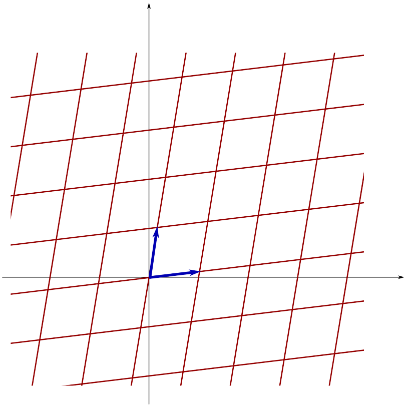
Lattices

Definition

A lattice (of rank d) is a subgroup Γ of $(\mathbb{R}^n; +)$ generated by d linearly independent vectors $\omega_1, \ldots, \omega_d$:

$$\Gamma = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_d = \{c_1\omega_1 + \cdots + c_d\omega_d : c_i \in \mathbb{Z}\}.$$

If $d = n$, then we say that Γ is a full-rank lattice.

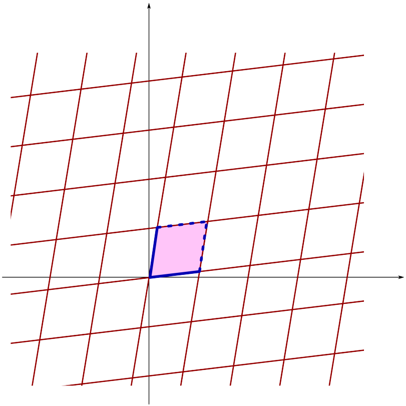


Definition

Let Γ be a full-rank lattice in $\mathbb{R}^n$ with basis $\omega_1, \ldots, \omega_n$. The set

$$P = \{x_1\omega_1 + \cdots + x_n\omega_n : 0 \leq x_i < 1\} \subseteq \mathbb{R}^n$$

is the fundamental parallelotope of L.



Fact

Translates of P cover $\mathbb{R}^n$ without overlaps:

$$\mathbb{R}^n = \dot{\bigcup_{\gamma \in \Gamma}} (\gamma + P) .$$

In other words, each coset of Γ (as a subgroup of $\mathbb{R}^n$) has a unique representative in P.

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Let Ω be the $n \times n$ matrix obtained by writing $\omega_1, \ldots, \omega_n$ next to each other as column vectors. Then the volume of P is

$$\mathrm{vol}\,(P) = |\det \Omega| = |\det (\omega_1, \ldots, \omega_n)| .$$

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The volume of the fundamental parallelotope is independent of the choice of the basis.

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Proof.

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$$\omega'_i = c_{1i}\omega_1 + \cdots + c_{ni}\omega_n \quad (i = 1, \ldots, n)\,.$$

In other words, we have $\Omega' = \Omega \cdot C$, where $C = (c_{ij}) \in \mathbb{Z}^{n \times n}$.

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Remark

By the above proposition, it makes sense to denote the volume of any/the fundamental parallelotope of Γ by $\mathrm{vol}\,(\Gamma)$.

Theorem

Let $\Gamma_1 \leq \Gamma \leq \mathbb{R}^d$ be full-rank lattices with fundamental parallelotopes P and P_1. Then Γ_1 is of finite index in Γ, and we have

$$[\Gamma : \Gamma_1] = |P_1 \cap \Gamma| = \frac{\mathrm{vol}\,(P_1)}{\mathrm{vol}\,(P)}.$$

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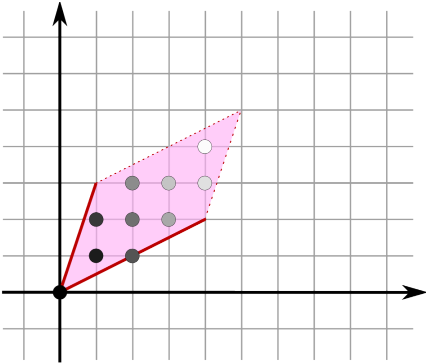
Proof.

The set $P_1 \cap \Gamma$ is finite (it is a compact discrete set), and it is a complete system of representatives of the cosets of Γ_1, hence $[\Gamma : \Gamma_1] = |P_1 \cap \Gamma| < \infty$.

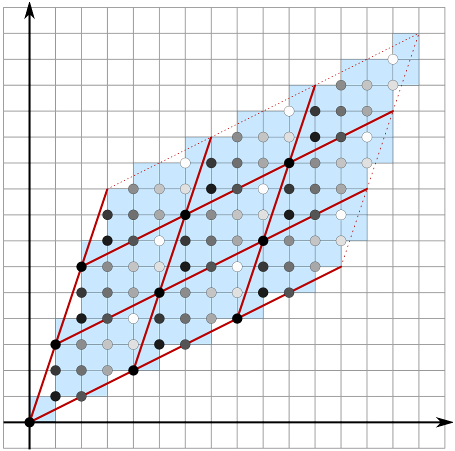


Proof. (cont.)

The union of the translates of P by the elements of $n \cdot P_1 \cap \Gamma$ provides an approximation for $n \cdot P_1$:

$$\mathrm{vol}\,(n \cdot P_1) \approx |n \cdot P_1 \cap \Gamma| \cdot \mathrm{vol}\,(P)\,.$$

(Note that $n \cdot P_1$ is the union of n^d copies of P_1.)



Proof. (cont.)

The error in this approximation is caused by the translates of P protruding and receding around the boundary of $n \cdot P_1$. Therefore, we can give the following estimate:

$$\operatorname{vol}(n \cdot P_1) = |n \cdot P_1 \cap \Gamma| \cdot \operatorname{vol}(P) + O\left(n^{d-1}\right).$$

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Observe that $\mathrm{vol}\left(n \cdot P_1\right) = n^d \cdot \mathrm{vol}\left(P_1\right)$ and $\left|n \cdot P_1 \cap \Gamma\right| = n^d \cdot \left|P_1 \cap \Gamma\right|$. Therefore, after dividing by n^d, we get

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Taking the limit as $n \to \infty$ we obtain the desired equality:

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Corollary

If Γ_1 is a sublattice of Γ, then $\mathrm{vol}\left(\Gamma_1\right)$ is a multiple of $\mathrm{vol}\left(\Gamma\right)$, and

$$\Gamma_1 = \Gamma \iff \mathrm{vol}\left(\Gamma_1\right) = \mathrm{vol}\left(\Gamma\right).$$

Definition

A set $S \subseteq \mathbb{R}^n$ is discrete if every element $s \in S$ has a neighborhood that contains no other elements from s. Formally:

$$\forall s \in S \; \exists \varepsilon > 0 : \; B_\varepsilon(s) \cap S = \{s\},$$

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Corollary

If $G \leq \mathbb{R}^n$ is a discrete group, then every bounded subset of $\mathbb{R}^n$ contains only finitely many elements of G.

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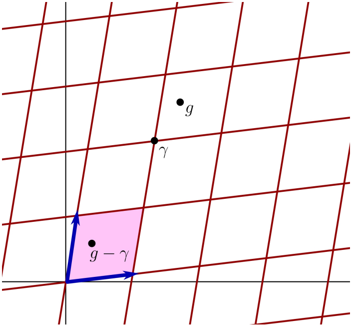
Let us choose linearly independent vectors $\omega_1, \ldots, \omega_n \in G$ with $|\det(\omega_1, \ldots, \omega_n)|$ minimal. Let $\Gamma = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$ and let P be the fundamental parallelotope of the lattice Γ.

In other words, $\Gamma \leq G$ is a sublattice of minimal volume. We claim that $\Gamma = G$.

Proof. (cont.)

Assume, on the contrary, that $\exists g \in G \setminus \Gamma$. Since the translates $\gamma + P$ $(\gamma \in \Gamma)$ cover $\mathbb{R}^n$, there exists $\gamma \in \Gamma$ such that $g \in \gamma + P$.



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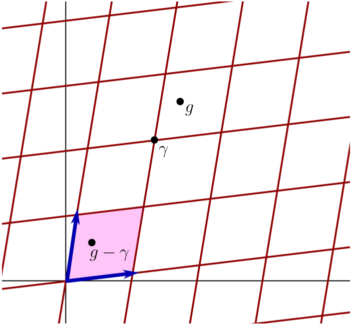

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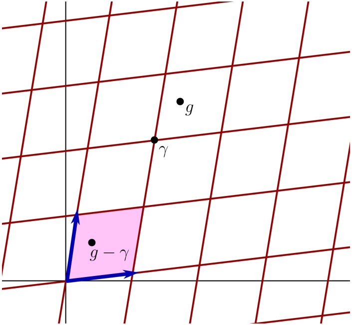
$$g - \gamma = x_1\omega_1 + \cdots + x_n\omega_n \ \ (0 \leq x_i < 1)\,,$$

where at least one of the x_i is nonzero, say (wlog) $x_1 \neq 0$.



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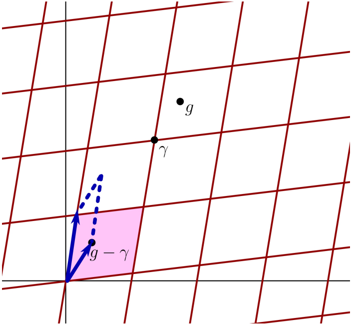
Let Γ_1 be the lattice obtained by replacing ω_1 by $g-\gamma$ in the basis: $\Gamma_1 = \mathbb{Z}(g-\gamma) + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_n$. We will prove that $\mathrm{vol}(\Gamma_1) < \mathrm{vol}(\Gamma)$.



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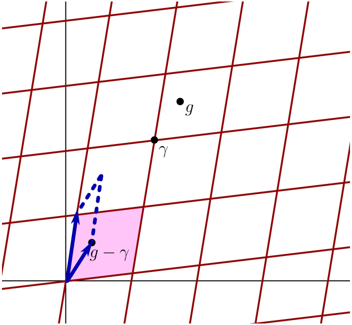
$$\mathrm{vol}(\Gamma_1) = |\det(g-\gamma, \omega_2, \ldots, \omega_n)| = \left|\det\left(\sum_{i=1}^{n} x_i\omega_i, \omega_2, \ldots, \omega_n\right)\right|$$

$$= \left|\sum_{i=1}^{n} x_i \det(\omega_i, \omega_2, \ldots, \omega_n)\right| = |x_1 \cdot \det(\omega_1, \omega_2, \ldots, \omega_n)| = x_1 \cdot \mathrm{vol}(\Gamma).$$

Hence $\mathrm{vol}(\Gamma_1) = x_1 \cdot \mathrm{vol}(\Gamma) < \mathrm{vol}(\Gamma)$, contradicting the minimality of $\mathrm{vol}(\Gamma)$. $\square$



Question

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Answer

We did not prove that there is a sublattice of minimal volume.

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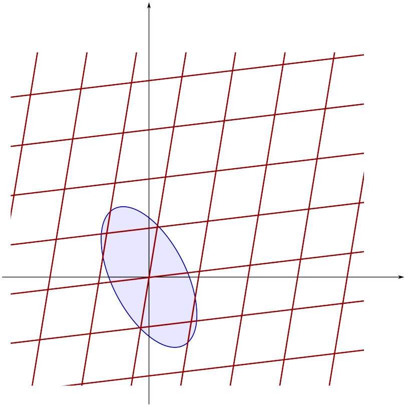
Theorem (Minkowski)

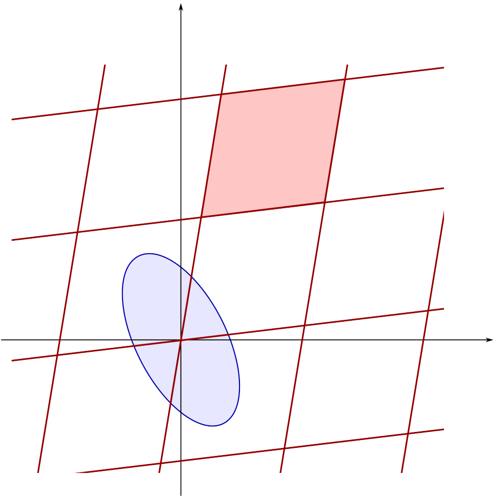
Let $\Gamma \leq \mathbb{R}^n$ be a full-rank lattice, and let $S \subseteq \mathbb{R}^n$ be a set such that

1. S is convex,
2. S is centrally symmetric with respect to the origin ($x \in S \implies -x \in S$),
3. $\mathrm{vol}\,(S) > 2^n \cdot \mathrm{vol}\,(\Gamma)$.

Then $S \cap \Gamma \neq \{0\}$, i.e., S contains at least one lattice point other than the origin.

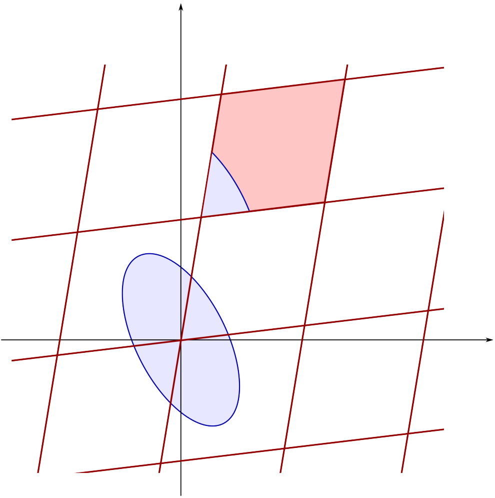


Proof of Minkowski's theorem



Let us pick one of the parallelotopes of 2Γ (its volume is $2^n \cdot \mathrm{vol}(\Gamma)$),

Proof of Minkowski's theorem



Let us pick one of the parallelotopes of 2Γ (its volume is $2^n \cdot \mathrm{vol}(\Gamma)$), and translate here all parallelotopes that intersect S.

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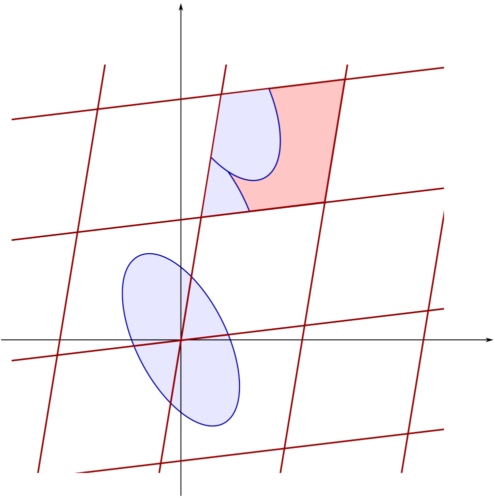

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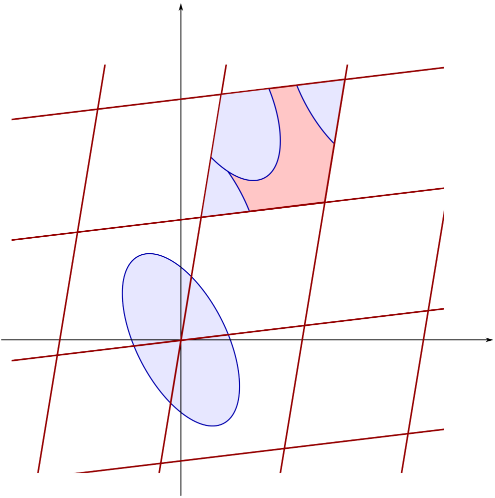

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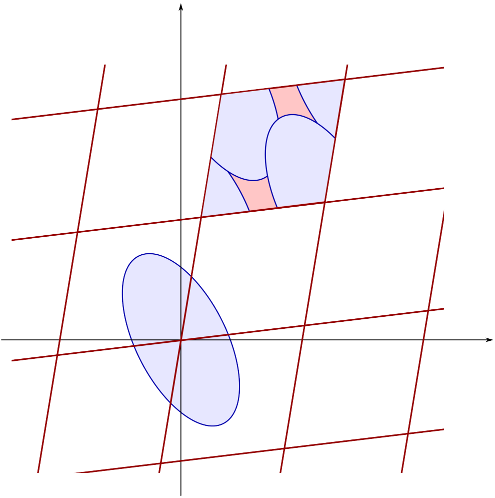

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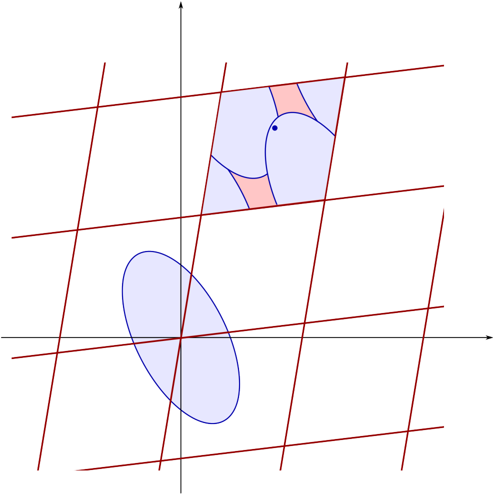

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Since $\mathrm{vol}(S) > 2^n \cdot \mathrm{vol}(\Gamma)$, there will be a point that is covered at least twice.

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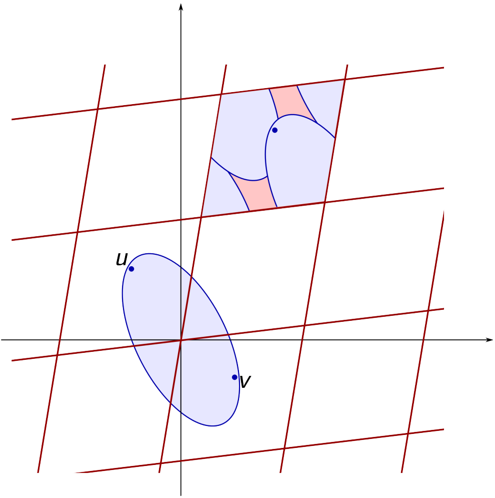

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Let u and v denote two different preimages of such a doubly covered point.

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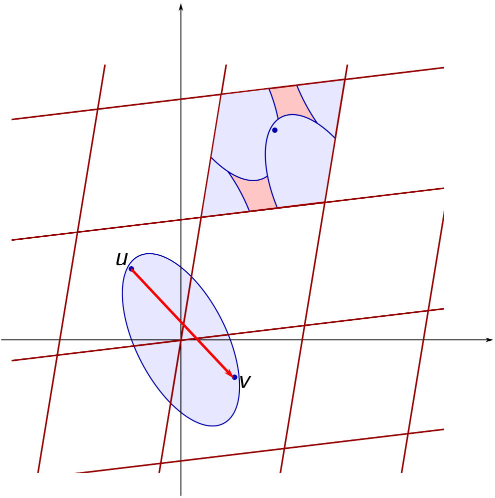

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Let u and v denote two different preimages of such a doubly covered point.

Then u and v are in the same coset of 2Γ, i.e., $u - v \in 2\Gamma$.

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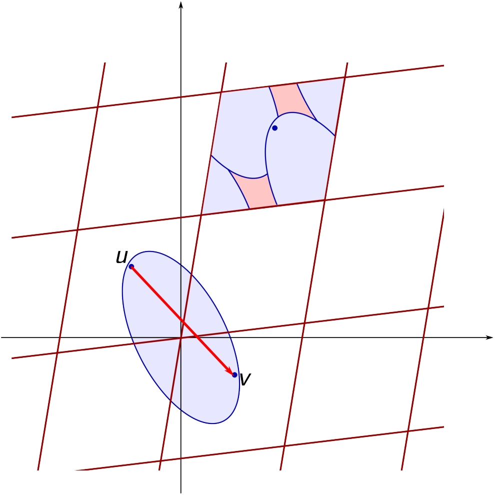

$$u, v \in S, \quad u - v \in 2\Gamma$$

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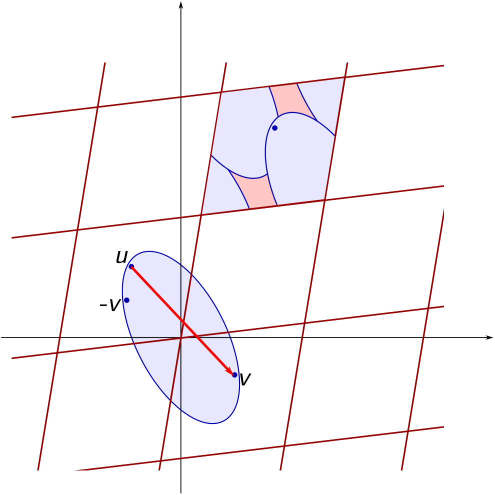

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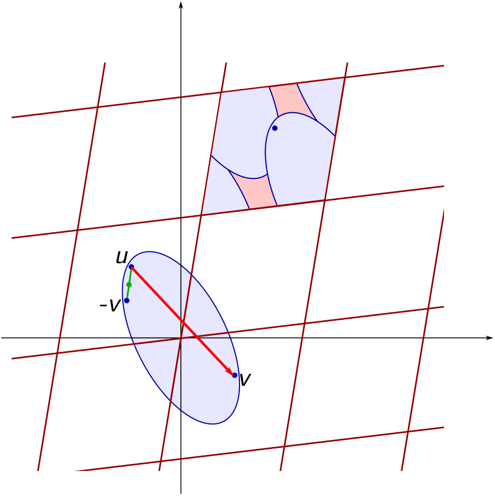

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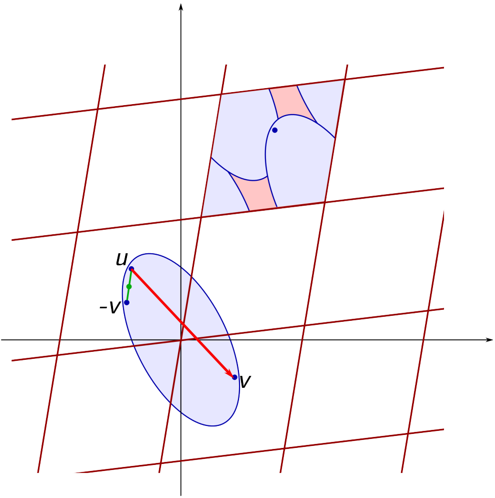

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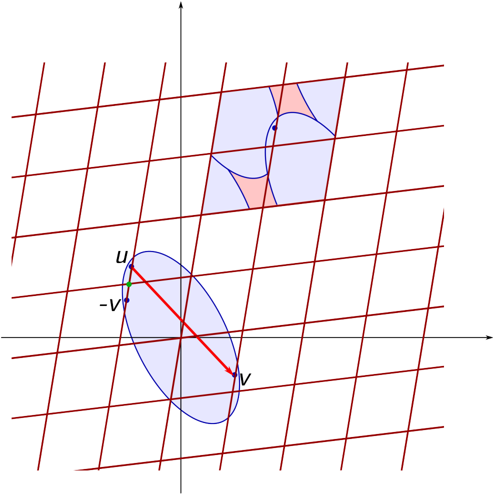

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Conclusion: $\dfrac{u-v}{2} \in S \cap \Gamma.$ $\square$