239

Tétel (Machin, 1706)

$$4 \cdot \operatorname{arctg} \frac{1}{5} - \operatorname{arctg} \frac{1}{239} = \frac{\pi}{4}$$

Bizonyítás.

A tangensfüggvény addíciós képlete:

$$\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \cdot \operatorname{tg} \beta}.$$

Ezt „kifordítva” kapjuk (az $a = \operatorname{tg} \alpha$, $b = \operatorname{tg} \beta$ jelöléssel), hogy

$$\operatorname{arctg} a + \operatorname{arctg} b = \operatorname{arctg} \frac{a + b}{1 - a \cdot b}.$$

Háromszor alkalmazva ezt az összefüggést, megkapjuk Machin képletét:

Tétel (Machin, 1706)

$$4 \cdot \operatorname{arctg} \frac{1}{5} - \operatorname{arctg} \frac{1}{239} = \frac{\pi}{4}$$

Bizonyítás (folyt.).

$$2 \cdot \operatorname{arctg} \frac{1}{5} = \operatorname{arctg} \frac{1}{5} + \operatorname{arctg} \frac{1}{5} = \operatorname{arctg} \frac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{5} \cdot \frac{1}{5}} = \operatorname{arctg} \frac{5}{12};$$

$$4 \cdot \operatorname{arctg} \frac{1}{5} = \operatorname{arctg} \frac{5}{12} + \operatorname{arctg} \frac{5}{12} = \operatorname{arctg} \frac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{5}{12} \cdot \frac{5}{12}} = \operatorname{arctg} \frac{120}{119};$$

$$\frac{\pi}{4} + \operatorname{arctg} \frac{1}{239} = \operatorname{arctg} 1 + \operatorname{arctg} \frac{1}{239} = \operatorname{arctg} \frac{1 + \frac{1}{239}}{1 - 1 \cdot \frac{1}{239}} = \operatorname{arctg} \frac{120}{119}.$$

□

$\operatorname{arctg} \frac{1}{239} = 0,239\,729\,896\ldots^\circ$

Tétel (Gregory, 1671)

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13} - \frac{x^{15}}{15} + \cdots$$

Következmény

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \cdots$$

Következmény

$$\frac{\pi}{4} = 4 \cdot \left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \cdots \right) - \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \cdots \right)$$

$$\tfrac{\pi}{4} \approx 4 \left(\tfrac{1}{5} - \tfrac{1}{3 \cdot 5^3} + \tfrac{1}{5 \cdot 5^5} - \tfrac{1}{7 \cdot 5^7} + \tfrac{1}{9 \cdot 5^9} - \tfrac{1}{11 \cdot 5^{11}} + \tfrac{1}{13 \cdot 5^{13}} - \tfrac{1}{15 \cdot 5^{15}} \right) - \left(\tfrac{1}{239} - \tfrac{1}{3 \cdot 239^3} \right)$$

239
040981
1254
36
57121 · 239
114242
171363
514089
13651919 · 3
40955757 = $3 \cdot 239^3$

1 : 40955757 = 0,000000024416591 78… (+)
100000000
81911514
180884860
163823028
170618320
163823028
67952920
40955757
269971630
245734542
242370880
204778785
375920950
368601813
73191370
40955757
322356130
286690299
356658310

$= \frac{1}{3 \cdot 239^3}$

1 : 239 = 0,0041841 (−) $= \frac{1}{239}$
1000
956
440
239
2010
1912
980
956
240
1

65536 : 3 = 21845,3
05
25
13
16
10
1

$\frac{2^{15}}{15 \cdot 10^{15}} = \frac{2^{16}}{3} \cdot 10^{-16} = \frac{65536}{3} \cdot 10^{-16} =$ 0,00000000000021845$\dot{3}$ (−)

8192 : 13 = 630,153846
39
020
70
50
110
60
80
2

$\frac{2^{13}}{13 \cdot 10^{13}} =$ 0,00000000000630153846 (+)

2048 : 11 = 186,18
94
68
20
90
2

$\frac{2^{11}}{11 \cdot 10^{11}} =$ 0,00000000018618 (−)

512 : 9 = 56,8
62
80
8

$\frac{2^{9}}{9 \cdot 10^{9}} =$ 0,0000000568 (+)

128 : 7 = 18,285714
58
20
60
40
50
10
30
2

$\frac{2^{7}}{7 \cdot 10^{7}} =$ 0,0000018285714 (−)

$\frac{2^{5}}{5 \cdot 10^{5}} = \frac{2^{6}}{10^{6}} =$ 0,000064 (+)

$\frac{2^{3}}{3 \cdot 10^{3}} =$ 0,0026 (−)

$\frac{2^{1}}{10^{1}} =$ 0,2 (+)

0,2000000000000000000
0,0000640000000000000
0,0000000056888888888
0,0000000000063015384
0,2000640569519042 6 (+)

0,0026666666666666666
0,0000018285714285 7
0,0000000001861818 18
0,0000000000000218453
0,0026684971020979 4 (−)

0,20006405695190426
− 0,00266849710209794
0,19739555984980632 · 4
0,78958223939922528
$\approx 4 \cdot \arctan \frac{1}{5}$

0,78958223939922528
0,00000000244165917 8
0,78958222638158170 6

0,78958222638158170 6
− 0,00418410041841004
0,78539816339740702
$\approx \frac{\pi}{4}$

0,78539816339740702 · 4
3,14159265358962808 $\approx \pi$

239
040981
1254
36

57121 · 239
114242
171363
514089

13651919 · 3

$40955757 = 3 \cdot 239^3$

$1 : 40955757 = \boxed{0{,}00000002441659178\ldots}$ (+)

$\| \;\; \dfrac{1}{3 \cdot 239^3}$

100000000
81911514
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170618320
163823028
67952920
40955757
269971630
245734542
242370880
204778785
375920950
368601813
73191370
40955757
322356130
286690299
356658310

$1 : 239 = \boxed{0{,}\dot{0}04184\dot{1}} = \dfrac{1}{239}$ (−)

1000
956
440
239
2010
1912
980
956
240
1

$65536 : 3 = 21845{,}\dot{3}$

05
25
13
16
10
1

$\frac{2^{15}}{15 \cdot 10^{15}} = \frac{2^{16}}{3} \cdot 10^{-16} = \frac{65536}{3} \cdot 10^{-16} = \boxed{0{,}0000000000021845\dot{3}}$ (−)

$8192 : 13 = 630{,}\dot{1}5384\dot{6}$

39
020
70
50
110
60
80
2

$\frac{2^{13}}{13 \cdot 10^{13}} = \boxed{0{,}0000000000630\dot{1}5384\dot{6}}$ (+)

$2048 : 11 = 186{,}\dot{1}\dot{8}$

94
68
20
90
2

$\frac{2^{11}}{11 \cdot 10^{11}} = \boxed{0{,}00000000186\dot{1}\dot{8}}$ (−)

$512 : 9 = 56{,}\dot{8}$

62
80
8

$\frac{2^{9}}{9 \cdot 10^{9}} = \boxed{0{,}0000000568\dot{8}}$ (+)

$128 : 7 = 18{,}\dot{2}8571\dot{4}$

58
20
60
40
50
10
30
2

$\frac{2^{7}}{7 \cdot 10^{7}} = \boxed{0{,}0000018\dot{2}8571\dot{4}}$ (−)

$\frac{2^{5}}{5 \cdot 10^{5}} = \frac{2^{6}}{10^{6}} = \boxed{0{,}000064}$ (+)

$\frac{2^{3}}{3 \cdot 10^{3}} = \boxed{0{,}002\dot{6}}$ (−)

$\frac{2^{1}}{10^{1}} = \boxed{0{,}2}$ (+)

0,2000000000000000000
0,0000640000000000000
0,0000000005688888888
0,0000000000063015388
0,20006405695190426 (+)

0,002666666666666666
0,000001828571428 57
0,0000000018618181 8
0,00000000000021845 3
0,00266849710209794 (−)

0,20006405695190426
− 0,00266849710209794
0,19739555984980632 · 4
0,78958223939922528
$\approx 4 \cdot \arctan \frac{1}{5}$

0,78958223939922528
0,000000602441653177
0,78958226381581706

0,78958226381581706
− 0,00418410041841004
0,78539816339740702
$\approx \frac{\pi}{4}$

0,78539816339740702 · 4
3,14159265358962808 $\approx \pi$

$$\frac{1}{1} = 1,000000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{2} = 0,5\ 0000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{3} = 0,333333333333333333333333333333333333\ldots \longrightarrow 1$$

$$\frac{1}{4} = 0,25\ 000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{5} = 0,2\ 0000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{6} = 0,1\ 6666666666666666666666666666666666\ldots \longrightarrow 1$$

$$\frac{1}{7} = 0,142857\ 142857\ 142857\ 142857\ 142857\ 14\ldots \longrightarrow 6$$

$$\frac{1}{8} = 0,125\ 00000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{9} = 0,111111111111111111111111111111111111\ldots \longrightarrow 1$$

$$\frac{1}{10} = 0,1\ 0000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{11} = 0,09\ 09\ 09\ 09\ 09\ 09\ 09\ 09\ 09\ 09\ 09\ 09\ 09\ 0\ldots \longrightarrow 2$$

$$\frac{1}{12} = 0,08\ 333333333333333333333333333333333\ldots \longrightarrow 1$$

$$\frac{1}{13} = 0,076923\ 076923\ 076923\ 076923\ 076923\ 076\ldots \longrightarrow 6$$

$$\frac{1}{14} = 0,0\ 714285\ 714285\ 714285\ 714285\ 714285\ 7\ldots \longrightarrow 6$$

$$\frac{1}{15} = 0,0\ 66666666666666666666666666666666666\ldots \longrightarrow 1$$

$$\frac{1}{16} = 0,0625\ 0000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{17} = 0,0588235294117647\ 0588235294117647\ 05\ldots \longrightarrow 16$$

$$\frac{1}{18} = 0,0\ 5555555555555555555555555555555555\ldots \longrightarrow 1$$

$$\frac{1}{19} = 0,052631578947368421\ 05263157894736842\ldots \longrightarrow 18$$

$$\frac{1}{20} = 0,05\ 000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{21} = 0,047619\ 047619\ 047619\ 047619\ 047619\ 04\ldots \longrightarrow 6$$

$$\frac{1}{22} = 0,0\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ 45\ldots \longrightarrow 2$$

$$\frac{1}{23} = 0,0434782608695652173913\ 0434782608695\ldots \longrightarrow 22$$

$$\frac{1}{24} = 0,041\ 66666666666666666666666666666666\ldots \longrightarrow 1$$

$$\frac{1}{25} = 0,04\ 000000000000000000000000000000000\ldots \longrightarrow 1$$

$$\frac{1}{26} = 0,0\ 384615\ 384615\ 384615\ 384615\ 384615\ldots \longrightarrow 6$$

$$\frac{1}{27} = 0,03\ 703\ 703\ 703\ 703\ 703\ 703\ 703\ 703\ 703\ldots \longrightarrow 3$$

$$\frac{1}{28} = 0,03\ 571428\ 571428\ 571428\ 571428\ 571428\ldots \longrightarrow 6$$

$$\frac{1}{29} = 0,0344827586206896551724137931\ 03448275\ldots \quad \longrightarrow \quad 28$$

$$\frac{1}{30} = 0,0\ 33333333333333333333333333333333333\ldots \quad \longrightarrow \quad 1$$

$$\frac{1}{31} = 0,032258064516129\ 032258064516129\ 03225\ldots \quad \longrightarrow \quad 15$$

$$\frac{1}{32} = 0,03125\ 0000000000000000000000000000000\ldots \quad \longrightarrow \quad 1$$

$$\frac{1}{33} = 0,03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ 03\ldots \quad \longrightarrow \quad 2$$

$$\frac{1}{34} = 0,0\ 2941176470588235\ 2941176470588235\ 29\ldots \quad \longrightarrow \quad 16$$

$$\frac{1}{35} = 0,0\ 285714\ 285714\ 285714\ 285714\ 285714\ 28\ldots \quad \longrightarrow \quad 6$$

$$\frac{1}{36} = 0,02\ 7777777777777777777777777777777777\ldots \quad \longrightarrow \quad 1$$

$$\frac{1}{37} = 0,027\ 027\ 027\ 027\ 027\ 027\ 027\ 027\ 027\ 027\ 0\ldots \longrightarrow \quad 3$$

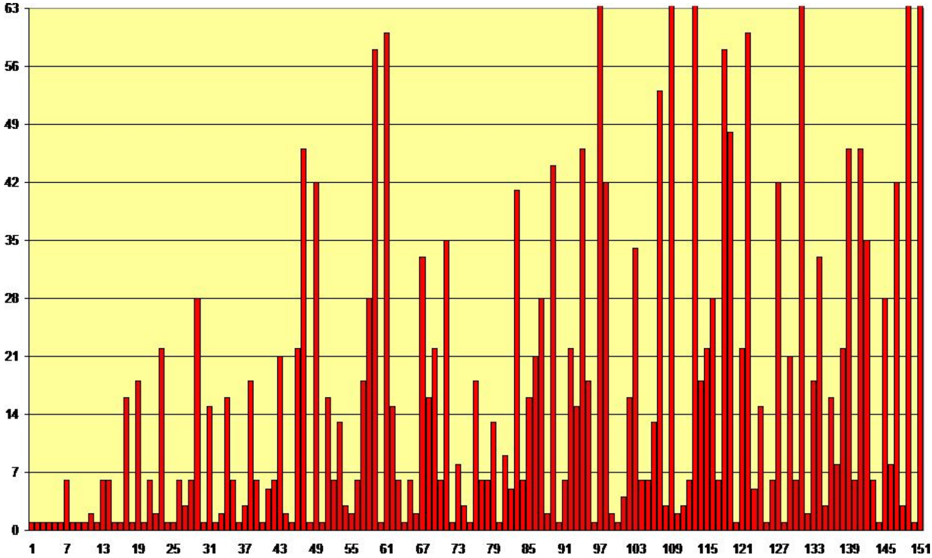
$$\frac{1}{38} = 0,0\ 263157894736842105\ 26315789473684210\ldots \longrightarrow \quad 18$$

$$\frac{1}{39} = 0,025641\ 025641\ 025641\ 025641\ 025641\ 0255\ldots \longrightarrow \quad 6$$

$$\frac{1}{40} = 0,025\ 0000000000000000000000000000000000\ldots \longrightarrow \quad 1$$

$$\frac{1}{41} = 0,02439\ 02439\ 02439\ 02439\ 02439\ 02439\ 024\ldots \quad \longrightarrow \quad 5$$

$$\frac{1}{42} = 0,0\ 238095\ 238095\ 238095\ 238095\ 238095\ 23\ldots \quad \longrightarrow \quad 6$$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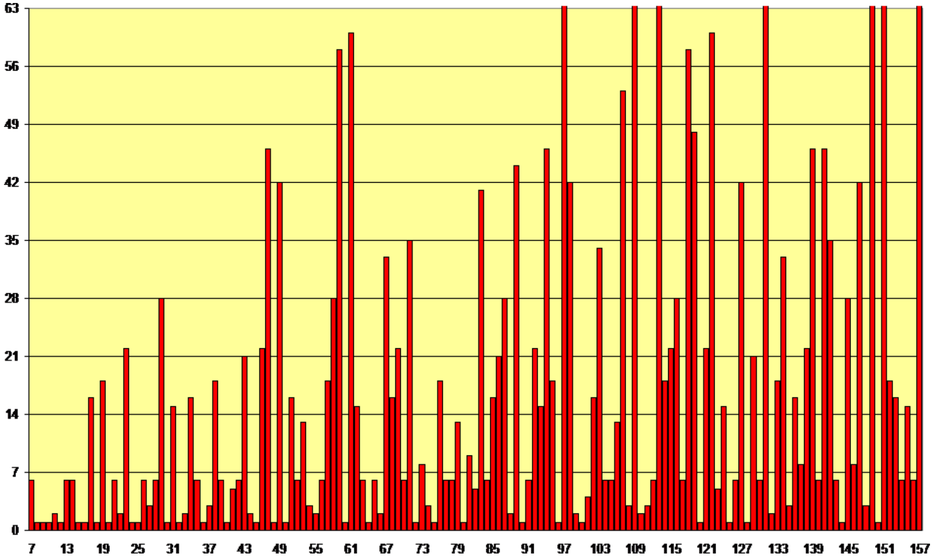
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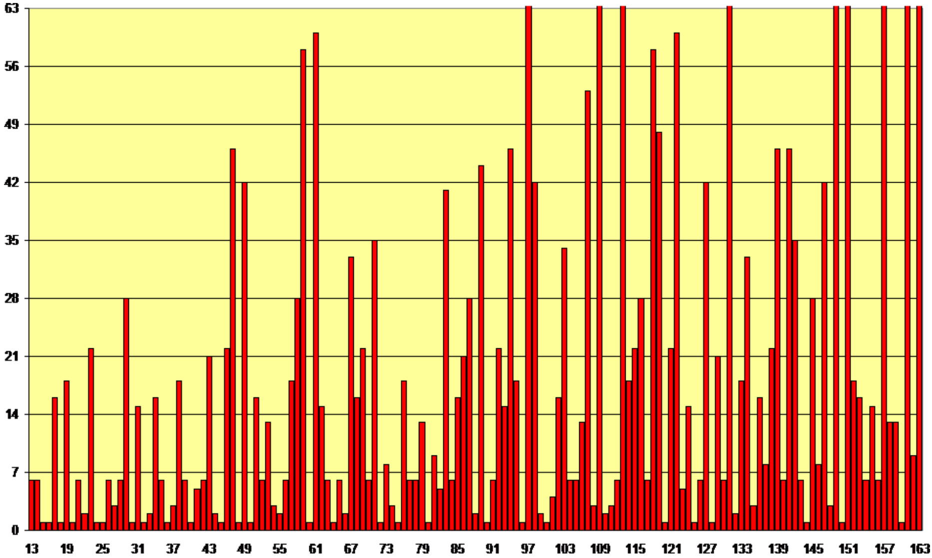

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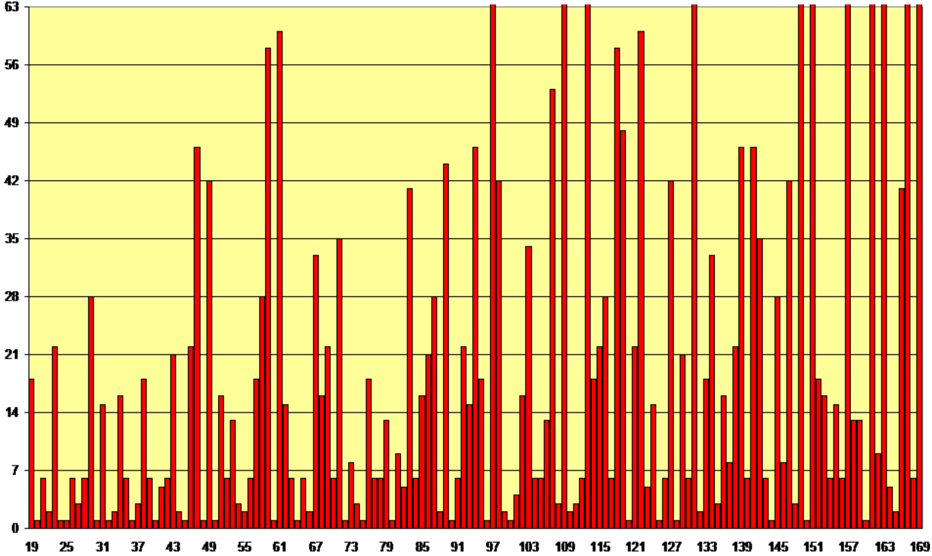
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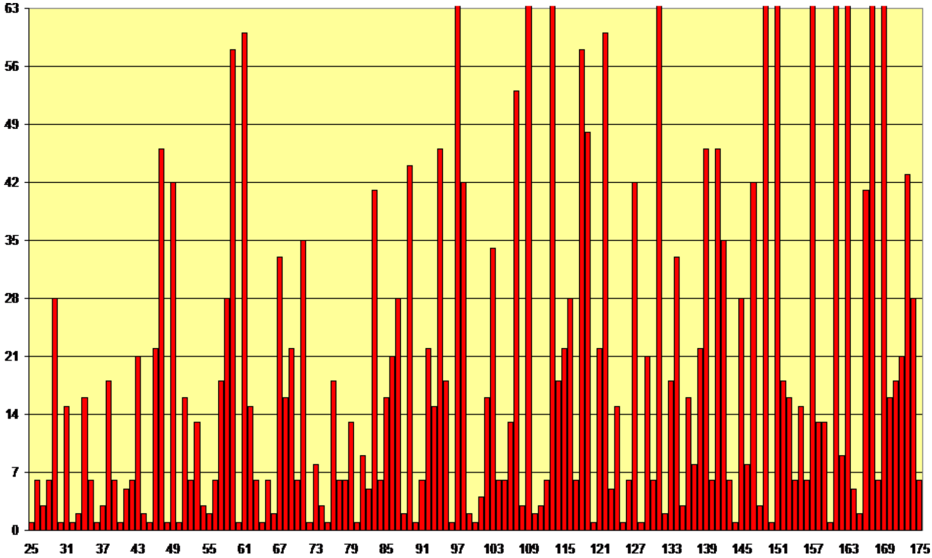
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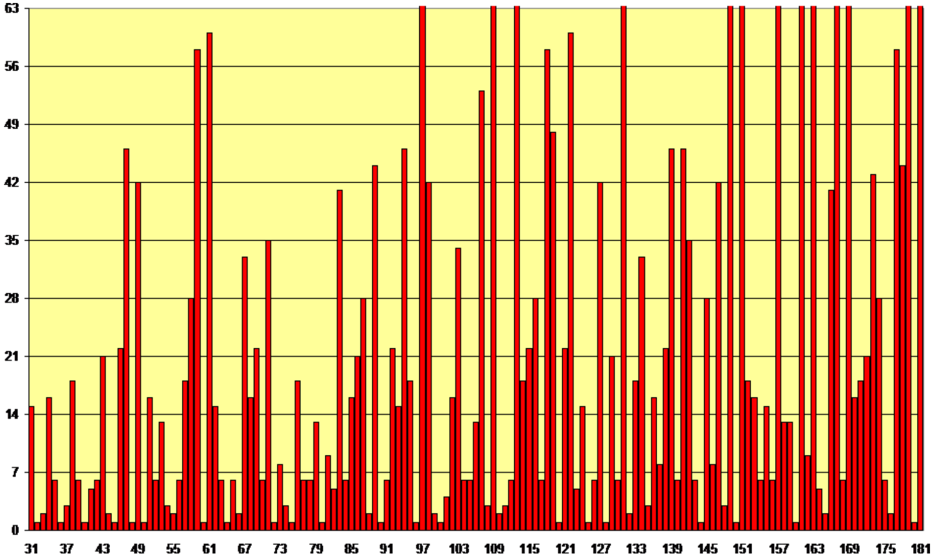
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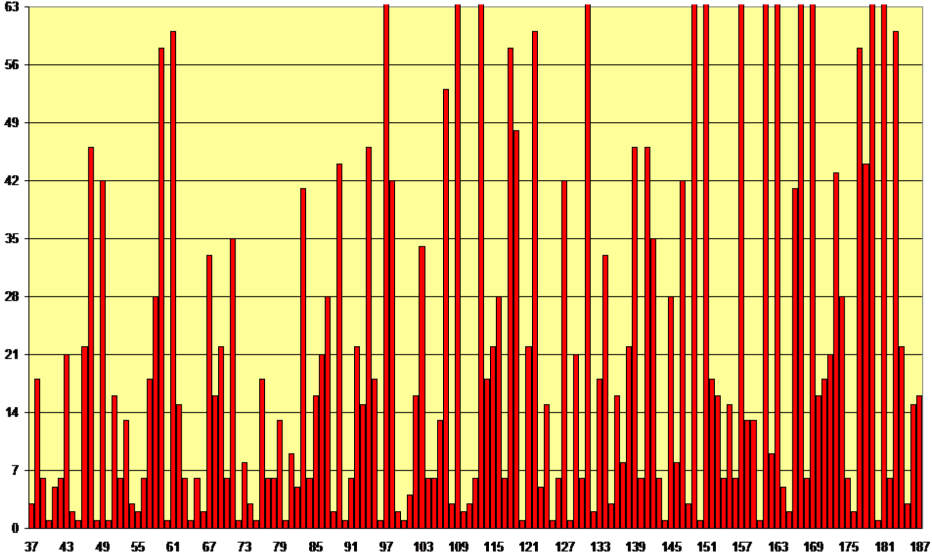

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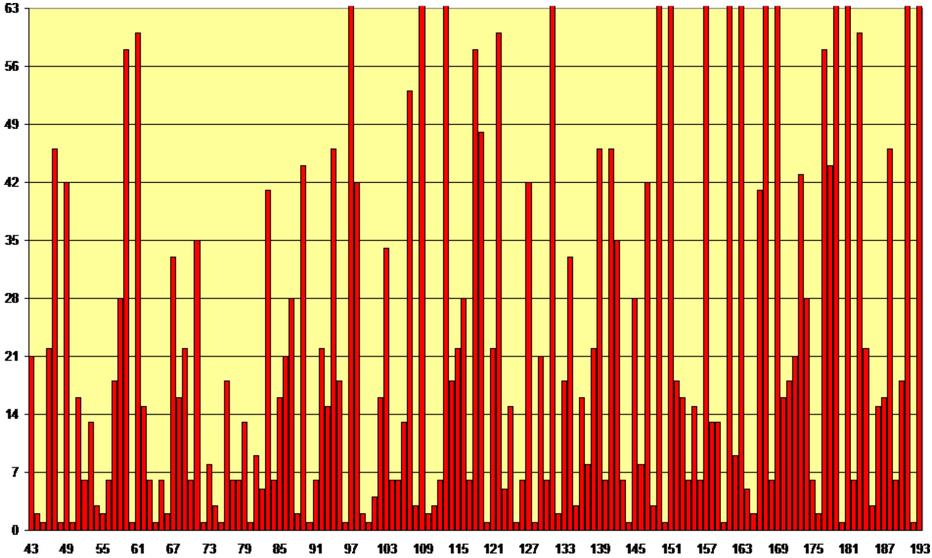
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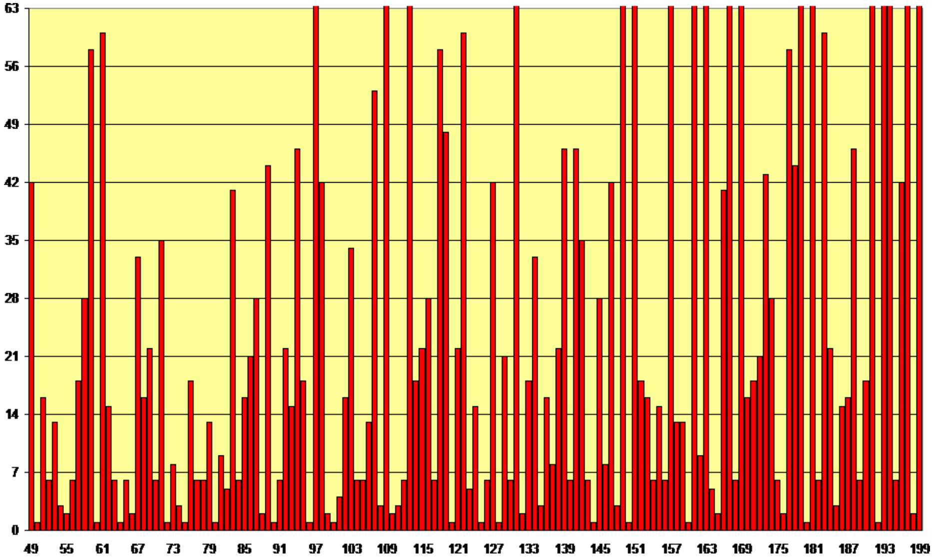
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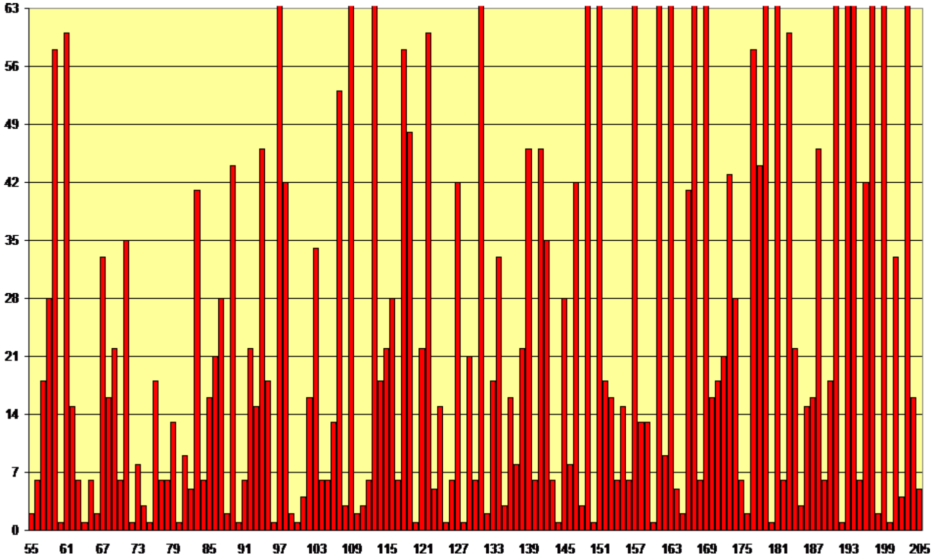
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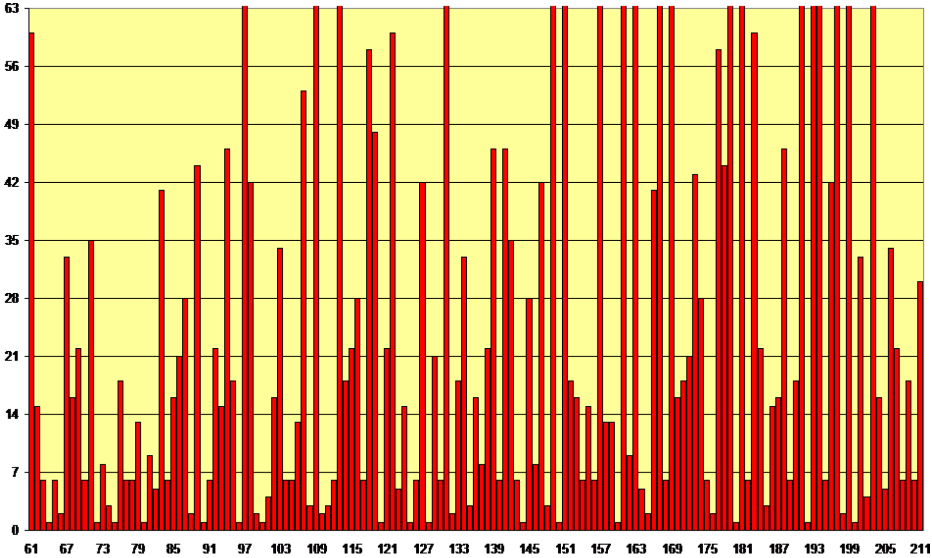
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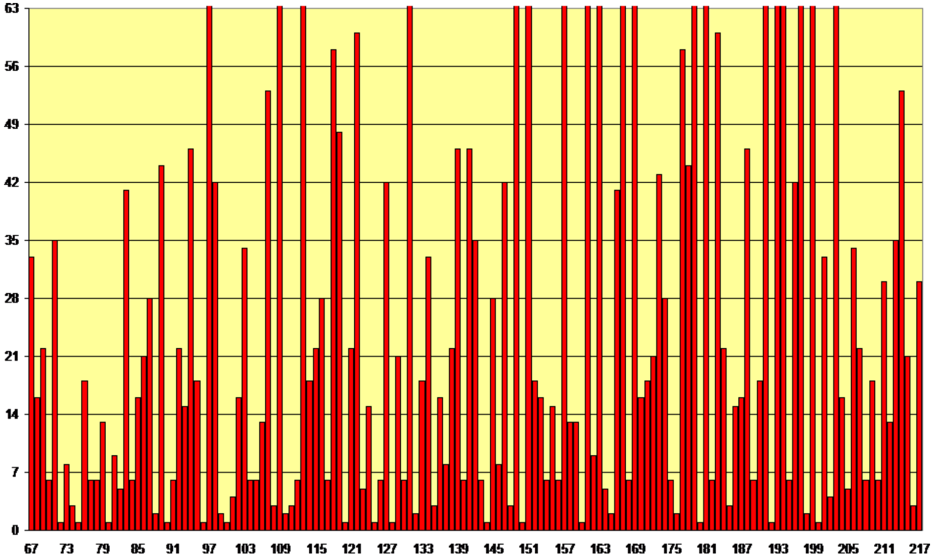

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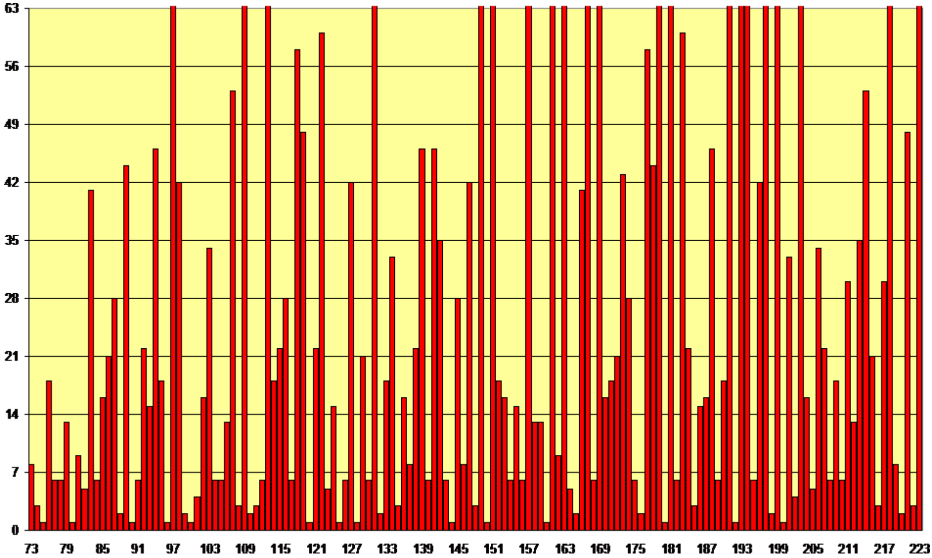
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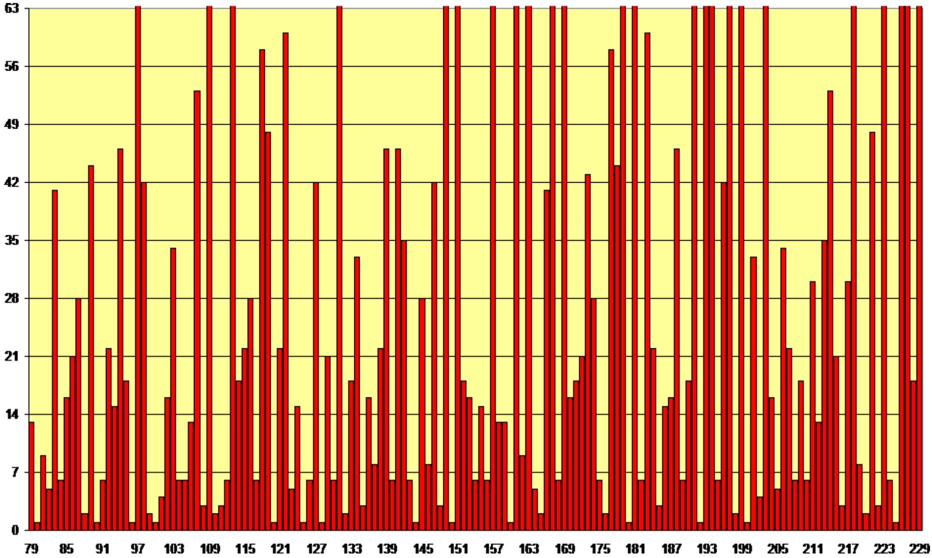
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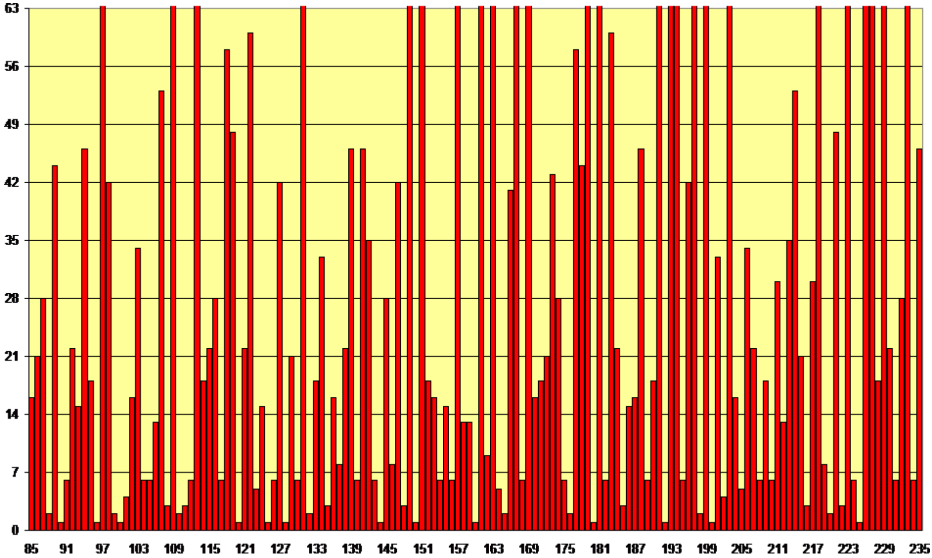
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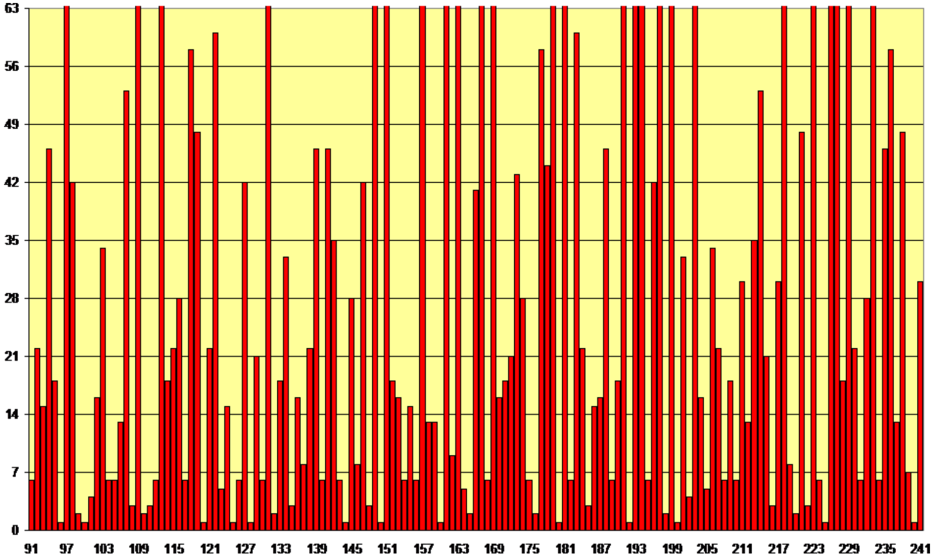

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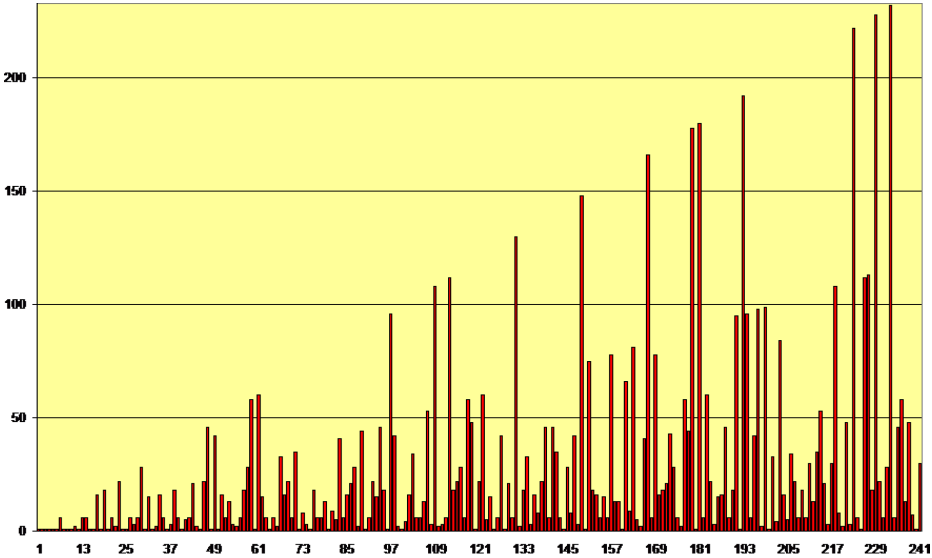

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$$\frac{1}{239} = \quad 0,0041841\ 0041841\ 0041841\ 0041841\ldots$$

$$\frac{10^7}{239} = 41841,0041841\ 0041841\ 0041841\ 0041841\ldots$$

$$\frac{10^7 - 1}{239} = 41841$$

$$10^7 - 1 = 9999999 = 3^2 \cdot 239 \cdot 4649$$

$$10^6 - 1 = 999999 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$$

$$10^5 - 1 = 99999 = 3^2 \cdot 41 \cdot 271$$

$$10^4 - 1 = 9999 = 3^2 \cdot 11 \cdot 101$$

$$10^3 - 1 = 999 = 3^3 \cdot 37$$

$$10^2 - 1 = 99 = 3^2 \cdot 11$$

$$10^1 - 1 = 9 = 3^2$$

$x = 239$, $y = 13^2$ megoldása az $x^2 - 2y^2 = -1$ Pell-egyenletnek

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2}}}}}} = \frac{239}{169}$$

$$\frac{239}{169} = 1,41420183\ldots \approx \sqrt{2}$$

$$239 = 4^3 + 4^3 + 3^3 + 3^3 + 3^3 + 3^3 + 1^3 + 1^3 + 1^3$$

Tétel (Wieferich, Kempner, 1912)

Minden szám előáll kilenc köbszám összegeként.

Tétel (Landau, 1909)

Majdnem minden szám előáll nyolc köbszám összegeként.

Tétel (Dickson, 1939)

A 23 és a 239 kivételével minden szám előáll nyolc köbszám összegeként.

Sejtés (Waring, 1770)

Minden szám előáll kilenc köbszám, tizenkilenc negyedik hatvány, stb. összegeként.

Tétel (Hilbert, 1909)

Minden szám előáll $g(k)$ darab k-adik hatvány összegeként.

Tétel

- $g(2) = 4$ *(Lagrange, 1770)*
- $g(3) = 9$ *(Wieferich, 1909, Kempner, 1912)*
- $g(4) = 19$ *(Balasubramanian, Dress, Deshouillers, 1986)*
- $g(5) = 37$ *(Chen, 1964)*
- $g(6) = 73$ *(Pillai, 1940)*

Tétel (Dickson, Pillai, Rubugunday, Niven, Mahler, 1957)

Majdnem minden k-ra

$$g\left(k\right) = 2^k + \left[\frac{3^k}{2^k}\right] - 2.$$

Tétel (Euler, 1772)

Minden k-ra

$$g\left(k\right) \geq 2^k + \left[\frac{3^k}{2^k}\right] - 2.$$

Bizonyítás.

Keressünk olyan n számot, ami sok kicsi k-adik hatvány összege.

$$n = \underbrace{1^k + \cdots + 1^k}_{i} + \underbrace{2^k + \cdots + 2^k}_{j}$$

Nem lehet csökkenteni az összeadandók számát, ha $i < 2^k$ és $n < 3^k$.

Bizonyítás (folyt.).

Tehát a legnagyobb i-t és j-t keressük, amelyre $i < 2^k$ és $n < 3^k$.

$i < 2^k \implies i := 2^k - 1$

$n = i + j \cdot 2^k = (2^k - 1) + j \cdot 2^k = (j+1) \cdot 2^k - 1 < 3^k$

$\implies j < \frac{3^k+1}{2^k} - 1 \implies j := \left[\frac{3^k}{2^k}\right] - 1$

Az n számhoz legalább $i + j = 2^k - 1 + \left[\frac{3^k}{2^k}\right] - 1 = 2^k + \left[\frac{3^k}{2^k}\right] - 2$ darab k-adik hatvány kell, ezért

$$g(k) \geq 2^k + \left[\frac{3^k}{2^k}\right] - 2.$$

□

Wolfgang Amadeus Mozart: **Serenata notturna** (D-dúr szerenád)

Köchel-jegyzékszáma: K239

Ez az egyetlen Mozart-darab, amely két együttesre íródott.