ON SOME PERMUTATIVE REPRESENTATIONS OF THE CUNTZ ALGEBRAS

MIKLÓS HARTMANN AND TAMÁS WALDHAUSER

Abstract. Bratteli and Jorgensen in [2] have defined certain permutative representations of the Cuntz algebras $\mathcal{O}_n$, and initiated their study. These ν-dimensional representations depend on a choice of an integer $\nu \times \nu$ matrix and a system of representatives of the cosets of the subgroup $N\mathbb{Z}^\nu$ of the group $\mathbb{Z}^\nu$. We continue the investigations of such one-dimensional representations by generalising some of the results from [2], by giving a (sharp) upper bound on the number of atoms in case one of the parameters tends to infinity and by presenting an infinite family of representations having only one atom.

1. Introduction

Let $d_1, \dots, d_n$ be a complete system of residues modulo n (i.e., integers that represent each residue class modulo n) and let $f_i \colon \mathbb{Z} \to n\mathbb{Z} + d_i$, $x \mapsto nx + d_i$. The union of the inverse maps f_i^{-1} constitutes a transformation $R \colon \mathbb{Z} \to \mathbb{Z}$ defined by $R(x) = \frac{x-d_i}{n}$, where d_i is the unique element of the set $\{d_1, \dots, d_n\}$ such that $x \equiv d_i \pmod{n}$. In this paper we investigate the set $B_\infty(d_1, \dots, d_n)$ of periodic points of R:

$$B_\infty(d_1, \dots, d_n) := \left\{ x \in \mathbb{Z} \colon R^\ell(x) = x \text{ for some } \ell \in \mathbb{N} \right\}. \tag{1}$$

Determining the number of periodic points is a highly nontrivial problem, which is of interest on its own, and it is also connected to the study of representations of the Cuntz C^*-algebras. We briefly describe these connections below, but in the rest of the paper we refer only to the elementary setup outlined above, in order to make the paper more accessible and self-contained. Our main results are characterizations of the cases where there are "many" periodic points (in a sense to be specified more precisely later) and asymptotic bounds on $|B_\infty|$ as one of the parameters d_i tends to infinity, and we also present infinite families of examples showing that these estimates are sharp.

The *Cuntz C^*-algebra $\mathcal{O}_n$* is *the* C^*-algebra generated by n pairwise orthogonal isometries on a Hilbert space. More precisely, let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space, and let S_i $(i = 1, \dots, n)$ be bounded linear operators on $\mathcal{H}$ such that $S_i^* S_i = I$ for every i and $S_1 S_1^* + \cdots + S_n S_n^* = I$. This implies that $S_i^* S_j = 0$ whenever $i \neq j$, and Cuntz [3] has proven that the C^* algebra $\mathcal{O}_n$ generated by $S_1, \dots, S_n$ is (up to isomorphism) independent of the choice of these isometries – hence the definite article in the first sentence of this paragraph. The so-called *permutative representations* of $\mathcal{O}_n$ are defined by $S_i e_j = e_{f_i(j)}$, where $\{e_j : j \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{H}$. These operators satisfy the defining relations of $\mathcal{O}_n$ if and only if the maps $f_i \colon \mathbb{Z} \to \mathbb{Z}$ are injective and their ranges form a partition of $\mathbb{Z}$. It is worth mentioning here that this happens exactly if $\{f_i \colon 1 \le i \le n\}$ constitute a strong representation of the polycyclic monoids $\mathcal{P}_n$, which were defined by Nivat and Perrot in [8]. This connection between permutative representations of $\mathcal{O}_n$

2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 20M30.

Key words and phrases. Cuntz inverse semigroups; permutative representations of Cuntz algebras; polycyclic monoids; atoms of representations.

and strong representations of $\mathcal{P}_n$ was discovered by Lawson in [7] and investigated further by Jones and Lawson in [5].

The special permutative one-dimensional representations of $\mathcal{O}_n$ defined in [2] arise by using the maps $f_i \colon \mathbb{Z} \to \mathbb{Z}$ defined in the first paragraph, that is, when

$$S_i(e_j) = e_{nj+d_i} \text{ for all } 1 \le i \le n \text{ and } j \in \mathbb{Z}. \tag{2}$$

The structure of these representations of $\mathcal{O}_n$ is to a large extent determined by $B_\infty(d_1, \ldots, d_n)$, the set of periodic points (see Section 2 for more details). Our work is motivated mainly by the papers [2] and [5], which contain a systematic study of the periodic points for $n = 2$. In this case one can assume without loss of generality that $d_1 = 0$ and $d_2 = p$ is an odd positive integer. Bratteli and Jorgensen [2] proved that $B_\infty(0, p) = \{-p, \ldots, -1, 0\}$ and that the period of an arbitrary $x \in B_\infty(0, p)$ equals the order of 2 modulo $p/\gcd(x, p)$. It is well known (and a nice exercise in number theory) that this order is the same as the period of the binary expansion of the fraction x/p. Jones and Lawson [5] showed that this is not a coincidence: the structure of the cycle containing x is strongly related to the digits in the binary expansion of x/p. Two other special cases are also considered in [2], namely, $B_\infty(0, 1, \ldots, n-1) = \{-1, 0\}$ and $B_\infty(1, 3, 5) = \{-2, -1\}$.

In all of the special cases mentioned above $d_1, \ldots, d_n$ is an arithmetic sequence. In Section 3 we generalize the results of [2, 5] by proving that $B_\infty(d_1, \ldots, d_n)$ consists of the integers in the interval $\left[-\frac{d_n}{n-1}, -\frac{d_1}{n-1}\right]$ whenever $d_1, \ldots, d_n$ is an arithmetic sequence, and we also relate the structure of the cycle containing the integer $x \in B_\infty(d_1, \ldots, d_n)$ to the n-ary expansion of the fraction $\frac{(n-1)x+d_1}{d_n-d_1}$. It is easy to see (cf. Section 2) that $B_\infty(d_1, \ldots, d_n) \subseteq \left[-\frac{d_n}{n-1}, -\frac{d_1}{n-1}\right]$, hence we can say that for arithmetic sequences the set of periodic points is as large as possible. We show in Section 4 that we may have such a large set of periodic points even if $d_1, \ldots, d_n$ is not an arithmetic sequence. We will characterize explicitly the sequences satisfying $B_\infty(d_1, \ldots, d_n) = \left[-\frac{d_n}{n-1}, -\frac{d_1}{n-1}\right] \cap \mathbb{Z}$, and we will see that these sequences are in some sense not far from being arithmetic. In Section 6 we study the asymptotic behaviour of the number of periodic points as one of the parameters, say d_n, tends to infinity, while $d_1, \ldots, d_{n-1}$ are fixed. We prove that $|B_\infty(d_1, \ldots, d_n)| = O\big(d_n^{\log_n 2}\big)$, and we also provide infinite series of examples showing that this upper bound cannot be improved. For a lower bound, we have the trivial estimate $|B_\infty(d_1, \ldots, d_n)| \ge 1$, and we will prove in Section 5 that in general one cannot have a nontrivial lower bound, since it is possible to let d_n tend to infinity in such a way that the number of periodic points stays constant 1.

2. Preliminaries

We recall some facts from [2, 5, 7] that we shall need in the sequel; we present these in terms of so-called branching functions systems, which provide a combinatorial framework for studying permutative representations of the Cuntz C^*-algebras $\mathcal{O}_n$ and strong representations of the polycyclic monoids $\mathcal{P}_n$, hence familiarity with the theory of operator algebras or semigroups is not assumed.

A *branching function system* is a tuple $(X; f_1, \ldots, f_n)$, where X is an infinite set and $f_i \colon X \to X$ $(i = 1, \ldots, n)$ are injective maps such that their ranges form a partition of X. One can visualize a branching function system as a directed graph with colored edges: the vertices are the elements of X, and an arrow of color i is drawn from x to y if $f_i(x) = y$; we shall frequently refer to this graph in the following. We say that two branching function systems are *equivalent* if the corresponding colored graphs are isomorphic (where the isomorphism is required to preserve colors); this corresponds to the usual notion of equivalence of representations of semigroups or C^*-algebras.

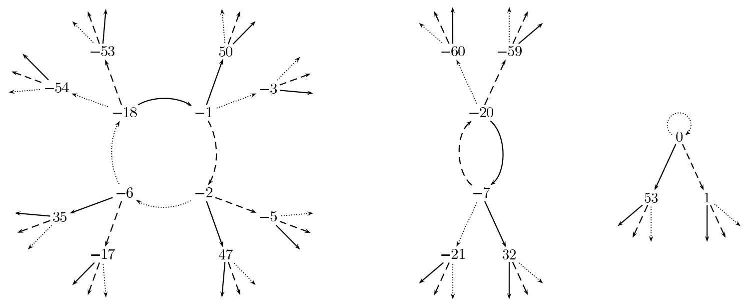

FIGURE 1. The graph of the branching function system with parameters $(d_1, d_2, d_3) = (0, 1, 53)$

Observe that the definition of a branching function system requires that for every vertex there are n outgoing edges (one of each of the n colors, leading to n different vertices) and that there is exactly one incoming edge. These two properties imply that there can be at most one cycle in every connected component of the graph and this cycle must be directed. (By a cycle we mean a closed path, as usual in graph theory. However, note that this terminology is different from that of [2, 5]; see Remark 1.) It is straightforward to verify that if a connected component contains a cycle, then the order of the colors appearing in the cycle determine that connected component up to isomorphism. (See Figure 1; it shows only a finite part of the graph of a branching function system with $n = 3$, but the missing parts are easy to imagine: an infinite ternary tree is rooted at the vertices $-54, -53, 50, -3$, etc. The numbering of the vertices will be explained later.) Let us note that cycle-free components can be more complicated; however, for the special branching function systems that we consider in this paper, each component contains a cycle.

The inverses of the maps f_i are partial bijections on X whose union is a surjective map $R\colon X \to X$, where $R(x)$ is the unique element $y \in X$ such that there exists $i \in \{1, \ldots, n\}$ with $f_i(y) = x$. The branching function system $(X; f_1, \ldots, f_n)$ can also be studied via the discrete dynamical system $(X; R)$. The *trajectory* $x, R(x), R^2(x), \ldots$ of a point x can be seen as a walk in the graph that starts from x and follows the arrows backwards. Let $\sigma(x) = (j_0, j_1, \ldots)$ be the sequence of the colors of the edges in this walk: $x = f_{j_0}(R(x))$, $R(x) = f_{j_1}\left(R^2(x)\right)$, etc.; this defines the so-called *coding map* $\sigma\colon X \to \{1, \ldots, n\}^{\mathbb{N}_0}$. We say that x is a *periodic point* if the trajectory of x is periodic, i.e., if x is a fixed point of $R^\ell(x)$ for some $\ell \in \mathbb{N}$. If ℓ is the least positive exponent such that $R^\ell(x) = x$, then (and only then) x lies on a cycle of length ℓ in the graph. Following [2], we denote the set of periodic points by B_∞. Using the terminology of discrete dynamical systems, we shall say that a set $A \subseteq X$ is *positively invariant* if $R(A) \subseteq A$, and A is said to be *absorbing* if for every $x \in X$, the trajectory of x lies eventually in A, i.e., we have $\left\{R^t(x), R^{t+1}(x), \ldots\right\} \subseteq A$ for some $t \in \mathbb{N}$.

Remark 1. Let us define two equivalence relations $\sim$ and $\approx$ on X by $x \sim y \iff \exists s, t : R^s(x) = R^t(y)$ and $x \approx y \iff \exists t : R^t(x) = R^t(y)$. In [2, 5] the main objects of study are the blocks of $\sim$ and $\approx$, which were called there cycles and atoms, respectively. Our terminology is different: we use the term cycle in the graph-theoretical sense, and we refer to the blocks of $\sim$ as connected components. For branching function systems corresponding to the representations of $\mathcal{O}_n$ defined in (2), there is a one-to-one correspondence between atoms and periodic points (see Scholium 3.9 of [2] and Proposition 4.9 of [5]). Since we deal only with this case, we work with the simpler concept of a periodic point instead of that of an atom.

Now let us turn to the special branching function systems mentioned at the beginning of Section 1. The notation that we introduce here will be used throughout the paper without further mention. Let $n \geq 2$ be a positive integer, let $d_1, \ldots, d_n$ be a complete system of residues modulo n, and let q_i and r_i denote the quotient and the remainder of d_i, when divided by n, i.e., $d_i = nq_i + r_i$ and $0 \leq r_i < n$. Clearly, we have $\{r_1, \ldots, r_n\} = \{0, \ldots, n-1\}$. The functions $f_i(x) := nx + d_i$ $(i = 1, \ldots, n)$ define a branching function system $(\mathbb{Z}; f_1, \ldots, f_n)$, and the corresponding dynamical system is $(\mathbb{Z}; R)$, where $R(x) = \frac{x-d_i}{n}$ with d_i being the uniqe element of $\{d_1, \ldots, d_n\}$ such that $x \equiv d_i \pmod{n}$. As an example, see Figure 1, which shows the graph of the branching function system corresponding to $n = 3$ and $(d_1, d_2, d_3) = (0, 1, 53)$, where dotted, dashed and solid arrows represent f_1, f_2 and f_3, respectively. Iterations of R can be seen in the figure by following the arrows backwards. For instance, the trajectory of 50 is $50, -1, -18, -6, -2, -1, -18, -6, -2, \ldots$ and $\sigma(50) = (3, 3, 1, 1, 2, 3, 1, 1, 2, \ldots)$.

In the following we collect some basic facts about these branching function systems. These results all appeared in [2, 5, 7]; for the reader's convenience we restate and reprove them. Unless otherwise mentioned, we will always assume that $d_1 < \cdots < d_n$.

Fact 2. *The following sequences give rise to equivalent branching function systems:*

a) $d_1, \ldots, d_n$*;*
b) $-d_1, -d_2, \ldots, -d_n$*;*
c) $d_1 + k(n-1), d_2 + k(n-1), \ldots, d_n + k(n-1)$*, for arbitrary* $k \in \mathbb{Z}$*.*

Proof. It is easy to check that the maps $\beta\colon \mathbb{Z} \to \mathbb{Z}, x \mapsto -x$ and $\gamma\colon \mathbb{Z} \to \mathbb{Z}, x \mapsto x - k$ establish isomorphisms from the graph corresponding to a) to the graph corresponding to b) and c), respectively. □

By this fact, one can always assume that the parameters $d_1 < \cdots < d_n$ are chosen such that $0 \leq d_1 < n - 1$. Let $\mathcal{I}$ denote the interval $\left[-\frac{d_n}{n-1}, -\frac{d_1}{n-1}\right]$, let $\mathcal{A}(d_1, \ldots, d_n) = \mathcal{I} \cap \mathbb{Z}$ and let $B_\infty(d_1, \ldots, d_n)$ be the set of periodic points, as defined in (1); sometimes we will write simply $\mathcal{A}$ and B_∞ whenever the parameters $d_1, \ldots, d_n$ are clear from the context. (From Figure 1 we see that $B_\infty(0, 1, 53) = \{-20, -18, -7, -6, -2, -1, 0\}$ and we have $\mathcal{A}(0, 1, 53) = \{-26, \ldots, 0\}$.)

Lemma 3. *The set $\mathcal{A}$ is a finite positively invariant absorbing set.*

Proof. Since $d_1 < \cdots < d_n$, we have $\frac{x-d_n}{n} \leq R(x) \leq \frac{x-d_1}{n}$ for every $x \in \mathbb{Z}$. From these inequalities one can deduce the following implications:

$$\begin{array}{llcl}
\text{a)} & x < \dfrac{-d_n}{n-1} & \Longrightarrow & x < R(x) < \dfrac{-d_1}{n-1}; \\
\text{b)} & \dfrac{-d_n}{n-1} \leq x \leq \dfrac{-d_1}{n-1} & \Longrightarrow & \dfrac{-d_n}{n-1} \leq R(x) \leq \dfrac{-d_1}{n-1}; \\
\text{c)} & \dfrac{-d_1}{n-1} < x & \Longrightarrow & \dfrac{-d_n}{n-1} < R(x) < x.
\end{array}$$

The second implication immediately yields that $\mathcal{A}$ is positively invariant. If $x \notin \mathcal{A}$ then either $x < R(x) < R^2(x) < \cdots < R^t(x) \in \mathcal{A}$ or $x > R(x) > R^2(x) > \cdots > R^t(x) \in \mathcal{A}$ for some $t \in \mathbb{N}$ by a) or c) depending on whether $x < -\frac{d_n}{n-1}$ or $x > -\frac{d_1}{n-1}$. In both cases b) shows that the rest of the trajectory of x stays in $\mathcal{A}$. □

Fact 4. *The graph corresponding to the branching function system determined by $d_1, \ldots, d_n$ has finitely many connected components, and each component contains a* *cycle. These cycles are all contained in* $\mathcal{A}(d_1,\dots,d_n)$*, i.e., we have*

$$B_\infty(d_1,\dots,d_n) \subseteq \mathcal{A}(d_1,\dots,d_n).$$

The trajectory of every integer $x \in \mathbb{Z}$ *is eventually periodic: there exist* $\ell, t \in \mathbb{N}$ *such that* $R^{i+\ell}(x) = R^i(x)$ *whenever* $i \geq t$*.*

Proof. For every $x \in \mathbb{Z}$, the trajectory of x is finite, since $R^i(x)$ belongs to the finite set $\mathcal{A}$ for almost every i, by Lemma 3. This implies that every trajectory is eventually periodic, hence each connected component contains a cycle. These cycles are contained in $\mathcal{A}$, thus there are finitely many cycles and finitely many connected components. □

According to Fact 4, B_∞ is a nonempty set consisting of finitely many cycles, and every trajectory eventually winds around one of these cycles. This implies that the sequence $\sigma(x)$ is also eventually periodic for every $x \in \mathbb{Z}$. We show below how to use these facts to see that $\sigma(x)$ uniquely determines x, i.e., that the coding map is injective. (Representations of $\mathcal{O}_n$ with an injective coding map are called multiplicity-free representations in [2].)

Fact 5. *Suppose that the trajectory of* x *becomes* ℓ*-periodic after* t *terms, i.e.,* $R^{t+\ell}(x) = R^t(x)$*, and let* $\sigma(x) = (j_0, j_1, \dots)$*. Then we have*

$$x = d_{j_0} + nd_{j_1} + \cdots + n^{t-1}d_{j_{t-1}} - \frac{d_{j_t}n^t + \cdots + d_{j_{t+\ell-1}}n^{t+\ell-1}}{n^\ell - 1}. \tag{3}$$

Proof. From the definition of R and σ one can deduce by induction on k that $x = d_{j_0} + nd_{j_1} + \cdots + n^{k-1}d_{j_{k-1}} + n^k R^k(x)$ holds for all $k \in \mathbb{N}$. Applying this formula for $k = t$ and $k = t + \ell$ we obtain

$$\begin{aligned} x &= d_{j_0} + nd_{j_1} + \cdots + n^{t-1}d_{j_{t-1}} + n^t R^t(x) \\ &= d_{j_0} + nd_{j_1} + \cdots + n^{t-1}d_{j_{t-1}} + n^t d_{j_t} + \cdots + n^{t+\ell-1}d_{j_{t+\ell-1}} + n^{t+\ell}R^{t+\ell}(x). \end{aligned}$$

By our assumption we have $R^{t+\ell}(x) = R^t(x)$, hence the equation $\left(1 - n^\ell\right) n^t R^t(x) = n^t d_{j_t} + \cdots + n^{t+\ell-1}d_{j_{t+\ell-1}}$ follows, and this proves (3). □

If $x \in B_\infty$ lies on a cycle of length ℓ, then the sequence $\sigma(x)$ is ℓ-periodic, that is, $\sigma(x) = (j_0, \dots, j_{\ell-1}, j_0, \dots, j_{\ell-1}, \dots)$, so we can describe it by a finite word $w = j_0 \cdots j_{\ell-1}$ over the alphabet $\{1, \dots, n\}$, which we will refer to as *the word corresponding to* x. This word can be read from the graph by recording the colors of the edges on the cycle, starting from x and following the arrows backwards. Words corresponding to periodic points on the same cycle are *conjugates*, i.e., they can be obtained from each other by cyclic shifts. (For example, the words corresponding to $-1, -2, -6$ and -18 on Figure 1 are 3112, 2311, 1231 and 1123, respectively.) Since our branching function systems have finitely many connected components, each containing a cycle (cf. Fact 4), their structure can be completely described by the words corresponding to periodic points (or cycles). Therefore, exploring periodic points and the corresponding words is essential in the study of representations of $\mathcal{O}_n$.

If a word w is not the repetition of a shorter word, i.e., it cannot be written in the form $w = u \cdots u = u^k$ $(k \geq 2)$, then w is said to be a *primitive* word. Observe that every word is the power of a unique primitve word. The next corollary of Fact 5 states that the word corresponding to a periodic point is always primitive, and we also get a characterization of those primitive words that correspond to a periodic point. (Note that primitivity of the word corresponding to x means that the shortest period of $\sigma(x)$ is ℓ. In an arbitrary branching function system it is possible that the shortest period of $\sigma(x)$ is a proper divisor of the shortest period of the sequence $x, R(x), R^2(x), \ldots$; however, this cannot happen for the special branching function systems considered here.)

Corollary 6. *A word $w = j_0 \cdots j_{\ell-1}$ over $\{1, \ldots, n\}$ corresponds to some periodic point if and only if w is primitive and*

$$x = -\frac{d_{j_{\ell-1}} n^{\ell-1} + \cdots + d_{j_1} n + d_{j_0}}{n^\ell - 1} \tag{4}$$

is an integer. If this holds, then the integer x defined by (4) is a periodic point on a cycle of length ℓ, and w is the word corresponding to x.

Proof. First we prove the second statement: let us assume that $w = j_0 \cdots j_{\ell-1}$ is a primitive word such that the number given by (4) is an integer. Then $x = d_{j_0} + d_{j_1} n + \cdots + d_{j_{\ell-1}} n^{\ell-1} + n^\ell x$, which implies that $R(x) = \frac{x - d_{j_0}}{n} = d_{j_1} + \cdots + d_{j_{\ell-1}} n^{\ell-2} + n^{\ell-1} x$, since $x \equiv d_{j_0} \pmod{n}$, and d_{j_0} is the only element of $\{d_1, \ldots, d_n\}$ with this property. Continuing this way we get $R^2(x) = \frac{R(x) - d_{j_1}}{n} = d_{j_2} + \cdots + d_{j_{\ell-1}} n^{\ell-3} + n^{\ell-2} x$, etc., and after ℓ steps we obtain $R^\ell(x) = x$. This means that x is indeed a periodic point; furthermore, if u is the word corresponding to x then $w = u^k$ with $\ell = km$, where m denotes the length of u (that is, the length of the cycle that contains x). Since w is a primitive word, we must have $k = 1$, thus the length of the cycle containing x is ℓ and the word corresponding to x is w.

The argument above proves not only the second statement of the corollary, but also the "if" part of the first statement, thus it remains to prove the "only if" part. Let $w = j_0 \cdots j_{\ell-1}$ be the word corresponding to some periodic point y. We can apply Fact 5 with $t = 0$ to obtain that the right hand side of the equality (4) equals y, therefore it is an integer. Finally, we prove primitivity of w: let $w = u^k$, where $u = j_m \cdots j_0$ is a primitive word and $\ell = km$. Then we have $j_i = j_{i \bmod m}$ for every $i \in \{0, \ldots, \ell - 1\}$, and this allows us to rewrite the right hand side of (4) as follows:

$$\begin{aligned} x &= -\frac{\left(d_{j_{m-1}} n^{m-1} + \cdots + d_{j_1} n + d_{j_0}\right) \cdot \left(1 + n^m + \cdots + n^{(k-1)m}\right)}{n^\ell - 1} \\ &= -\frac{\left(d_{j_{m-1}} n^{m-1} + \cdots + d_{j_1} n + d_{j_0}\right) \cdot \frac{n^{km}-1}{n^m-1}}{n^\ell - 1} = -\frac{d_{j_{m-1}} n^{m-1} + \cdots + d_{j_1} n + d_{j_0}}{n^m - 1}. \end{aligned}$$

Applying the first paragraph of this proof to the primitive word u, we see that x belongs to a cycle of length m (and the word corresponding to x is u). Consequently, $k = 1$ (and $u = w$) proving that w is indeed primitive. $\square$

Remark 7. The proof of Corollary 6 shows that if the number x given by (4) is an integer, then it is a periodic point even if the word w is not primitive. Moreover, if $w = u^k$, where u is a primitive word, then the word corresponding to x is u. Thus, the periodic points are exactly the integers of the form

$$\frac{a_{\ell-1} n^{\ell-1} + \cdots + a_1 n + a_0}{1 - n^\ell} \qquad (\ell \in \mathbb{N}, a_0, \ldots, a_{\ell-1} \in \{d_1, \ldots, d_n\}).$$

Two such expressions yield the same number if and only if the words describing the coefficients $a_0, \ldots, a_{\ell-1}$ are the powers of the same primitive word. In particular, different expressions of the same length ℓ give different periodic points.

Fact 8. *The words corresponding to periodic points on different cycles are not conjugate.*

Proof. Let v and w be words corresponding to periodic points x and y that lie on different cycles, and suppose that v and w are conjugates. Then there exists a point x' on the same cycle as x such that the word corresponding to x' is w. However, according to Corollary 6, a periodic point is uniquely determined by the corresponding word by (4). This implies that $x' = y$, hence x and y belong to the same cycle, contrary to our assumption. □

Summarizing the above facts, we can say that the structure of the branching function system corresponding to $d_1, \ldots, d_n$ is determined by a finite set of primitive words over the alphabet $\{1, \ldots, n\}$, each considered up to conjugacy. For the sake of canonicity, it is customary to choose the lexicographically least word from each conjugacy class; these words are called *Lyndon words*. (The Lyndon words describing the three cycles of Figure 1 are 1123, 23 and 1.) One possible approach to find these Lyndon words is to follow the trajectories of the elements of $\mathcal{A}(d_1, \ldots, d_n)$. Another possibility is to determine all numbers of the form $a_{\ell-1}n^{\ell-1} + \cdots + a_1 n + a_0$ with $a_0, \ldots, a_{\ell-1} \in \{d_1, \ldots, d_n\}$ that are divisible by $n^\ell - 1$ (cf. Remark 7). Since $\mathcal{A}(d_1, \ldots, d_n)$ is finite, both searches can be completed in a finite number of steps (note that we must have $\ell \leq |B_\infty| \leq |\mathcal{A}|$).

Corollary 9. *It is decidable whether two representations of $\mathcal{O}_n$ defined as in (2) are equivalent.*

Remark 10. Let $\sigma(x) = (j_0, j_1, \ldots)$ and let us write out the representations of x considered in Fact 5 for $k = 1, 2, 3, \ldots$:

$$x = d_{j_0} + nR(x) = d_{j_0} + nd_{j_1} + n^2 R^2(x) = d_{j_0} + nd_{j_1} + n^2 d_{j_2} + n^3 R^3(x) = \cdots .$$

It would be natural to extend this to an infinite expansion of x:

$$x = d_{j_0} + nd_{j_1} + n^2 d_{j_2} + n^3 d_{j_3} + \cdots . \tag{5}$$

Of course, this infinite series does not converge in general. However, we can infer from (5) that $x \equiv d_{j_0} \pmod{n}$, $x \equiv d_{j_0} + nd_{j_1} \pmod{n^2}$, $x \equiv d_{j_0} + nd_{j_1} + n^2 d_{j_2} \pmod{n^3}$, etc. Moreover, since the expansion is periodic, we can sum the right hand side of (5) *formally* by letting $1 + n^\ell + n^{2\ell} + \cdots = \frac{1}{1-n^\ell}$. Although this geometric series is clearly divergent, this formal evaluation gives actually the correct value of x. Indeed, let $a = d_{j_0} + nd_{j_1} + \cdots + n^{t-1} d_{j_{t-1}}$ and $b = d_{j_t} n^t + \cdots + d_{j_{t+\ell-1}} n^{t+\ell-1}$ (with t and ℓ being the same as in Fact 5); then we can rewrite (5) as $x = a + b + bn^\ell + bn^{2\ell} + \cdots = a + \frac{b}{1-n^\ell}$, and this is the same as (3).

One can regard (5) as a representation of x in a number system with radix n and digits $d_1, \ldots, d_n$. As mentioned in Section 1, $B_\infty(0, \ldots, n-1) = \{-1, 0\}$ (see also Theorem 12). This means that using the standard digits $0, \ldots, n-1$, the sequence of digits in the expansion of every integer is eventually constant 0 or constant $n-1$, namely for nonnegative integers the digits are eventually 0 (as it is well known), and for negative integers the digits are eventually $n-1$. For instance, the representation of $x = -1$ is $-1 = (n-1) + (n-1)n + (n-1)n^2 + \cdots$, which can be verfied using the formal summation $1 + n + n^2 + \cdots = \frac{1}{1-n}$.

As another example, we can read from Figure 1 that 50 can be represented in the number system with radix 3 and digits $0, 1, 53$ as

$$\begin{aligned} 50 &= d_3 + nd_3 + n^2 d_1 + n^3 d_1 + n^4 d_2 + n^5 d_3 + n^6 d_1 + n^7 d_1 + n^8 d_2 + \cdots \\ &= 53 + 53 \cdot 3 + 0 \cdot 3^2 + 0 \cdot 3^3 + 1 \cdot 3^4 + 53 \cdot 3^5 + 0 \cdot 3^6 + 0 \cdot 3^7 + 1 \cdot 3^8 + \cdots \\ &= 53 + 240 \cdot \left(1 + 3^4 + 3^8 + \cdots\right). \end{aligned}$$

Evaluating $1 + 3^4 + 3^8 + \cdots$ formally as $\frac{1}{1-3^4}$, we indeed get $53 + 240 \cdot \frac{1}{-80} = 53 - 3 = 50$.

Finally, we state an elementary identity about integer parts that will be needed later; its proof is left to the reader.

Lemma 11. *For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we have $\sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor$.*

3. Arithmetic sequences

In this section we investigate periodic points and cycles in the case when the parameters form an arithmetic sequence. For notational convenience, we shift the indices by 1 (as it is done in Section 5 of [5]): instead of $d_1, \dots, d_n$ we shall work with $d_0, \dots, d_{n-1}$, and we will use words over $\{0, \dots, n-1\}$ to describe the cycles. Thus, we assume throughout this section that $d_0, \dots, d_{n-1}$ is an arithmetic sequence: $d_i = d_0 + ih$ for $i = 0, \dots, n-1$, where $h \in \mathbb{N}$ is relatively prime to n and $0 \le d_0 < n-1$. Observe that $\mathcal{A} = \{-h, \dots, -1, 0\}$ if $d_0 = 0$ and $\mathcal{A} = \{-h, \dots, -1\}$ if $0 < d_0 < n-1$.

Theorem 12. *If $d_0 < \cdots < d_{n-1}$ is an arithmetic sequence, then $B_\infty = \mathcal{A}$, therefore the number of periodic points is $h+1$ or h (depending on whether $d_0 = 0$ or not).*

Proof. Since $\mathcal{A}$ is a positively invariant set (Lemma 3) and $B_\infty \subseteq \mathcal{A}$ (Fact 4), it suffices to prove that the restriction of R to $\mathcal{A}$ is bijective. In fact, it is sufficient to establish injectivity, as $\mathcal{A}$ is finite. Let $x, y, z \in \mathcal{A}$ such that $R(x) = R(y) = z$ and $x < y$. Then we have $x = nz + d_i, y = nz + d_j$ for some $i, j \in \{1, \dots, n\}$, therefore $y - x = d_j - d_i = h(j-i) \ge h$. If $d_0 \ne 0$ then this is already a contradiction, as the diameter of the set $\mathcal{A}$ is $h-1$ in this case. If $d_0 = 0$, then the diameter of $\mathcal{A}$ is h, and this implies that $x = \min \mathcal{A} = -h$ and $y = \max \mathcal{A} = 0$. However, $R(-h) = -h$ and $R(0) = 0$, contradicting the assumption $R(x) = R(y)$. $\square$

The next lemma is essentially just a reformulation of Corollary 6 for arithmetic sequences.

Lemma 13. *Assume that $d_0 < \cdots < d_{n-1}$ is an arithmetic sequence, let $w = j_0 \cdots j_{\ell-1}$ be a primitive word over $\{0, \dots, n-1\}$, and let $\overleftarrow{w} \in \{0, 1, \dots, n^\ell - 1\}$ be the nonnegative integer given by the base-n representation determined by the reverse of w:*

$$\overleftarrow{w} = \overline{j_{\ell-1} \cdots j_1 j_0}_{\,n} = \sum_{i=0}^{\ell-1} j_i n^i.$$

Then w corresponds to a cycle if and only if

$$n^\ell - 1 \,\Bigg|\, \frac{d_0(n^\ell - 1)}{n-1} + h\overleftarrow{w}\ .$$

Proof. Substituting $d_0 + j_i h$ for d_{j_i} in the numerator of the right hand side of (4),we obtain

$$\sum_{i=0}^{\ell-1} d_0 n^i + \sum_{i=0}^{\ell-1} j_i h n^i = \frac{d_0(n^\ell - 1)}{n-1} + h\overleftarrow{w}.$$

By Corollary 6, w corresponds to a periodic point if and only if the above number is divisible by $n^\ell - 1$. $\square$

Now we are ready to generalize Theorem 5.5 of [5] by relating the words corresponding to periodic points to base-n expansions of certain fractions. (Note that for $n = 2$ we must have $d_0 = 0$, hence the following theorem indeed contains Theorem 5.5 of [5] as a special case.)

Theorem 14. *Assume that $d_0 < \cdots < d_{n-1}$ is an arithmetic sequence. If $d_0 = 0$ then the words corresponding to the periodic points are the reverses of the periods of the base-n representations of the fractions $\frac{0}{h}, \dots, \frac{h}{h}$. If $0 < d_0 < n-1$, then the corresponding words are the reverses of the periods of the base-n representations of $\frac{1}{h} - \frac{d_0}{h(n-1)}, \dots, \frac{h}{h} - \frac{d_0}{h(n-1)}$.*

Proof. Let $x \in B_\infty$ and let $w = j_0 \cdots j_{\ell-1}$ be a primitive word over $\{0, \ldots, n-1\}$. By (the proof of) Lemma 13, w corresponds to x if and only if $-x = \frac{d_0}{n-1} + \frac{h\overleftarrow{w}}{n^\ell - 1}$, which is equivalent to

$$\begin{aligned}-\frac{x + \frac{d_0}{n-1}}{h} &= \frac{\overleftarrow{w}/n^\ell}{1 - 1/n^\ell} = \overleftarrow{w}\left(\frac{1}{n^\ell} + \frac{1}{n^{2\ell}} + \cdots\right) \\ &= j_{\ell-1}\frac{1}{n} + \cdots + j_0\frac{1}{n^\ell} + j_{\ell-1}\frac{1}{n^{\ell+1}} + \cdots + j_0\frac{1}{n^{2\ell}} + \cdots \\ &= 0.\overline{j_{\ell-1}\cdots j_0}_{\,n}.\end{aligned}$$

Thus, the word corresponding to $x \in B_\infty$ is the reverse of the period in the base-n expansion of $-\frac{x}{h} - \frac{d_0}{h(n-1)}$. Taking into account that $B_\infty = \{-h, \ldots, -1, 0\}$ if $d_0 = 0$ and $B_\infty = \{-h, \ldots, -1\}$ if $0 < d_0 < n-1$, we obtain the (negatives of the) fractions listed in the statement of the theorem. $\square$

Corollary 15. *The length of the cycle containig the periodic point*

$$x \in B_\infty(d_1, \ldots, d_n)$$

equals the multiplicative order of n modulo $\frac{h(n-1)}{\gcd\left(x(n-1)+d_0,\, h(n-1)\right)}$.

Proof. It is well known that if $a, b \in \mathbb{N}$ are relatively prime and b is also relatively prime to n, then the length of the period of the base-n representation of $\frac{a}{b}$ is the order of n modulo b. We have seen in the proof of Theorem 14 that the length of the cycle containing $x \in B_\infty$ is the length of the period of the base-n expansion of $\frac{x(n-1)+d_0}{h(n-1)}$; therefore, it only remains to observe that after simplification the denominator becomes $\frac{h(n-1)}{\gcd\left(x(n-1)+d_0,\, h(n-1)\right)}$, which is clearly relatively prime to n, as both h and $n-1$ are. $\square$

Corollary 16. *For any finite set C of primitive words over the alphabet $\{0, \ldots, n-1\}$ there exists an arithmetic sequence $d_0 < \cdots < d_{n-1}$ such that the set of words corresponding to the elements of $B_\infty(d_0, \ldots, d_{n-1})$ contains C.*

Proof. By Lemma 13, it suffices to find d_0 and h such that

$$\frac{d_0(n^{|w|} - 1)}{n-1} + h\overleftarrow{w} \equiv 0 \pmod{n^{|w|} - 1}$$

for every $w \in C$, where $|w|$ denotes the length of w. Clearly, for $d_0 = 0$ and $h = \operatorname{lcm}\left\{n^{|w|} - 1 \colon w \in C\right\}$ each of these congruences are satisfied (note that h is relatively prime to n). $\square$

We finish this section by showing that the only equivalences between representations (or branching function systems) given by arithmetic sequences are the trivial ones of Fact 2.

Theorem 17. *Let $0 \le d_0, d_0' < n-1$ and let h, h' be relatively prime to n. The representations (of $\mathcal{P}_n$ or $\mathcal{O}_n$) arising from the arithmetic sequences $d_0, d_0+h, \ldots, d_0+(n-1)h$ and $d_0', d_0'+h', \ldots, d_0'+(n-1)h'$ are equivalent if and only if $d_0 = d_0'$ and $h = h'$.*

Proof. Let the two representations given in the statement of the theorem be equivalent, and first let us assume that $h \ne h'$; without loss of generality we can suppose that $h < h'$. Since equivalent representations have the same number of periodic points, Theorem 12 implies that $d_0 = 0, d_0' \ne 0$ and $h' = h+1$. Then $0 \in B_\infty(d_0, \ldots, d_{n-1})$ and the corresponding word is 0 (of length one). However, since $n-1 \nmid d_0$, this word corresponds to no element of $B_\infty\left(d_0', \ldots, d_{n-1}'\right)$, contradicting the equivalence of the representations.

Now let us assume that $h = h'$ but $d_0 \neq d_0'$. Recall that two representations are equivalent if and only if the sets of words determined by their periodic points are the same. Let w be a word of length ℓ corresponding to some element of $B_\infty(d_0, \ldots, d_{n-1})$; then w also corresponds to some element of $B_\infty\left(d_0', \ldots, d_{n-1}'\right)$. Lemma 13 implies that

$$\frac{d_0(n^\ell - 1)}{n-1} + h\overleftarrow{w} \equiv \frac{d_0'(n^\ell - 1)}{n-1} + h'\overleftarrow{w} \equiv 0 \pmod{n^\ell - 1},$$

which in turn implies that $d_0 \equiv d_0' \pmod{n-1}$, as $h = h'$. However, this is impossible, since $0 \leq d_0, d_0' < n-1$ and $d_0 \neq d_0'$. ☐

4. Many periodic points

Arithmetic sequences give rise to branching function systems where the set of periodic points is "as large as possible", i.e., $B_\infty = \mathcal{A}$. In this section we characterize sequences $d_1, \ldots, d_n$ that have the same property; as we shall see, these are "almost arithmetic sequences".

Theorem 18. *Let $d_1 < \cdots < d_n$, let $\mathcal{I}$ denote the interval $\left[-\frac{d_n}{n-1}, -\frac{d_1}{n-1}\right]$ as before, and let $\mathcal{I}_i = \frac{1}{n}\mathcal{I} - \frac{1}{n}d_i$ for $i = 1, \ldots, n$. The following conditions are equivalent:*

(i) $B_\infty = \mathcal{A}$;
(ii) $\mathcal{A} \subseteq \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n$;
(iii) *for all $i \in \{1, \ldots, n-1\}$ we have*

$$\left\lfloor \frac{d_1}{n(n-1)} + \frac{d_{i+1}}{n} \right\rfloor = \left\lfloor \frac{d_n}{n(n-1)} + \frac{d_i}{n} \right\rfloor.$$

Proof. Just as in the proof of Theorem 12, we can see that $B_\infty = \mathcal{A}$ if and only if the restriction of R to $\mathcal{A}$ is bijective. Since $\mathcal{A}$ is finite, bijectivity is equivalent to surjectivity in this case, thus (i) holds if and only if $R^{-1}(x) \cap \mathcal{A} = R^{-1}(x) \cap \mathcal{I} \neq \emptyset$ for all $x \in \mathcal{A}$. This latter condition means that for every $x \in \mathcal{A}$ there exists an $i \in \{1, \ldots, n\}$ such that $nx + d_i \in \mathcal{I}$, i.e., $x \in \mathcal{I}_i$. This proves the equivalence of (i) and (ii).

The intervals $\mathcal{I}_i$ are all translates of the interval $\frac{1}{n}\mathcal{I}$. The leftmost one of these intervals is $\mathcal{I}_n$, and the left endpoint of $\mathcal{I}_n$ coincides with the left endpoint of $\mathcal{I}$; similarly, the right endpoint of $\mathcal{I}_1$ coincides with the right endpoint of $\mathcal{I}$. Therefore, condition (ii) holds if and only if there is no integer between the right endpoint of $\mathcal{I}_{i+1}$ and the left endpoint of $\mathcal{I}_i$ for every $i \in \{1, \ldots, n-1\}$:

$$\nexists x \in \mathbb{Z}: \ -\frac{d_1}{n(n-1)} - \frac{d_{i+1}}{n} \leq x \leq -\frac{d_n}{n(n-1)} - \frac{d_i}{n}. \tag{6}$$

Since $1 \neq i+1$, we have $d_1 \not\equiv d_{i+1} \pmod{n}$, and this implies that $-\frac{d_1}{n(n-1)} - \frac{d_{i+1}}{n}$ is not an integer; similarly, the right hand side of (6) is not an integer either, as $n \neq i$. Hence (6) is equivalent to $\left\lfloor -\frac{d_1}{n(n-1)} - \frac{d_{i+1}}{n} \right\rfloor \geq \left\lfloor -\frac{d_n}{n(n-1)} - \frac{d_i}{n} \right\rfloor$, which, in turn, is equivalent to

$$\left\lfloor \frac{d_1}{n(n-1)} + \frac{d_{i+1}}{n} \right\rfloor \leq \left\lfloor \frac{d_n}{n(n-1)} + \frac{d_i}{n} \right\rfloor. \tag{7}$$

This shows that condition (ii) is satisfied if and only if the inequality (7) is true for $i = 1, \ldots, n-1$. In order to prove the equivalence of (ii) and (iii), we just need to verify that if (7) holds for all $i \in \{1, \ldots, n-1\}$, then we actually have equality in (7) for every i. This will follow immediately from the observation that adding the inequalities (7) for $i = 1, \ldots, n-1$, we obtain the same values on the left hand side and on the right hand side. Indeed, summing the left hand sides of (7) for $i = 1, \ldots, n-1$ we get

$$\sum_{i=1}^{n-1} \left\lfloor \frac{d_1}{n\,(n-1)} + \frac{d_{i+1}}{n} \right\rfloor = \sum_{j=1}^{n} \left\lfloor \frac{d_1}{n\,(n-1)} + \frac{nq_j + r_j}{n} \right\rfloor - \left\lfloor \frac{d_1}{n\,(n-1)} + \frac{d_1}{n} \right\rfloor$$
$$\overset{(a)}{=} \sum_{j=1}^{n} q_j + \sum_{k=0}^{n-1} \left\lfloor \frac{d_1}{n\,(n-1)} + \frac{k}{n} \right\rfloor - \left\lfloor \frac{d_1}{n-1} \right\rfloor$$
$$\overset{(b)}{=} \sum_{j=1}^{n} q_j,$$

where in step (a) we used the fact that $\{r_1, \ldots, r_n\} = \{0, \ldots, n-1\}$, as $d_1, \ldots, d_n$ is a complete system of residues modulo n, and in step (b) we applied Lemma 11 with $x = \frac{d_1}{n(n-1)}$. A similar calculation shows that the sum of the right hand sides of (7) for $i = 1, \ldots, n-1$ is also $\sum_{j=1}^{n} q_j$. □

Remark 19. Assuming (without loss of generality) that $0 \leq d_1 < n-1$, condition (iii) of Theorem 18 yields

$$(8) \qquad q_{i+1} - q_i = \left\lfloor \frac{d_n}{n\,(n-1)} + \frac{r_i}{n} \right\rfloor \in \left\{ \left\lfloor \frac{d_n}{n\,(n-1)} \right\rfloor, \left\lfloor \frac{d_n}{n\,(n-1)} \right\rfloor + 1 \right\}.$$

Thus in this case $q_1, \ldots, q_n$ is almost an arithmetic sequence: the difference of consecutive entries can assume at most two different values (which are consecutive integers), and then the numbers $d_i = nq_i + r_i$ are also quite evenly distributed in the interval $[d_1, d_n]$. So we can say informally that $B_\infty = \mathcal{A}$ if and only if $d_1, \ldots, d_n$ is not far from being an arithmetic sequence. (Note that this intuitive interpretation of Theorem 18 can be misleading; see Example 20.) It is straightforward to verify that $d_1, \ldots, d_n$ form an arithmetic sequence if and only if the equality in condition (iii) holds even without taking integer parts.

From (8) we also obtain the following explicit formula for q_i (taking into account that $q_1 = 0$):

$$(9) \qquad q_i = \left\lfloor \frac{d_n}{n\,(n-1)} + \frac{r_1}{n} \right\rfloor + \cdots + \left\lfloor \frac{d_n}{n\,(n-1)} + \frac{r_{i-1}}{n} \right\rfloor.$$

This means that if we prescribe the residues $r_1, \ldots, r_n$ and we also fix d_1 and d_n, then there is at most one possibility for the numbers $d_2, \ldots, d_{n-1}$ such that $B_\infty(d_1, \ldots, d_n) = \mathcal{A}(d_1, \ldots, d_n)$. However, it may happen that the numbers $d_2, \ldots, d_{n-1}$ calculated from (9) do not form an increasing sequence (cf. Example 21). In this case we can conclude from Theorem 18 that there is no branching function system with the given values for the residues and for d_1 and d_n with $B_\infty = \mathcal{A}$

Example 20. The following table gives the data for a branching function system with $n = 10$. We can observe that $q_{i+1} - q_i \in \{1, 2\}$ for $i = 1, \ldots, 9$, in accordance with (8). The last line of the table gives the members of the arithmetic sequence s_i with $s_1 = d_1 = 0$ and $s_{10} = d_{10} = 141$, rounded to the nearest integer, i.e., $s_i = \left\lfloor (i-1) \cdot \frac{141}{9} \right\rceil$. Theorem 18 and Remark 19 might give the impression that d_i should be one of the integers that are closest to s_i and congruent to r_i modulo n. However, this is not true; for instance, one would expect that d_5 is either 56 or 66, but the actual value is $d_5 = 76$. The case $i = 7$ is even more counterintuitive: $s_7 = 94$ is already congruent to $r_7 = 4$ modulo 10 (and we do not even need to round when computing s_7, as $6 \cdot \frac{141}{9}$ is an integer), yet $d_7 = 114$ is quite far from

s_7.

i	1	2	3	4	5	6	7	8	9	10
r_i	0	9	8	7	6	5	4	3	2	1
q_i	0	1	3	5	7	9	11	12	13	14
d_i	0	19	38	57	76	95	114	123	132	141
s_i	0	16	31	47	63	78	94	110	125	141

Example 21. Let us try to find a branching function system with $B_\infty = \mathcal{A}$ for $n = 4$ such that it satisfies $r_1 = 2$, $r_2 = 3$, $r_3 = 1$, $r_4 = 0$ and $d_1 = 2$, $d_4 = 8$. The values computed from (9) are given in the table below. We can see that $d_3 > d_4$, and this means that no branching function system with the given parameters satisfies $B_\infty = \mathcal{A}$. (Note that if we switch r_2 and r_3 then we have $B_\infty = \mathcal{A}$ with $d_2 = 5$, $d_3 = 7$.)

i	1	2	3	4
r_i	2	3	1	0
q_i	0	1	2	2
d_i	2	7	9	8

5. A single periodic point

In [2], the authors raised the question when a representation of $\mathcal{O}_n$ has a single periodic point (these representations are necessarily irreducible). In this section we provide an infinite family of representations having a single periodic point and give some sporadic examples indicating that there are far more such representations.

If $d_1, \ldots, d_n$ are consecutive integers, then the number of periodic points is either one or two, by Theorem 12. In the next theorem we investigate how the number of periodic points changes when one of the d_i's is replaced by $d_i + n^k$. We will see that for most choices of d_i the number of periodic points does not change, but in certain special cases the number of periodic points increases exponentially with k. By the remark following Fact 2, we may assume without loss of generality that $(d_1, \ldots, d_n) = (b, \ldots, b + n - 1)$ with $0 \leq b \leq n - 2$.

Theorem 22. *Let $0 \leq b \leq n - 2$, let $r \in \{b, b+1, \ldots, b+n-1\}$ and $k \in \mathbb{N}$. The number of periodic points corresponding to*

$$(d_1, \ldots, d_n) = \left(b, \ldots, r-1, r+n^k, r+1, \ldots, b+n-1\right)$$

is given by Table 1.

Proof. Let $(d_1, \ldots, d_n) = \left(b, \ldots, r-1, r+n^k, r+1, \ldots, b+n-1\right)$, i.e., $d_i = b + i - 1$ for $i \neq r - b + 1$ and $d_i = b + i - 1 + n^k$ for $i = r - b + 1$. Let x be an arbitrary periodic point on a cycle of length ℓ, let $\sigma(x) = (j_0, j_1, \ldots)$, and let $a_i = d_{j_i}$. By Corollary 6, we have $x = -y/\left(n^\ell - 1\right)$, where $y = n^{\ell-1}a_{\ell-1} + \cdots + na_1 + a_0$; moreover, $\{a_i\}$ is a periodic sequence whose shortest period is ℓ.

Let us define the following two modified sequences of coefficients (here, and in the sequel, $\ominus$ and $\oplus$ stand for subtraction and addition modulo ℓ, respectively):

$$a'_i := \begin{cases} a_i - n^k, & \text{if } a_i = r + n^k; \\ a_i, & \text{otherwise;} \end{cases} \qquad a''_i := \begin{cases} a'_i + 1, & \text{if } a'_{i\ominus k} = r; \\ a'_i, & \text{otherwise.} \end{cases}$$

Note that $0 \leq a''_i \leq b + n \leq 2n - 2$ for all $i \in \{0, \ldots, \ell - 1\}$. We can now express y modulo $n^\ell - 1$ with these new coefficients, using the observation $s \equiv t \pmod{\ell} \implies$

		$\lvert B_\infty \rvert$
$b=0$	$r=0$	1
$b=0$	$r\in\{1,\dots,n-3\}$	2
$b=0$	$r\in\{n-2,n-1\}$	2^k+1
$1\le b\le n-3$	$r\in\{n-2,n-1\}$	2^k
$1\le b\le n-3$	$r\notin\{n-2,n-1\}$	1
$b=n-2$	$r\in\{n-2,n-1\}$	2^k
$b=n-2$	$r\in\{n,\dots,2n-4\}$	1
$b=n-2$	$r=2n-3$	2

TABLE 1. The number of periodic points for some special branching function systems

$n^s \equiv n^t \pmod{n^\ell - 1}$:

$$\begin{aligned} y &= \sum_{i=0,\dots,\ell-1} a'_i n^i + \sum_{\substack{i=0,\dots,\ell-1\\ a_i=r+n^k}} n^k n^i \equiv \sum_{i=0,\dots,\ell-1} a'_i n^i + \sum_{\substack{i=0,\dots,\ell-1\\ a_i=r+n^k}} n^{i\oplus k} \\ &= \sum_{j=0,\dots,\ell-1} a'_j n^j + \sum_{\substack{j=0,\dots,\ell-1\\ a'_{j\ominus k}=r}} n^j = \sum_{j=0,\dots,\ell-1} a''_j n^j =: y'' \pmod{n^\ell - 1}. \end{aligned}$$

Since $0 \le a''_i \le 2n-2$, we have $0 \le y'' \le (2n-2)\sum_{j=0}^{\ell-1} n^j = 2\left(n^\ell - 1\right)$. On the other hand, $y'' \equiv y \equiv 0 \pmod{n^\ell - 1}$, therefore $y'' \in \left\{0, n^\ell - 1, 2\left(n^\ell - 1\right)\right\}$. We examine these three cases separately.

1) If $y''=0$, then $a''_i = 0$ for all i, hence $a'_i = 0$ and $a'_{i\ominus k} \ne r$ for all i. This happens if and only if $b=0, r\ne 0$ and $a_i = 0$ for all i. This gives us the periodic point $x=0$ with $\ell = 1$.
2) If $y'' = n^\ell - 1$, then $a''_i = n-1$ for all i. Indeed, $-1 \equiv n^\ell - 1 = y'' \equiv a''_0 \pmod{n}$ and $0 \le a''_0 \le 2n-2$ imply that $a''_0 = n-1$. Then we have $-1 \equiv n^{\ell-1} - 1 = \left(y'' - a''_0\right)/n \equiv a''_1 \pmod{n}$ and $0 \le a''_1 \le 2n-2$, hence $a''_1 = n-1$. Continuing this way one can prove by induction on i that $a''_i = n-1$ for every i. This happens if and only if for each i we have either $a'_i = n-1, a'_{i\ominus k} \ne r$ or $a'_i = n-2, a'_{i\ominus k} = r$. Here we can distinguish three subcases.
   2a) If $r \notin \{n-2, n-1\}$, then $a'_i = a_i = n-1$ for all i, therefore $\ell = 1$ and we obtain the periodic point $x = -1$.
   2b) If $r = n-2$, then $a'_i \in \{n-1, n-2\}$ and $a'_{i\ominus k} = a'_i$ for all i. Thus $\{a_i\}$ is a k-periodic sequence with entries $n-1$ and $n-2$. (Recall that the shortest period of $\{a_i\}$ was ℓ, hence $\ell \mid k$.) By Remark 7, we obtain 2^k different periodic points in this case.
   2c) If $r = n-1$, then $a'_i \in \{n-1, n-2\}$ and $a'_{i\ominus k} \ne a'_i$ for all i, which implies $a'_{i\ominus 2k} = a'_i$. Thus $\{a_i\}$ is a $2k$-periodic sequence with entries $n-1$ and $n-2$ such that the first half of the period uniquely determines the second half: $a'_{i\oplus k} = 2n-3-a'_i$. (Here we must have $\ell \mid 2k$ and $\ell \nmid k$.) Just like in case 2b), we get 2^k periodic points.
3) If $y'' = 2\left(n^\ell - 1\right)$, then $a''_i = 2n-2$ for all i, hence $a'_i = 2n-3$ and $a'_{i\ominus k} = r$ for all i. This happens if and only if $b = n-2, r = 2n-3$ and

		1)	2a)	2b)	2c)	3)	$\lvert B_\infty \rvert$
$b=0$	$r=0$		1				1
$b=0$	$r \in \{1,\dots,n-3\}$	1	1				2
$b=0$	$r=n-2$	1		2^k			2^k+1
$b=0$	$r=n-1$	1			2^k		2^k+1
$1 \le b \le n-3$	$r=n-2$			2^k			2^k
$1 \le b \le n-3$	$r=n-1$				2^k		2^k
$1 \le b \le n-3$	$r \notin \{n-2,n-1\}$		1				1
$b=n-2$	$r=n-2$			2^k			2^k
$b=n-2$	$r=n-1$				2^k		2^k
$b=n-2$	$r \in \{n,\dots,2n-4\}$		1				1
$b=n-2$	$r=2n-3$		1			1	2

Table 2. The number of periodic points in the different cases in the proof of Theorem 22

$a_i = 2n-3+n^k$ for all i. Just as in the first case, this implies $\ell = 1$, and the corresponding periodic point is $x = -\frac{2n-3+n^k}{n-1}$.

Now only some bookkeeping is needed to finish the proof: one has to count the number of periodic points in the above cases, and determine which cases can occur for given values of b and r. The results are given in Table 2. ☐

Remark 23. The previous theorem concentrated on just one family of branching function systems, however, there are many more cases with a single periodic point. Here are some examples for $n=7$. We conjecture that the 7-powers appearing in these sequences can be replaced arbitrarily by other 7-powers:

- $8, 2, 10, 7^9+4, 5, 6, 7$;
- $1, 51, 3, 7^9+4, 5, 6, 21$;
- $7a, 1{+}7b, 2{+}7c, 3{+}7d, 4{+}7^i, 5, 6$ for $a \in \{1,2,3\}, b,c,d \in \{0,1,2\}, 0 \le i \le 7$.

Example 24. Applying Theorem 22 for $b=r=0$ we get

$$\left|B_\infty\left(n^k, 1, \dots, n-1\right)\right| = \left|B_\infty\left(1,\dots,n-1,n^k\right)\right| = 1,$$

while for $b=r=n-2$ we obtain (also taking Fact 2 into account)

$$\begin{aligned}\left|B_\infty\left(n-2+n^k, n-1, \dots, 2n-3\right)\right| &= \left|B_\infty\left(n-1,\dots,2n-3,n-2+n^k\right)\right| \\ &= \left|B_\infty\left(0,\dots,n-2,n^k-1\right)\right| = 2^k.\end{aligned}$$

These examples show that there may be a significant difference between the cardinalities $\left|B_\infty\left(d_1,d_2,\dots,d_n\right)\right|$ and $\left|B_\infty\left(d_1+1,d_2+1,\dots,d_n+1\right)\right|$, at least in certain special cases. (We will see in Theorem 30 that for fixed $d_1,\dots,d_{n-1}$ we have $\left|B_\infty\left(d_1,\dots,d_n\right)\right| = O\big(d_n^{\log_n 2}\big)$, which yields $\left|B_\infty\left(0,\dots,n-2,n^k-1\right)\right| = O\left(2^k\right)$. Thus, the difference between the cardinalities $\left|B_\infty\left(d_1,d_2,\dots,d_n\right)\right|$ and $\left|B_\infty\left(d_1+1,d_2+1,\dots,d_n+1\right)\right|$ cannot be much larger compared to d_n than in this example.)

We know that $\left|B_\infty\left(0,1,\dots,n-1\right)\right| = 2$ and $\left|B_\infty\left(0,1,\dots,n-2,n-1+n^k\right)\right| = 2^k+1 \ge 3$. The following proposition generalizes these results by giving a lower estimate for $\left|B_\infty\left(0,1,\dots,n-2,d_n\right)\right|$.

Proposition 25. *If $d_n \equiv n-1 \pmod n$ and $d_n > n-1$, then we have*

$$|B_\infty(0,1,\dots,n-2,d_n)| \geq 3.$$

If, in addition, $n-1 \mid d_n$, then $|B_\infty(0,1,\dots,n-2,d_n)| \geq 4$.

Proof. Assume first that $n-1 \nmid d_n$. Clearly, $0 \in B_\infty(0,1,\dots,n-2,d_n)$ and this is the only periodic point with cycle length 1. The set $\mathcal{A} \setminus \{0\}$ is easily seen to be nonempty and closed under R, hence it contains a cycle, which must have length at least 2. Thus, we have at least two more periodic points besides 0.

If $n-1 \mid d_n$, then we have two periodic points with cycle length 1, namely 0 and $-d_n/(n-1)$. Now $\mathcal{A} \setminus \{0, -d_n/(n-1)\}$ is closed under R, and it is nonempty, since $d_n > n-1$. Therefore, we have at least two periodic points in this set, hence $|B_\infty(0,1,\dots,n-2,d_n)| \geq 4$ in this case. □

Remark 26. The estimates given in the previous proposition are sharp: for example $B_\infty(0,1,5) = \{-2,-1,0\}$, $B_\infty(0,1,32) = \{-16,-12,-4,0\}$, $B_\infty(0,1,8) = \{-4,-3,-1,0\}$ and $B_\infty(0,1,1181) = \{-443,-148,0\}$.

6. Asymptotic behaviour of the number of periodic points

In this section we are interested in the number of periodic points when one or all of $d_1,\dots,d_n$ tend to infinity. If we let all the d_i's go to infinity in such a way that their differences stay the same, then we obtain the sequence $|B_\infty(d_1+s,\dots,d_n+s)|$ for the number of periodic points. The asymptotic behavior of this sequence as $s \to \infty$ is not very interesting, since it is periodic with period at most $n-1$ by Fact 2:

$$|B_\infty(d_1+n-1,\dots,d_n+n-1)| = |B_\infty(d_1,\dots,d_n)|.$$

In some cases this sequence is constant, for instance we obtain the sequence $4,4,\dots$ for $(d_1,d_2,d_3) = (0,1,17)$ and also for $(d_1,d_2,d_3) = (0,1,257)$. In some other cases, there might be big oscillations, as we have seen in Example 24. For example, we obtain the sequence $16,1,16,1,\dots$ for $(d_1,d_2,d_3) = (0,1,80)$, and we get $16,1,1,16,1,1,\dots$ for $(d_1,d_2,d_3,d_4) = (0,1,2,255)$.

Next we consider the case when $d_1,\dots,d_n$ tend to infinity in such a way that their quotients stay the same; as before, we assume that $d_1 < \dots < d_n$. We shall see that the number of periodic points increases linearly in this case (see Theorem 27 in the special case $c=0$). We will make use of the fact that $B_\infty = -\mathbb{T} \cap \mathbb{Z}$ (see Proposition 3.12 in [2]), where the set $\mathbb{T} = \mathbb{T}(d_1,\dots,d_n)$ is given by

$$\mathbb{T}(d_1,\dots,d_n) = \left\{ \sum_{i=1}^{\infty} n^{-i} a_i : a_i \in \{d_1,\dots,d_n\} \right\}.$$

Observe that the least element of $\mathbb{T}$ is $\sum_{i=1}^{\infty} n^{-i} d_1 = \frac{d_1}{n-1}$; similarly, the greatest element of $\mathbb{T}$ is $\frac{d_n}{n-1}$, hence $\mathbb{T} \subseteq \left[\frac{d_1}{n-1}, \frac{d_n}{n-1}\right] = -\mathcal{I}$. It was shown by Bandt in [1] that $\mathbb{T}$ is a compact set with non-empty interior.

Theorem 27. *Let $d_1 < \dots < d_n$, let c be an arbitrary integer and let $s \to \infty$ through integers relatively prime to n. Then we have*

$$\frac{|B_\infty(d_1 s + c,\dots,d_n s + c)|}{|\mathcal{A}(d_1 s + c,\dots,d_n s + c)|} \to \frac{n-1}{d_n - d_1}\mu(\mathbb{T}),$$

where $\mu(\mathbb{T})$ is the Lebesgue measure of $\mathbb{T}(d_1,\dots,d_n)$.

Proof. Let $\mathbb{T} = \mathbb{T}(d_1, \dots, d_n)$, and let us observe that $\mathbb{T}(d_1 s + c, \dots, d_n s + c) = s\mathbb{T} + \frac{c}{n-1}$. Therefore we have

$$\begin{aligned} B_\infty (d_1 s + c, \dots, d_n s + c) &= \{x \in \mathbb{Z} \colon -x \in \mathbb{T}(d_1 s + c, \dots, d_n s + c)\} \\ &= \Big\{x \in \mathbb{Z} \colon \frac{-x - \frac{c}{n-1}}{s} \in \mathbb{T}\Big\}, \end{aligned}$$

which means that the elements of $B_\infty (d_1 s + c, \dots, d_n s + c)$ are in a one-to-one correspondence with the elements of $\mathbb{T} \cap \frac{1}{s}\big(\mathbb{Z} - \frac{c}{n-1}\big)$. Since the latter set partitions the real line into intervals of length $\frac{1}{s}$, we see that

$$|B_\infty (d_1 s + c, \dots, d_n s + c)| \cdot \frac{1}{s} = \left|\mathbb{T} \cap \frac{1}{s}\Big(\mathbb{Z} - \frac{c}{n-1}\Big)\right| \cdot \frac{1}{s}$$

is a Riemann sum of the characteristic function of $\mathbb{T}$. Keesling has shown in [6] that the boundary of $\mathbb{T}$ has Lebesgue measure 0, hence $\mathbb{T}$ is Jordan measurable and then its characteristic function is Riemann-integrable. It follows that $|B_\infty (d_1 s + c, \dots, d_n s + c)| \cdot \frac{1}{s} \to \mu(\mathbb{T})$ as $s \to \infty$.

Since $\mathcal{A}(d_1 s + c, \dots, d_n s + c)$ consists of the integers in the interval

$$\Big[-\frac{d_n s + c}{n-1}, -\frac{d_1 s + c}{n-1}\Big] = s\mathcal{I} - \frac{c}{n-1},$$

the cardinality of $\mathcal{A}(d_1 s + c, \dots, d_n s + c)$ and $\frac{(d_n - d_1)s}{n-1}$ (which is the length of $s\mathcal{I} - \frac{c}{n-1}$) differ by at most one. As a consequence, we have that

$$\begin{aligned} &\lim_{s \to \infty} \frac{|B_\infty (d_1 s + c, \dots, d_n s + c)|}{|\mathcal{A}(d_1 s + c, \dots, d_n s + c)|} = \\ &= \lim_{s \to \infty} |B_\infty (d_1 s + c, \dots, d_n s + c)| \frac{n-1}{(d_n - d_1)s} = \frac{n-1}{d_n - d_1} \mu(\mathbb{T}). \end{aligned}$$

□

Our next goal is to study the number of periodic points when only one of the parameters, say d_n, tends to infinity, while the others are fixed. As a lower estimate we have the trivial inequality $|B_\infty (d_1, \dots, d_n)| \geq 1$, and in general we cannot have a nontrivial lower bound, since it is possible to let $d_n \to \infty$ in such a way that the number of periodic points stays constant 1 (cf. Theorem 22).

To establish an upper estimate, we will need the box-counting dimension of certain fractals. We briefly recall the necessary definitions and facts; for more background we refer the reader to [4]. Let $\mathcal{K}$ be a bounded subset of the real line. For $\delta > 0$, let $N_\delta(\mathcal{K})$ be the number of intervals of the form $[k\delta, (k+1)\delta)$ with $k \in \mathbb{Z}$ that contain at least one point from $\mathcal{K}$. The box-counting dimension (or Minkowski dimension) of $\mathcal{K}$ is defined as the limit

$$\lim_{\delta \to 0} \frac{\log N_\delta(\mathcal{K})}{\log(1/\delta)},$$

provided it exists.

Let $\mathcal{C}_n$ be the set defined similarly to the Cantor set, successively removing the middle $\frac{n-2}{n}$ part of the intervals ($n = 3$ gives the usual Cantor set). A real number $c \in [0, 1]$ belongs to $\mathcal{C}_n$ if and only if the base-n expansion of c contains only digits 0 and $n-1$. Just like the Cantor set, $\mathcal{C}_n$ is a self-similar fractal, and it has box-counting dimension $\log_n 2$. Thus we have $N_\delta(\mathcal{C}_n) \leq (1/\delta)^{\varepsilon + \log_n 2}$ for small enough δ; however, for our purposes the following stronger estimate will be necessary.

Proposition 28. *For all* $0 < \delta < 1$ *we have*

$$N_\delta(\mathcal{C}_n) \leq 4 \cdot (1/\delta)^{\log_n 2}.$$

Proof. It is easy to verify that $N_{1/n^t}(\mathcal{C}_n) = 2^t$ for all natural numbers t. Now let $\delta < 1$ be a positive real number, and let $t \in \mathbb{N}$ such that $1/n^t \leq \delta \leq 1/n^{t-1}$. Then we have $N_\delta(\mathcal{C}_n) \leq 2 \cdot N_{1/n^t}(\mathcal{C}_n)$, as any interval of length $1/n^t$ is covered by (at most) two intervals of length δ. Taking into account that $t \leq 1 + \log_n(1/\delta)$, we obtain the desired inequality:

$$N_\delta(\mathcal{C}_n) \leq 2 \cdot N_{1/n^t}(\mathcal{C}_n) = 2^{t+1} \leq 2^{2+\log_n(1/\delta)} = 4 \cdot (1/\delta)^{\log_n 2}.$$

□

The set $\frac{1}{n-1} \cdot \mathcal{C}_n$ consists of all numbers of the interval $[0,1]$ (actually, they are from the interval $\left[0, \frac{1}{n-1}\right]$) that have only 0 and 1 in their base-n representation. Obviously, $\frac{1}{n-1} \cdot \mathcal{C}_n$ also has box-counting dimension $\log_n 2$, and we have a similar estimate for N_δ as for $\mathcal{C}_n$.

Corollary 29. *There is a positive constant K depending only on n such that for all $0 < \delta < \frac{1}{n-1}$ we have*

$$N_\delta\left(\frac{1}{n-1} \cdot \mathcal{C}_n\right) \leq K \cdot \left(\frac{1}{\delta}\right)^{\log_n 2}.$$

Proof. Clearly, $N_\delta\left(\frac{1}{n-1} \cdot \mathcal{C}_n\right) = N_{(n-1)\delta}(\mathcal{C}_n)$, hence

$$N_\delta\left(\frac{1}{n-1} \cdot \mathcal{C}_n\right) \leq 4 \cdot \left(\frac{1}{(n-1)\,\delta}\right)^{\log_n 2}.$$

□

Theorem 30. *If $d_1 < \cdots < d_{n-1}$ are fixed and $d_n \to \infty$, then*

$$|B_\infty(d_1, \ldots, d_n)| = O\left(d_n^{\log_n 2}\right).$$

Proof. Let us decompose an arbitrary $x = \sum_{i=1}^{\infty} \frac{a_i}{n^i} \in \mathbb{T}$ as $x = \eta_x + \gamma_x$, where

$$\eta_x = \sum_{a_i \neq d_n} \frac{a_i}{n^i} \quad \text{and} \quad \gamma_x = \sum_{a_i = d_n} \frac{a_i}{n^i}.$$

Then we have $0 \leq \eta_x \leq d_{n-1} \cdot \sum_{i=1}^{\infty} \frac{1}{n^i} = \frac{d_{n-1}}{n-1}$, and $\gamma_x = d_n \cdot \sum_{a_i = d_n} \frac{1}{n^i} = d_n \cdot c_x$, where $c_x \in [0,1]$ has only digits 0 and 1 in its base-n expansion, i.e., $c_x \in \frac{1}{n-1} \cdot \mathcal{C}_n$. If x is an integer, then it must be one of the numbers $\lfloor \gamma_x \rfloor, \lfloor \gamma_x \rfloor + 1, \ldots, \lfloor \gamma_x \rfloor + \left\lfloor \frac{d_{n-1}}{n-1} \right\rfloor + 1$, thus we obtain the following "upper estimate" for the (negative of the) set of periodic points:

$$-B_\infty(d_1, \ldots, d_n) = \mathbb{T} \cap \mathbb{Z} \subseteq \bigcup_{c \in \frac{1}{n-1} \cdot \mathcal{C}_n} \left\{ \lfloor d_n \cdot c \rfloor, \lfloor d_n \cdot c \rfloor + 1, \ldots, \lfloor d_n \cdot c \rfloor + \left\lfloor \frac{d_{n-1}}{n-1} \right\rfloor + 1 \right\}.$$

Each set of this union has $\left\lfloor \frac{d_{n-1}}{n-1} \right\rfloor + 2$ elements, and the sets corresponding to c and c' coincide if and only if $\lfloor d_n \cdot c \rfloor = \lfloor d_n \cdot c' \rfloor$, hence

$$|B_\infty(d_1, \ldots, d_n)| = |\mathbb{T} \cap \mathbb{Z}| \leq \left(\left\lfloor \frac{d_{n-1}}{n-1} \right\rfloor + 2 \right) \cdot \left| \left\{ \lfloor d_n \cdot c \rfloor : c \in \frac{1}{n-1} \cdot \mathcal{C}_n \right\} \right|.$$

An integer k appears as $\lfloor d_n \cdot c \rfloor$ in the formula above if and only if there exists $c \in \frac{1}{n-1} \cdot \mathcal{C}_n$ such that $d_n \cdot c \in [k, k+1)$. The latter is equivalent to $c \in \left[\frac{k}{d_n}, \frac{k}{d_n} + \frac{1}{d_n}\right)$, therefore

$$|B_\infty(d_1, \ldots, d_n)| = |\mathbb{T} \cap \mathbb{Z}| \leq \left(\left\lfloor \frac{d_{n-1}}{n-1} \right\rfloor + 2 \right) \cdot N_{1/d_n}\left(\frac{1}{n-1} \cdot \mathcal{C}_n\right). \tag{10}$$

From Corollary 29 it follows that for $d_n > n-1$ we have

$$N_{1/d_n}\left(\frac{1}{n-1}\cdot\mathcal{C}_n\right)\leq K\cdot d_n^{\log_n 2}$$

for some constant K, and this together with (10) implies that $|B_\infty| = O\big(d_n^{\log_n 2}\big)$. □

The next theorem shows that the upper estimate obtained above cannot be sharpened: for fixed $d_2,\ldots,d_{n-1}$ it is possible to let $d_n\to\infty$ in such a way that $|B_\infty(0,d_2,\ldots,d_n)|$ is bounded from below by a constant multiple of $d_n^{\log_n 2}$.

Theorem 31. *Let $0,d_2,\ldots,d_{n-1},r_n$ be a complete system of residues modulo n. Then there is a sequence $d_{n,1},d_{n,2},\ldots$ of positive integers tending to ∞ such that for all $\ell\in\mathbb{N}$, $d_{n,\ell}\equiv r_n \pmod{n}$, and $|B_\infty(0,d_2,\ldots,d_{n-1},d_{n,\ell})| = \Theta\big(d_{n,\ell}^{\log_n 2}\big)$ as $\ell\to\infty$.*

Proof. Let $d_{n,\ell}$ be the least positive integer such that $d_{n,\ell}\equiv r_n \pmod{n}$ and $d_{n,\ell}\equiv 0 \pmod{n^\ell - 1}$. Since the moduli are relatively prime, such a $d_{n,\ell}$ exists, and $d_{n,\ell}\leq n\left(n^\ell-1\right) < n^{\ell+1}$. (Note that $d_{n,\ell}\geq n^\ell - 1$, therefore $d_{n,\ell}$ indeed tends to ∞ as $\ell\to\infty$.) We are going to prove that there are at least 2^ℓ periodic points for $0,d_2,\ldots,d_{n-1},d_{n,\ell}$. This suffices to prove the theorem, since

$$2^\ell = \frac{1}{2}\left(n^{\ell+1}\right)^{\log_n 2} > \frac{1}{2}d_{n,\ell}^{\log_n 2},$$

thus the number of periodic points is bounded from below by a constant multiple of $d_{n,\ell}^{\log_n 2}$, and we have seen in Theorem 30 that it is also bounded from above by a constant multiple of $d_{n,\ell}^{\log_n 2}$.

Let us choose a sequence $a_0,\ldots,a_{\ell-1}$ such that $a_i\in\{0,d_{n,\ell}\}$ for $i=1,\ldots,\ell$. There are 2^ℓ such sequences, and each of them gives a periodic point

$$x = -\frac{a_0 + a_1 n + \cdots + a_\ell n^{\ell-1}}{n^\ell - 1}.$$

Indeed, by Remark 7, we only need to verify that x is an integer, which is clearly the case, as $a_i\in\{0,d_{n,\ell}\}$ and $d_{n,\ell}\equiv 0 \pmod{n^\ell-1}$. □

7. Some open problems

Concluding the paper, we list some open problems that seem worthwhile investigating.

Theorem 17 characterizes equivalence of branching function systems corresponding to arithmetic sequences. Here we required that the equivalence preserves the colors of the edges of the underlying graphs, i.e., d_i corresponds to d_i'. It would be interesting to consider a weaker notion of equivalence, where we are allowed to permute the colors (i.e., d_i may correspond to some d_j' with $j\neq i$). We conjecture that Theorem 17 remains valid in this more general setting, too. Another natural extension of Theorem 17 would be to classify the branching function systems arising from the "almost arithmetic" sequences of Section 4 up to equivalence.

Corollary 16 also raises a natural question: Does every conjugation-closed set of primitive words arise as the set of words corresponding to the periodic points of a branching function system determined by a complete system of residues $d_1,\ldots,d_n$? (The answer is obviously "yes" if one allows arbitrary branching function systems.)

In Section 5 we presented examples with a single periodic point; however, there are probably many other such branching function systems, and it is still an open problem to characterize these. Proposition 25 gives a negative result: setting $d_1 = 0,\ldots,d_{n-1}=n-2$, there is no d_n such that $|B_\infty(d_1,\ldots,d_n)| = 1$. Is it possible

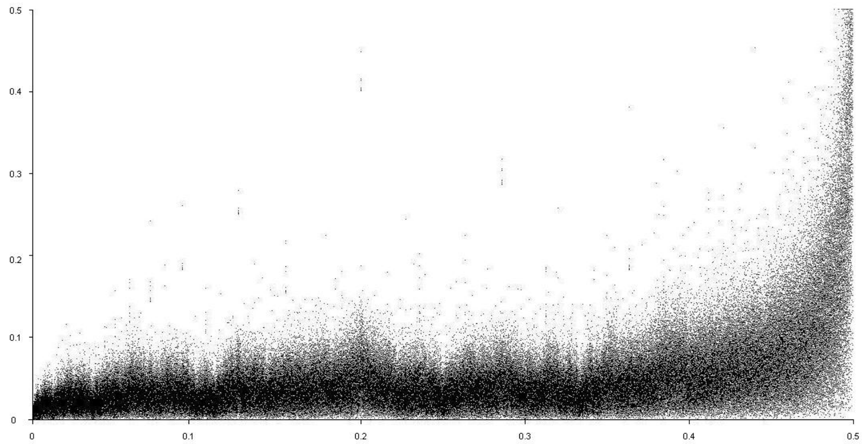

FIGURE 2. A graph of points of the form $\left(\frac{a}{b}, \min(0.5, \frac{B_\infty(0,a,b)}{\mathcal{A}(0,a,b)})\right)$

to fix $n-2$ parameters in such a way that the number of periodic points is at least 2, no matter what the remaining two parameters are?

We had to assume $d_1 = 0$ in Theorem 31 in order to make our construction work. Is it possible to remove this assumption?

The following figure shows the distribution of the pairs $\left(\frac{a}{b}, \min(0.5, \frac{|B_\infty|}{|\mathcal{A}|})\right)$ for triples of the form $(0, a, b)$ where $9000 < b < 9300$ and $0 < a < \frac{b}{2}$. (We have cut the graph at $y = 0.5$ in order to get a better resolution for smaller ratios.) The data has been calculated by a C++ program.

The graph allows us to formulate some conjectures:

- Theorem 27 shows that for example triples of the form $(0, a, 5a)$ yield proportionately the same amount of periodic points for large a's. The graph shows furthermore that triples of the form $(0, a+\epsilon, 5a+\epsilon)$ where a is large and ϵ is small compared to a yield considerable fewer periodic points, and they yield fewer as ϵ gets larger. This suggests that triples of the form $(0, a, b)$ yield locally the most periodic points when the greatest common divisor of a and b is big, and they also induce more periodic points in their neighbourhood, but the radius of this influence seems to be sublinear in b (the spikes on the same graph corresponding to bigger b's are narrower).
- If $\frac{a}{b}$ is close to $\frac{1}{2}$ then the ratio $\frac{|B_\infty|}{|\mathcal{A}|}$ is closer to 1, however, not for all such a's. Recall that the ratio equals 1 if $a = \frac{b}{2}$ by Theorem 12.

8. ACKNOWLEDGEMENTS

The authors would like to thank Dr. Mark Lawson for initiating this research, for his inspiring questions and for his help and hospitality during the first author's visits to Edinburgh. The first author acknowledges the support of the Engineering and Physical Sciences Research Council (EPSRC) grant no. EP/I032312/1. Research is partially supported by the Hungarian Scientific Research Fund (OTKA) grants no. K83219 and K104251, by the European Union and co-funded by the European Social Fund under the project "Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences" of project number "TÁMOP-4.2.2.A-11/1/KONV-2012-0073". The second author is supported by the János Bolyai Research Scholarship.

References

[1] Bandt, C.: 'Self-similar sets V: Integer matrices and fractal tilings of $\mathbb{R}^n$', *Proc. Amer. Math. Soc.* **112** (1991), 549–562.

[2] Bratteli, O., Jorgensen, E.T.: *Iterated function systems and permutation representations of the Cuntz algebra*, Memoirs of the A.M.S. No. 663, Amer. Math. Soc., 1999.

[3] Cuntz, J.: 'Simple C^*-algebras generated by isometries', *Commun. Math. Phys.*, **57** (1977), 173–185.

[4] Falconer, K.: 'Fractal Geometry: Mathematical Foundations and Applications', John Wiley, 2nd edition, 2003.

[5] Jones, D.G., Lawson, M. V.: 'Strong representations of the polycyclic inverse monoids: cycles and atoms', *Periodica Math. Hungar.* **64** (2012), 53 – 78.

[6] Keesling, J.: 'The boundaries of sel-similar tiles in $\mathbb{R}^n$', *Topology Appl.* **94** (1999), 195–205.

[7] Lawson, M.V.: 'Primitive partial permutation representations of the polycyclic monoids and branching funtion systems', *Period. Math. Hungar.*, **58** (2009), 189–207.

[8] Nivat, M., Perrot, J.-F.: 'Une gènèralisation du monoïde bicyclique', *C. R. Acad. Sci. Paris Sèr. A-B*, **271** (1970), 824–827 (in French).

Department of Mathematics, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

E-mail address: hartm@math.u-szeged.hu

Department of Mathematics, University of Szeged, Aradi vértanúk tere 1, H-6720 Szeged, Hungary

E-mail address: twaldha@math.u-szeged.hu