Maximal and minimal closed classes in multiple-valued logic

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***Abstract*—We consider classes of operations in multiple-valued logic that are closed under composition as well as under permutation of variables, identification of variables (diagonalization) and introduction of inessential variables (cylindrification). Such closed classes on a given finite set form a complete lattice that includes the lattice of clones as the principal filter above the trivial clone. We determine all maximal closed classes; it turns out that there is only one family of closed classes besides Rosenberg's six families of maximal clones. For minimal closed classes we prove an analogon of Rosenberg's five-type classification of minimal clones and we describe explicitly the unary closed classes.**

I. Introduction

Let A be a nonempty finite set, and let $\mathcal{C}$ be a class of finitary operations on A. If $\mathcal{C}$ is closed under composition of operations and contains the projections, then $\mathcal{C}$ is called a clone on A. There are countably infinitely many clones on a two-element base set, and all such clones were determined by Post [1]. For $|A| \geq 3$ there exists a continuum of clones on A (see [2]), and it is widely accepted that an explicit description of clones is an extremely difficult task even for $|A| = 3$. The set of all clones on A is a complete lattice, and many authors have investigated different parts of these lattices of clones. Here we focus on two special classes of clones: maximal and minimal clones. Maximal clones (i.e., coatoms in the lattice of clones) have been determined by Rosenberg [3] on arbitrary finite sets. The description of minimal clones (i.e., atoms in the lattice of clones) seems to be a considerably harder problem; a full description is available only for $|A| \leq 4$ (see [4], [5], [6], [7]). However, Rosenberg classified minimal clones into five types, and for two of the types he found necessary and sufficient conditions for minimality over arbitrary finite sets [8].

In this paper we generalize these two theorems of Rosenberg about maximal and minimal clones to more general classes of operations. We consider classes that are closed under composition but do not necessarily contain projections. However, we assume that our classes are closed under permutation of variables, identification of variables and introduction of inessential variables; in the case of clones, these properties are guaranteed by the presence of projections. The set of these closed classes on a given base set A forms a complete lattice under inclusion, in which the lattice of clones appears as the principal filter generated by the trivial clone (i.e., the clone that consists of projections only). We determine the coatoms in the lattice of closed classes, thus extending Rosenberg's theorem about maximal clones to these more general classes of operations. We will see that the coatoms are exactly the maximal clones and one more family of closed classes without projections. Since the bottom element of the lattice of closed classes is the empty class, the atoms are quite trivial to determine (one of them is the trivial clone). The true analogues of minimal clones turn out to be the closed classes on the "second floor" of the lattice, i.e., the covers of atoms, hence we will refer to these classes as minimal closed classes. These include minimal clones, hence describing them is at least as difficult as describing minimal clones. We provide a classification of minimal closed classes in the spirit of Rosenberg's classification of minimal clones, and we determine minimal closed classes for one of the five types, namely for the unary type.

In the next section we recall the definitions and results about classes of operations, composition, clones and relations that will be used in the sequel. For more background on these topics we refer the reader to the monographs [9] and [10]. Then in Section III we state and prove the above mentioned generalization of Rosenberg's theorem on maximal clones, and in Section IV we establish a generalization of Rosenberg's classification of minimal clones.

II. Preliminaries

Throughout this paper, A is a nonempty finite set, and $\mathcal{O}_A = \bigcup_{n\geq 1} A^{A^n}$ denotes the set of all finitary operations on A. For a class $\mathcal{K} \subseteq \mathcal{O}_A$ of operations, let $\mathcal{K}^{(n)}$ stand for the n-ary part of $\mathcal{K}$, i.e., $\mathcal{K}^{(n)} := \mathcal{K} \cap A^{A^n}$. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$.

A. Composition of operations and classes of operations

For $f \in \mathcal{O}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{O}_A^{(k)}$, the *composition* of f by $g_1, \ldots, g_n$ is the operation $f(g_1, \ldots, g_n) \in \mathcal{O}_A^{(k)}$ defined by

$$(f(g_1, \ldots, g_n))(\mathbf{x}) = f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) \text{ for all } \mathbf{x} \in A^k.$$

We can extend this definition to *composition of classes of operations*: for $\mathcal{K}, \mathcal{L} \subseteq \mathcal{O}_A$ let $\mathcal{K} \circ \mathcal{L}$ denote the set

$$\left\{ f(g_1, \ldots, g_n) : k, n \in \mathbb{N}, f \in \mathcal{K}^{(n)}, g_1, \ldots, g_n \in \mathcal{L}^{(k)} \right\}.$$

This composition is a binary operation on the power set of $\mathcal{O}_A$. In general, it is not associative, but it becomes associative when restricted to equational classes (see, e.g., [11], [12]).

B. Subfunctions and equational classes

The i-th n-ary *projection* for $1 \leq i \leq n \in \mathbb{N}$ is the operation $e_i^{(n)} \in \mathcal{O}_A^{(n)}$ such that $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ for all $(x_1, \ldots, x_k) \in A^n$. For $f, g \in \mathcal{O}_A$, we say that g is a *subfunction* of f (notation: $g \preceq f$) if g belongs to the class composition

$$\{f\} \circ \{e_i^{(n)} : n \in \mathbb{N}, i \in [n]\},$$

i.e., if g can be obtained from f by permutation of variables, identification of variables (diagonalization) and introduction of inessential variables (cylindrification). The subfunction relation is a quasiorder on $\mathcal{O}_A$, and the corresponding equivalence is defined by $f \equiv g \iff f \preceq g$ and $g \preceq f$. It is easy to see that two operations are equivalent if and only if they can be obtained from each other by permutation of variables and introduction or deletion of inessential variables. In particular, denoting the identity function on A by id, we have $f \equiv \mathrm{id}$ if and only if f is a projection. Therefore, we will simply write $\{\mathrm{id}\}$ for the set of projections.

A class $\mathcal{K} \subseteq \mathcal{O}_A$ of operations on A is called an *equational class* if it is an order ideal in the subfunction quasiorder. This terminology is motivated by the fact that definability by certain types of functional equations is equivalent to being closed under forming subfunctions [13], [14]. Although composition of classes of operations is not associative in general, equational classes form a semigroup under composition [11]. Every clone is an idempotent element in this semigroup, and every idempotent is a composition-closed equational class (see the formal definition in the next subsection). In [15] the study of the semigroup of equational classes was initiated with the intention of obtaining a better understanding of composition of operations and composition-closed classes such as clones.

C. Clones and closed classes

A *clone* on A is a class $\mathcal{K} \subseteq \mathcal{O}_A$ that is closed under composition and contains all projections:

$$\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K} \text{ and } \{\mathrm{id}\} \subseteq \mathcal{K}. \tag{1}$$

The least clone containing a given class $\mathcal{K} \subseteq \mathcal{O}_A$ is denoted by $[\mathcal{K}]$; it consists of those operations that can be built from members of $\mathcal{K}$ and from projections by means of composition. The set of all clones on a fixed base set A constitutes a lattice under inclusion, with the lattice operations being $\mathcal{C}_1 \wedge \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_1 \vee \mathcal{C}_2 = [\mathcal{C}_1 \cup \mathcal{C}_2]$. The least element of the lattice of clones over A is the clone containing only projections, which is called the *trivial clone*, and the greatest element is $\mathcal{O}_A$, the clone of all operations on A. Atoms and coatoms of the lattice of clones are called *minimal clones* and *maximal clones*, respectively. As mentioned in the introduction, maximal clones on finite sets are completely known [3], while for minimal clones only a classification is available, with two of the five types completely described [8].

We consider in this paper equational classes that are closed under composition, that is, classes $\mathcal{K} \subseteq \mathcal{O}_A$ such that $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$ and $f \in \mathcal{K}, g \preceq f \implies g \in \mathcal{K}$ for all $g \in \mathcal{O}_A$, or, more compactly,

$$\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K} \text{ and } \mathcal{K} \circ \{\mathrm{id}\} \subseteq \mathcal{K}. \tag{2}$$

For brevity, in the following we will simply write *closed class* instead of composition-closed equational class. For $\mathcal{K} \subseteq \mathcal{O}_A$, we denote by $\lfloor \mathcal{K} \rfloor$ the least closed class containing $\mathcal{K}$. The set of all closed classes on A forms a lattice under inclusion, with the lattice operations being $\mathcal{K}_1 \wedge \mathcal{K}_2 = \mathcal{K}_1 \cap \mathcal{K}_2$ and $\mathcal{K}_1 \vee \mathcal{K}_2 = \lfloor \mathcal{K}_1 \cup \mathcal{K}_2 \rfloor$. Note that the least element of this lattice is the empty class. For all $\mathcal{K} \subseteq \mathcal{O}_A$, we have $[\mathcal{K}] = \lfloor \mathcal{K} \cup \{\mathrm{id}\} \rfloor$, and $\mathcal{C} \subseteq \mathcal{O}_A$ is a clone if and only if $\mathcal{C}$ is a closed class and $\mathrm{id} \in \mathcal{C}$. Therefore, the lattice of clones appears in the lattice of closed classes as the principal filter generated by the trivial clone. The lattice of closed classes over a two-element base set has already continuum cardinality; this lattice has been described in [16].

Remark 1: Iterative algebras provide another generalization of clones. These classes are usually defined by the five Mal'tsev operations ζ, τ, Δ, ∇, $\star$ (see [9], [10]), but they can also be defined by means of class composition as follows. A class $\mathcal{K} \subseteq \Omega$ is an iterative algebra iff

$$\mathcal{K} \circ (\mathcal{K} \cup \{\mathrm{id}\}) \subseteq \mathcal{K}. \tag{3}$$

It is clear that $(1) \implies (3) \implies (2)$, hence every clone is an iterative algebra, and every iterative algebra is a closed class.

D. Relations and constraints

For an m-ary relation $R \subseteq A^m$ and a matrix $N \in A^{m \times n}$, we say that N is an R-matrix if each column of N belongs to R. If $f \in \mathcal{O}_A^{(n)}$, then fN stands for the m-tuple that is obtained by applying f row-wise to N, and let $fR = \{fN : N \in A^{m \times n} \text{ is an } R\text{-matrix}\}$. If $fR \subseteq R$, then we say that f *preserves* the relation R. For a set $\mathcal{Q}$ of relations, let $\operatorname{Pol} \mathcal{Q}$ denote the set of all operations that preserve every member of $\mathcal{Q}$:

$$\operatorname{Pol} \mathcal{Q} = \{f \in \mathcal{O}_A : \forall R \in \mathcal{Q} \;\; fR \subseteq R\}.$$

Preservation of relations induces a Galois connection between operations and relations on A. The corresponding Galois closed sets are exactly the clones and the so-called relational clones (see [17], [18]). Thus, $\mathcal{C} \subseteq \mathcal{O}_A$ is a clone if and only if there exists a set of relations $\mathcal{Q}$ such that $\mathcal{C} = \operatorname{Pol} \mathcal{Q}$. Relational clones can also be characterized as sets of relations closed under certain constructions; since we will not need this description, we do not give the details here. We will only use the fact that the relations in the smallest relational clone (generated by the unary total relation A) are exactly relations of the form

$$S = \{\mathbf{a} \in A^m : a_i = a_j \text{ whenever } (i, j) \in \varepsilon\}, \tag{4}$$

where ε is an equivalence relation on $[m]$. In other words, a relation is preserved by all operations on A if and only if S is of form (4).

Composition-closed equational classes can be described by *relational constraints*, i.e., pairs (R, S) of relations, where R and S have the same arity. For a set $\mathcal{Q}$ of relational constraints, we define $\operatorname{Pol}^* \mathcal{Q} \subseteq \mathcal{O}_A$ as follows:

$$\operatorname{Pol}^* \mathcal{Q} = \{f \in \mathcal{O}_A \colon \forall (R, S) \in \mathcal{Q} \;\; fR \subseteq S \text{ and } fS \subseteq S\}.$$

It was shown in [16] that a class $\mathcal{K} \subseteq \mathcal{O}_A$ is a closed class if and only if there exists a set of relational constraints $\mathcal{Q}$ such that $\mathcal{C} = \operatorname{Pol}^* \mathcal{Q}$.

Remark 2: Iterative algebras (cf. Remark 1) also admit a characterization in terms of relational constraints, as shown by Harnau in [19]. Here one uses pairs of relations (R, S) with $S \subseteq R$, and an operation f is said to preserve such a pair if $fR \subseteq S$.

III. Maximal closed classes

First let us recall Rosenberg's description of maximal clones. We do not define the types of relations appearing in the theorem, as we will not need them.

Theorem 1 (Rosenberg's theorem [3]): A clone $\mathcal{C} \subseteq \mathcal{O}_A$ is a maximal clone if and only if $\mathcal{C} = \operatorname{Pol} R$ for some relation R satisfying one of the following six conditions:

(i) R is a bounded partial order;
(ii) R is the graph of a permutation of prime order;
(iii) R is a nontrivial equivalence relation;
(iv) R is a prime-affine relation;
(v) R is a central relation;
(vi) R is an h-regular relation.

Proposition 2: Every composition-closed equational class of $\mathcal{O}_A$ is contained in a maximal composition-closed equational class.

Proof: It is well known that $\mathcal{O}_A$ is a finitely generated clone. In fact, $\mathcal{O}_A$ can be generated by a single operation; such a generator is called a *Sheffer operation*. For instance, the operation defined by

$$g(x, y) = \begin{cases} x \oplus 1, & \text{if } x = y; \\ 0, & \text{if } x \neq y. \end{cases}$$

is a Sheffer operation on $A = \{0, 1, \ldots, n-1\}$, where $\oplus$ stands for addition modulo n (see [20]). Thus, we have $[g] = \mathcal{O}_A$, hence $\lfloor g, \mathrm{id} \rfloor = \mathcal{O}_A$. This shows that $\mathcal{O}_A$ is a finitely generated closed class, and then by Zorn's lemma we have that each closed class is contained in a maximal one. ∎

We will prove that the maximal closed subclasses of $\mathcal{O}_A$ are exactly the maximal clones together with the classes

$$\mathcal{M}_{ab} := \{f \in \mathcal{O}_A \colon f(a, \ldots, a) = f(b, \ldots, b)\}$$

for $a, b \in A, a \neq b$.

It is easy to verify directly that $\mathcal{M}_{ab}$ is a closed class; alternatively, one can observe that $\mathcal{M}_{ab} = \operatorname{Pol}^* (\{(a, b)\}, =)$.

Lemma 3: For any $a, b \in A, a \neq b$, the clone generated by $\mathcal{M}_{ab}$ is $\mathcal{O}_A$.

Proof: It would be sufficient to verify that $\mathcal{M}_{ab}$ is not contained in any of the maximal clones given in Theorem 1. Alternatively, since $\mathcal{M}_{ab}$ contains all constants, one could use one of the criteria for functional completeness (e.g., the Werner-Wille theorem). However, it seems easier to give a direct proof as follows. Let $f \in \mathcal{O}_A$ be an arbitrary operation, and let n denote the arity of f. Choose any operation $g \in \mathcal{O}_A$ of arity $n+2$ such that

$$g(x_1, \ldots, x_n, a, b) = f(x_1, \ldots, x_n) \tag{5a}$$

for all $(x_1, \ldots, x_n) \in A^n$, and

$$g(x, \ldots, x) = a \tag{5b}$$

for all $x \in A$. (Clearly, there are many such operations g; the values of g on tuples not listed above are irrelevant.) Condition (5b) guarantees that $g \in \mathcal{M}_{ab}$, and then by (5a) we conclude $f \in [\mathcal{M}_{ab}]$, as the constants a and b also belong to $\mathcal{M}_{ab}$. ∎

Theorem 4: The maximal composition-closed equational classes on A are the maximal clones (see Theorem 1) and the classes $\mathcal{M}_{ab}$ with $a, b \in A, a \neq b$.

Proof: Let $\mathcal{K}$ be a maximal closed class that is not a clone. Since $\mathcal{K}$ is closed, there exists a set $\mathcal{Q}$ of relational constraints such that $\mathcal{K} = \operatorname{Pol}^* (\mathcal{Q})$. For each $(R, S) \in \mathcal{Q}$ we have $\operatorname{Pol}^* (R, S) \supseteq \mathcal{K}$, hence $\operatorname{Pol}^* (R, S) = \mathcal{K}$ or $\operatorname{Pol}^* (R, S) = \mathcal{O}_A$ by the maximality of $\mathcal{K}$. From $\mathcal{K} = \bigcap_{(R,S) \in \mathcal{Q}} \operatorname{Pol}^* (R, S)$ it follows that there exists $(R, S) \in \mathcal{Q}$ such that $\operatorname{Pol}^* (R, S) = \mathcal{K}$. Then we have $\mathcal{K} = \operatorname{Pol}^* (R, S) \subseteq \operatorname{Pol}(S)$, hence either $\operatorname{Pol}(S) = \mathcal{K}$ or $\operatorname{Pol}(S) = \mathcal{O}_A$, again by the maximality of $\mathcal{K}$. However, the first case is impossible, as $\mathcal{K}$ is not a clone. Thus $\operatorname{Pol}(S) = \mathcal{O}_A$, which means that S belongs to the smallest relational clone (generated by the unary total relation A), thus S is of the form (4) for some $m \in \mathbb{N}$ and an equivalence relation ε on $[m]$.

Next we show that $R \nsubseteq S$. Suppose for contradiction that $R \subseteq S$, and let $f \in \mathcal{O}_A$ be an arbitrary operation. Then $fR \subseteq fS \subseteq S$, since S is preserved by all operations. This implies that $\operatorname{Pol}^* (R, S) = \mathcal{O}_A$, which contradicts $\operatorname{Pol}^* (R, S) = \mathcal{K}$.

Thus $R \nsubseteq S$, and then there exists a tuple $\mathbf{r} \in R \setminus S$. Taking into account that S is given by (4), $\mathbf{r} \notin S$ implies that there exist $i, j \in [m]$ such that $(i, j) \in \varepsilon$ but $r_i \neq r_j$. We claim that $\mathcal{K} = \mathcal{M}_{ab}$ with $a = r_i, b = r_j$.

Let $f \in \mathcal{K}$ be an arbitrary operation, and let N be the $m \times n$ matrix such that each column vector of N is $\mathbf{r}$ and n is the arity of f. Clearly, N is an R-matrix. Since $f \in \mathcal{K} = \operatorname{Pol}^* (R, S)$, we have $fN \in S$. The i-th and j-th entries of the tuple fN are $f(a, \ldots, a)$ and $f(b, \ldots, b)$, respectively. Therefore, $fN \in S$ and $(i, j) \in \varepsilon$ imply that $f(a, \ldots, a) = f(b, \ldots, b)$ according to (4). This shows that $f \in \mathcal{M}_{ab}$ for every $f \in \mathcal{K}$, i.e., $\mathcal{K} \subseteq \mathcal{M}_{ab}$. By the maximality of $\mathcal{K}$ we can conclude that $\mathcal{K} = \mathcal{M}_{ab}$.

We have proved that every maximal closed class is either a maximal clone or one of the classes $\mathcal{M}_{ab}$ with $a \neq b$. It remains to prove that each of these classes is indeed maximal. By Proposition 2, every closed class is contained in a maximal one, hence it suffices to prove that the aforementioned classes are pairwise incomparable. It is clear that maximal clones are pairwise incomparable as well as the classes $\mathcal{M}_{ab}$ $(a, b \in A, a \neq b)$. Now let $\mathcal{C}$ be a maximal clone and let let $a, b \in A, a \neq b$. There are no projections in $\mathcal{M}_{ab}$, hence $\mathcal{C} \nsubseteq \mathcal{M}_{ab}$, and $\mathcal{M}_{ab} \nsubseteq \mathcal{C}$ follows immediately from Lemma 3. ■

Corollary 5: There are finitely many maximal composition-closed equational classes on A.

Proof: It follows from Theorem 1 that there are finitely many maximal clones on A, and it is clear that there are finitely many classes $\mathcal{M}_{ab}$. ■

Example 6: All closed classes on $\{0,1\}$ have been described in [16]; in particular, the maximal closed classes turned out to be the five maximal clones of Boolean functions (0-preserving functions, 1-preserving functions, monotone functions, linear functions, selfdual functions) and the class $\Omega_= = \{f \in \mathcal{O}_{\{0,1\}} : f(0,\ldots,0) = f(1,\ldots,1)\}$. Theorem 4 indicates that the situation is similar over arbitrary finite sets, as the classes $\mathcal{M}_{ab}$ are immediate generalizations of $\Omega_=$.

IV. Minimal closed classes

First we determine the atoms of the lattice of closed classes on A. We say that a unary operation $u \in \mathcal{O}_A^{(1)}$ is *idempotent*, if $u^2 = u$, i.e., $u(u(x)) = u(x)$ holds for all $x \in A$. Observe that u is idempotent if and only if $u(x) = x$ for all $x \in \operatorname{ran} u := \{u(x) : x \in A\}$. (Note that it is also customary to say that $f \in \mathcal{O}_A$ is idempotent if $f(x,\ldots,x) = x$ for all $x \in A$. We will not use this notion of idempotence in this paper.)

Proposition 7: A class $\mathcal{K} \subseteq \mathcal{O}_A$ is an atom in the lattice of composition-closed equational classes on A if and only if $\mathcal{K} = \lfloor u \rfloor$ for some idempotent unary operation $u \in \mathcal{O}_A^{(1)}$.

Proof: Let $\mathcal{K}$ be an atom in the lattice of closed classes, and let $f \in \mathcal{K}$ be an arbitrary operation. Since $\mathcal{K}$ is an equational class, the unary operation $g(x) := f(x,\ldots,x)$ belongs to $\mathcal{K}$. Finiteness of A implies that some power of g is idempotent, i.e., there exists $k \in \mathbb{N}$ such that $u := g^k$ satisfies $u^2 = u$. Clearly, $u \in \mathcal{K}$, as $\mathcal{K}$ is closed under composition. Therefore, $\emptyset \subsetneqq \lfloor u \rfloor \subseteq \mathcal{K}$, and then $\lfloor u \rfloor = \mathcal{K}$, since $\mathcal{K}$ has no proper nonempty closed subclasses. ■

Remark 3: If u is an idempotent unary operation, then $\lfloor u \rfloor$ contains only the operations that are equivalent to u, i.e., essentially unary operations f of the form $f(x_1,\ldots,x_n) = u(x_i)$. For $u = \mathrm{id}$ we obtain the trivial clone $\lfloor \mathrm{id} \rfloor = [\mathrm{id}] = \{e_i^{(n)} : n \in \mathbb{N}, i \in [n]\}$. In the following we will refer to the atoms described in Proposition 7 as *trivial closed classes*. The proof of Proposition 7 shows that every nontrivial nonempty closed class contains a trivial closed class.

To be in accordance with the terminology of clone theory, we shall say that a nonempty nontrivial closed class $\mathcal{K}$ is a *minimal closed class*, if the only nonempty nontrivial closed subclass of $\mathcal{K}$ is $\mathcal{K}$ itself. If $\mathcal{K}$ is a minimal closed class and $\lfloor u \rfloor \subseteq \mathcal{K}$ is a trivial closed class, then $\mathcal{K}$ covers $\lfloor u \rfloor$ in the lattice of closed classes, and we will briefly express this fact by saying that $\mathcal{K}$ is a minimal closed class above $\lfloor u \rfloor$. Note that it may happen that $\mathcal{K}$ covers two trivial closed classes; see Remark 4.

Now we recall Rosenberg's theorem on minimal clones, and then we present the corresponding result for minimal closed classes.

Theorem 8 (Rosenberg's theorem [8]): Let $\mathcal{C} \subseteq \mathcal{O}_A$ be a minimal clone, and let f be an operation in $\mathcal{C} \setminus [\mathrm{id}]$ of minimum arity. Then $\mathcal{K} = [f]$ and one of the following five conditions hold for f:

(I) f is a unary operation;
(II) f is a binary operation such that for all $x \in A$, $f(x,x) = x$;
(III) f is a ternary operation such that for all $x, y \in A$, $f(x,x,y) = f(x,y,x) = f(y,x,x) = x$;
(IV) f is a ternary operation such that for all $x, y \in A$, $f(x,x,y) = f(x,y,x) = f(y,x,x) = y$;
(V) f is of arity n with $3 \leq n \leq |A|$, and there exists an $i \in [n]$ such that $f(x_1,\ldots,x_n) = x_i$ whenever $|\{x_1,\ldots,x_n\}| < n$.

Theorem 9: Let $\mathcal{K} \subseteq \mathcal{O}_A$ be a minimal composition-closed equational class above $\lfloor u \rfloor$, where $u \in \mathcal{O}_A^{(1)}$ is an idempotent unary operation. Let f be an operation in $\mathcal{K} \setminus \lfloor u \rfloor$ of minimum arity. Then $\mathcal{K} = \lfloor f, u \rfloor$, and one of the following five conditions hold for f:

(I) f is a unary operation;
(II) f is a binary operation such that for all $x \in A$, $f(x,x) = u(x)$;
(III) f is a ternary operation such that for all $x, y \in A$, $f(x,x,y) = f(x,y,x) = f(y,x,x) = u(x)$;
(IV) f is a ternary operation such that for all $x, y \in A$, $f(x,x,y) = f(x,y,x) = f(y,x,x) = u(y)$;
(V) f is of arity n with $3 \leq n \leq |A|$, and there exists an $i \in [n]$ such that $f(x_1,\ldots,x_n) = u(x_i)$ whenever $|\{x_1,\ldots,x_n\}| < n$.

Proof: Since $\mathcal{K}$ is minimal, it is clear that $\mathcal{K} = \lfloor f, u \rfloor$ for any $f \in \mathcal{K} \setminus \lfloor u \rfloor$. Let $f \in \mathcal{K} \setminus \lfloor u \rfloor$ be of minimum arity, and let us denote this minimal arity by n. If $n = 1$, then f is of type (I). From now on we shall assume that $n \geq 2$. If g is any operation that is obtained from f by identifying some of its variables, then $g \in \lfloor u \rfloor$ by the minimality of the arity of f, hence $g \equiv u$. If $n = 2$, then this immediately implies that (II) holds.

If $n \geq 4$, then by a generalization of Świerczkowski's lemma (see Theorem 7 in [21]) there exists an index $i \in [n]$ such that $f(x_1,\ldots,x_n) = u(x_i)$ whenever $x_1,\ldots,x_n \in A$ are not pairwise different. If $n > |A|$, then this implies $f(x_1,\ldots,x_n) = u(x_i)$ for all $x_1,\ldots,x_n \in A$, i.e., $f \equiv u$. However, this contradicts the assumption $f \in \mathcal{K} \setminus \lfloor u \rfloor$. Therefore, we must have $n \leq |A|$, and we can conclude that f is of type (V).

It only remains to consider the case $n = 3$. By the above arguments, there exist $r, s, t \in \{x, y\}$ such that for all $x, y \in A$ we have

$$f(x,x,y) = u(r),$$
$$f(x,y,x) = u(s),$$
$$f(y,x,x) = u(t).$$

The cases $(r,s,t) = (x,x,x)$ and $(r,s,t) = (y,y,y)$ correspond to types (III) and (IV), respectively, while the cases

$(r,s,t) \in \{(x,x,y), (x,y,x), (y,x,x)\}$ correspond to type (V).

In the remaining three cases we can assume (up to a permutation of variables) that $(r,s,t) = (y,x,y)$. If u is a constant operation, then f belongs to types (III) and (IV), which coincide in this case. Thus we may assume without loss of generality that the range of u contains two different elements, say a and b. In particular, we have

$$f(a,a,b) = u(b) = b. \tag{6}$$

Using the idempotence of u, it is easy to see that the operation $g(x,y,z) := f(u(x), f(x,y,z), u(z)) \in \mathcal{K}$ satisfies

$$\forall x,y \in A: g(x,x,y) = g(x,y,x) = g(y,x,x) = u(x). \tag{7}$$

Moreover, it is also straightforward to verify (by term induction) that every ternary operation in $\lfloor g,u \rfloor \setminus \lfloor u \rfloor$ satisfies (7) as well. The minimality of $\mathcal{K}$ implies that $\mathcal{K} = \lfloor g,u \rfloor$. Therefore, we have $f \in \lfloor g,u \rfloor \setminus \lfloor u \rfloor$, hence f satisfies (7), too. In particular, we have $f(a,a,b) = a$, which contradicts (6). Thus, the case $(r,s,t) = (y,x,y)$ is impossible whenever u is not constant, and this completes the proof. ■

Corollary 10: There are finitely many minimal composition-closed equational classes in $\mathcal{O}_A$, and every nonempty nontrivial composition-closed equational class contains a minimal one.

Proof: Since A is finite, there are finitely many operations on A of arity at most $|A|$, hence there are finitely many minimal closed classes by Theorem 9.

To prove the second statement of the theorem, let us denote by T the set of all closed classes of $\mathcal{O}_A$ that are of the form $\lfloor f,u \rfloor$, where u is an idempotent unary operation and $f \notin \lfloor u \rfloor$ satisfies one of the five conditions listed in Theorem 9. (Note that we do *not* require here that $\lfloor f,u \rfloor$ is a minimal closed class.) We will show that every nontrivial closed class contains a closed subclass that belongs to T. To this extent, let $\mathcal{K}$ be a nontrivial closed subclass of $\mathcal{O}_A$, and let $u \in \mathcal{K}^{(1)}$ with $u^2 = u$. If $\mathcal{K}$ contains a unary operation $f \neq u$, then $\mathcal{K}$ contains the closed subclass $\lfloor f,u \rfloor \in T$, which corresponds to type (I). Now let us assume that u is the only unary operation in $\mathcal{K}$, and let $f \in \mathcal{K} \setminus \lfloor u \rfloor$ be of minimum arity. Then the proof of Theorem 9 shows that f satisfies one of the conditions (II)–(V), hence $\lfloor f,u \rfloor \in T$ is the desired closed subclass of $\mathcal{K}$, and this completes the proof of our claim.

Now let $\mathcal{K}$ be an arbitrary nonempty nontrivial closed class, and let P be the set of closed subclasses of $\mathcal{K}$ that belong to T. Then $(P; \subseteq)$ is a finite nonempty partially ordered set, hence it contains at least one minimal element $\mathcal{K}_0$. We claim that $\mathcal{K}_0$ is a minimal closed class. Suppose for contradiction that $\mathcal{K}_1 \subsetneqq \mathcal{K}_0$ is a proper nonempty nontrivial closed subclass of $\mathcal{K}_0$. Then, by the second paragraph of this proof, there exists a class $\mathcal{K}_2 \subseteq \mathcal{K}_1$ such that $\mathcal{K}_2 \in P$. Therefore, we have $\mathcal{K}_2 \subsetneqq \mathcal{K}_0$ and $\mathcal{K}_0, \mathcal{K}_2 \in P$, which contradicts the fact that $\mathcal{K}_0$ is a minimal element of the poset P. Thus, $\mathcal{K}_0$ is indeed a minimal closed subclass of $\mathcal{K}$. ■

Minimal clones of type (I) and type (IV) have been explicitly described by Rosenberg in [8]. A unary operation f generates a minimal clone if and only if either f is idempotent ($f^2 = f$) or f is a permutation of prime order ($f^p = \mathrm{id}$ for some prime p). A ternary operation f of type (IV) generates a minimal clone if and only if there exists a binary operation $+$ on A such that $(A;+)$ is an Abelian group of exponent 2 and $f(x,y,z) = x+y+z$. In the following theorem we describe minimal closed classes $\mathcal{K}$ of type (I). In this case all operations in $\mathcal{K}$ are essentially unary (and equivalent to some member of $\mathcal{K}^{(1)}$), hence it suffices to describe the unary part $\mathcal{K}^{(1)}$.

Theorem 11: Let $\mathcal{K} \subseteq \mathcal{O}_A$ be a minimal composition-closed equational class of type (I) above $\lfloor u \rfloor$, where $u \in \mathcal{O}_A^{(1)}$ is an idempotent unary operation. Then there exists $f \in \mathcal{K}^{(1)} \setminus \{u\}$ such that $\mathcal{K} = \lfloor f,u \rfloor$ and one of the following three conditions hold:

($\mathrm{I_a}$) there exists a prime p such that $f^p = u, fu = uf = f$; in this case we have that $\mathcal{K}^{(1)} = \{f, f^2, \ldots, f^p\}$;

($\mathrm{I_b}$) $f^2 = f,\ fu, uf \in \{f,u\}$; in this case we have that $\mathcal{K}^{(1)} = \{f,u\}$;

($\mathrm{I_c}$) $f^2 = fu = uf = u$; in this case we have that $\mathcal{K}^{(1)} = \{f, f^2\}$.

Proof: Since every member of $\mathcal{K}$ is essentially unary, we may work with its unary part, which constitutes a subsemigroup of the transformation semigroup $\mathcal{O}_A^{(1)}$. The minimality of $\mathcal{K}$ means that $\mathcal{K}^{(1)}$ has exactly two subsemigroups containing u, namely $\{u\}$ and $\mathcal{K}^{(1)}$. Let $f \in \mathcal{K}^{(1)} \setminus \{u\}$ be an arbitrary operation, then $\mathcal{K} = \lfloor f,u \rfloor$ and $\mathcal{K}^{(1)}$ (as a semigroup) is generated by f and u. Since $\mathcal{K}^{(1)}$ is a finite semigroup, each of its elements has an idempotent power. In particular, there exists $k \in \mathbb{N}$ such that f^k is idempotent. We separate two cases on whether $f^k = u$ or not.

Case 1: $f^k = u$. In this case $\mathcal{K}^{(1)}$ is generated by f, hence it is a cyclic semigroup. If the index of this cyclic semigroup is at least 2, then $\{f^2, f^3, \ldots\}$ is a proper subsemigroup of $\mathcal{K}^{(1)}$. By minimality, this implies $\{f^2, f^3, \ldots\} = \{u\}$, hence $f^2 = u$ and $f^3 = fu = uf = u$, and thus the conditions of ($\mathrm{I_c}$) are fulfilled (in this case $\mathcal{K}^{(1)}$ is a two-element zero semigroup). If the index of $\mathcal{K}^{(1)}$ is 1, then $\mathcal{K}^{(1)}$ is a group with identity element u. Then it is clear that $\mathcal{K}$ is minimal if and only if $\mathcal{K}^{(1)}$ is a cyclic group of prime order, hence ($\mathrm{I_a}$) is satisfied.

Case 2: $f^k \neq u$. In this case f^k and u generate a subsemigroup of $\mathcal{K}^{(1)}$ that properly contains $\{u\}$. By minimality, this subsemigroup must be all of $\mathcal{K}^{(1)}$. Therefore, $\mathcal{K}^{(1)}$ is generated by two idempotents, hence we may assume without loss of generality that f is idempotent (we replace the generating set $\{f,u\}$ with $\{f^k,u\}$). It is clear that $\mathcal{K}^{(1)} \circ \{u\} := \{gu \colon g \in \mathcal{K}^{(1)}\}$ is a subsemigroup of $\mathcal{K}^{(1)}$ that contains $\{u\}$. Therefore, we have either $\mathcal{K}^{(1)} \circ \{u\} = \mathcal{K}^{(1)}$ or $\mathcal{K}^{(1)} \circ \{u\} = \{u\}$. In the former case $f = gu$ for some $g \in \mathcal{K}^{(1)}$, hence $fu = gu^2 = gu = f$, while in the latter case $fu = u$. A similar argument, using the subsemigroup $\{u\} \circ \mathcal{K}^{(1)}$, shows that $uf = f$ or $uf = u$, hence the conditions of ($\mathrm{I_b}$) are satisfied. (Note that we have four possibilities for the pair (fu, uf): two of them yield a two-element semilattice, and the other two possibilities correspond to $\mathcal{K}^{(1)}$ being a two-element left or right zero semigroup.) ■

Remark 4: Note that if f belongs to types (II)–(V), then $f(x,\ldots,x) = u(x)$, hence $\lfloor f,u \rfloor = \lfloor f \rfloor$. Also, if condition ($\mathrm{I_a}$) or ($\mathrm{I_c}$) of Theorem 11 holds, then $\lfloor f,u \rfloor = \lfloor f \rfloor$. However, if f corresponds to type ($\mathrm{I_b}$), then $\lfloor f,u \rfloor \neq \lfloor f \rfloor$, and in this case $\lfloor f,u \rfloor$ cannot be generated by a single operation. Observe also that in all types except ($\mathrm{I_b}$), there is only one idempotent unary operation in a minimal closed class, hence the operation u in Theorems 9 and 11 is unique, and then our minimal class is join-irreducible in the lattice of closed classes. A minimal class of type ($\mathrm{I_b}$) contains exactly two idempotent unary operations, hence it has two lower covers in the lattice of closed classes, and therefore it is not join-irreducible.

Example 12: There are three atoms in the lattice of closed classes of Boolean functions: $\lfloor \mathrm{id} \rfloor, \lfloor 0 \rfloor, \lfloor 1 \rfloor$. The minimal closed classes above $\lfloor \mathrm{id} \rfloor$ are the seven minimal clones: $[0]$ (type ($\mathrm{I_b}$)), $[1]$ (type ($\mathrm{I_b}$)), $[\neg x]$ (type ($\mathrm{I_a}$)), $[x \wedge y]$ (type (II)), $[x \vee y]$ (type (II)), $[xy \vee xz \vee yz]$ (type (III)) and $[x+y+z]$ (type (IV)). The results of [16] imply that the minimal closed classes covering $\lfloor 0 \rfloor$ are $\lfloor 0,1 \rfloor$ (type ($\mathrm{I_b}$)), $\lfloor x+y \rfloor$ (type (II)), $\lfloor xy+y \rfloor$ (type (II)). The minimal closed classes covering $\lfloor 1 \rfloor$ are the duals of the latter classes, namely $\lfloor 0,1 \rfloor$ (type ($\mathrm{I_b}$)), $\lfloor x+y+1 \rfloor$ (type (II)), $\lfloor \rightarrow \rfloor$ (type (II)). As observed in Remark 4, minimal closed classes of type ($\mathrm{I_b}$) cover two atoms: $[0]$ covers $\lfloor 0 \rfloor$ and $\lfloor \mathrm{id} \rfloor$, $[1]$ covers $\lfloor 1 \rfloor$ and $\lfloor \mathrm{id} \rfloor$ and $\lfloor 0,1 \rfloor$ covers $\lfloor 0 \rfloor$ and $\lfloor 1 \rfloor$.

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