha$

$\alpha \sim 1 \iff \alpha \mid 1$ (és $1 \mid \alpha$) $\iff \exists \alpha^{-1} \in R : \alpha \alpha^{-1} = 1 \iff \alpha$ egység

$(R^*; \cdot)$ csoport

$R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$ Sűrű

$\alpha = a + b\sqrt{2}$ konjugált: $\overline{\alpha} = a - b\sqrt{2}$

$\alpha + \overline{\alpha} = 2a$

$\alpha \cdot \overline{\alpha} = a^2 - 2b^2 = N(\alpha) \in \mathbb{Z}$

$b \neq 0$ ($\Leftrightarrow \alpha \notin \mathbb{Q}$) esetén $m_\alpha = (x - \alpha)(x - \overline{\alpha}) =$

$= x^2 - 2a \cdot x + N(\alpha)$.

$\overline{\alpha \cdot \beta} = \overline{\alpha} \cdot \overline{\beta}$

$N(\alpha \cdot \beta) = \alpha\beta \cdot \overline{\alpha\beta} = \alpha\beta\overline{\alpha}\overline{\beta} = \alpha\overline{\alpha} \cdot \beta\overline{\beta} = N(\alpha) \cdot N(\beta)$

Áll: $\forall \alpha \in \mathbb{Z}[\sqrt{2}]$: $\alpha \in \mathbb{Z}[\sqrt{2}]^{*} \iff N(\alpha) = \pm 1$.

Biz: ($\Longrightarrow$) Tfh. α egység: $\exists\, \alpha^{-1} \in \mathbb{Z}[\sqrt{2}]$: $\alpha \cdot \alpha^{-1} = 1$

$$N(\alpha\alpha^{-1}) = N(1) = 1$$

$$\underbrace{N(\alpha)}_{\in \mathbb{Z}}\, \underbrace{N(\alpha^{-1})}_{\in \mathbb{Z}} = 1$$

$$\Longrightarrow \underbrace{N(\alpha)}= N(\alpha^{-1}) = \underbrace{\pm 1}$$

($\Longleftarrow$) Tfh. $\underbrace{N(\alpha)}_{\alpha\overline{\alpha}} = \pm 1$

$$\alpha\overline{\alpha} = \pm 1 \quad / \cdot \pm 1$$

$$\alpha \underbrace{(\pm\overline{\alpha})}_{\alpha^{-1}} = 1$$

$\blacksquare$