ASSOCIATIVE SPECTRA OF LINEAR QUASIGROUPS

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Abstract. We describe the associative spectra of linear quasigroups in terms of linear congruences for left and right depth sequence of binary trees.

1. Introduction

The associative spectrum of a binary operaition (or of a groupoid) is a sequence of natural numers that measures – in some sense – how far the operation is from being associative. In this paper we focus on linear quasigroups, i.e., groupoids of the form $\mathbf{A} = (A, \circ)$ with $x \circ y = \varphi(x) + \psi(y)$, where $+$ is a group operation on A and φ, ψ are automorphisms of the group $(A, +)$. Our main result (Theorem 4.10) is the description of the associative spectra of linear quasigroups.

To this end, we provide necessary and sufficient conditions for the satisfaction of a bracketing identity $t \approx t'$ by a groupoid in terms of a condition on the binary trees T and T' corresponding to t and t' that is easy to verify directly from the trees. Such conditions are typically based on the (left, right) depths of leaves in the trees modulo an integer, or some variants of this idea. Each such condition yields an equivalence relation on binary trees, and the n-th term of the associative spectrum of the groupoid under consideration is then given by the number of equivalence classes of binary trees with n leaves.

These sequences of numbers of equivalence classes of binary trees are interesting variants of the ubiquitous Catalan numbers. Many of these variants are new and do not appear in the OEIS, and we believe they may be of interest on their own right, as they are based on simple and fundamental relationships between binary trees. We have computed the first few members of the sequences, but unfortunately we were not able to explicitly describe the entire sequences. Finding explicit formulas for the n-th member of such sequences remains an intriguing open problem.

2. Preliminaries

2.1. **Generalities.** We assume the reader is familiar with basic concepts in abstract algebra: algebras, terms, identities, etc.

We will use the following notation for familiar sets of natural numbers. Let $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. For any $a, b \in \mathbb{N}$, let $[a, b] := \{i \in \mathbb{N} \mid a \leq i \leq b\}$, the *interval* from a to b; in particular, $[a, b] = \emptyset$ if $a > b$. For $n \in \mathbb{N}$, let $[n] := [1, n]$.

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We will denote tuples by bold letters and their components by the corresponding italic letters with subscripts, e.g., $\mathbf{a} = (a_1, \dots, a_n)$.

An *operation* on A is a mapping $f \colon A^n \to A$ for some number $n \in \mathbb{N}$, called the *arity* of f. The i-th n-ary *projection* on A is the operation $\mathrm{pr}_i^{(n)} \colon A^n \to A$, $(a_1, \dots, a_n) \mapsto a_i$.

2.2. Groupoids, groups, quasigroups. Recall that a *groupoid* is an algebra $\mathbf{A} = (A, \circ)$ with a single binary operation $\circ$, often referred to as *multiplication* and usually written simply as juxtaposition. A *semigroup* is a groupoid with associative multiplication. A *monoid* is a semigroup with a neutral element. A *group* is a monoid in which every element is invertible.

The *opposite groupoid* of a groupoid $\mathbf{A} = (A, \circ)$ is the groupoid $\mathbf{A}^{\mathrm{op}} = (A, \diamond)$ where the multiplication is defined as $a \diamond b := b \circ a$.

Given a groupoid $\mathbf{A} = (A, \circ)$, multiplication by a fixed element $a \in A$ gives two self-maps on A: the *left translation* by a, which is the map $L_a \colon A \to A$, $x \mapsto ax$, and the *right translation* by a, which is the map $R_a \colon A \to A$, $x \mapsto xa$.

A *quasigroup* is a groupoid $(A, \circ)$ such that for all $a, b \in A$, there exist unique elements $x, y \in A$ such that $a \circ x = b$ and $y \circ a = b$. In other words, the left and right translations L_a and R_a of $(A, \circ)$ are bijections for all $a \in A$. In other words, the multiplication table of $(A, \circ)$ (for a finite set A) is a Latin square. We can thus define the the *right division* $/$ and the *left division* $\backslash$ as follows: $b/x := a$ if $a \circ x = b$, and $y \backslash b := a$ if $y \circ a = b$. A quasigroup with a neutral element is called a *loop.* An associative loop is a group.

2.3. Bracketings, associative spectrum. In this paper, we consider terms in the language of groupoids over the set $X := \{x_i \mid i \in \mathbb{N}_+\}$, the so-called *standard set of variables.* Such *terms* can be defined by the following recursion: every variable $x_i \in X$ is a term, and if t_1 and t_2 are terms, then $(t_1 t_2)$ is a term; every term is obtained by a finite number of applications of these rules. Denote by $W(X)$ the set of all terms over X. A subword of a term t that is itself a term is called a *subterm* of t.

For a subset $S \subseteq \mathbb{N}_+$, let $X_S := \{x_i \mid i \in S\}$. The set of variables occurring in a term t is denoted by $\mathrm{var}(t)$. A term t is *n-ary* if $\mathrm{var}(t) \subseteq X_{[n]}$. Note that an n-ary term is also m-ary for every $m \geq n$. In order to emphasize the fact that $\mathrm{var}(t) \subseteq X_{[p,q]}$, we may write t as $t(x_p, x_{p+1}, \dots, x_q)$.

Let $\mathbf{A} = (A, \circ)$. Any *assignment* $h \colon X \to A$ of values from A for the variables extends to a *valuation* $h' \colon W(X) \to A$ of terms in $\mathbf{A}$ by the following recursion: $h'(x_i) := h(x_i)$ for $x_i \in X$, and if $t = (t_1 t_2) \in W(X)$, then $h'(t) := h'(t_1) \circ h'(t_2)$. Of course, for the valuation of a term $t \in W(X)$, we only need to consider a partial assignment $h \colon \mathrm{var}(t) \to A$. We will simplify the notation and write $h(t)$ instead of $h'(t)$.

Let $\mathbf{A} = (A, \circ)$ be a groupoid, and let t be an n-ary term. Define the operation $t^{\mathbf{A}} \colon A^n \to A$ by the following recursion: if $t = x_i \in X$, then $t^{\mathbf{A}} := \mathrm{pr}_i^{(n)}$; if $t = (t_1 t_2)$, then $t^{\mathbf{A}}(\mathbf{a}) := t_1^{\mathbf{A}}(\mathbf{a}) \circ t_2^{\mathbf{A}}(\mathbf{a})$ for all $\mathbf{a} \in A^n$. The operation $t^{\mathbf{A}}$ is called the *term operation* induced by t on $\mathbf{A}$. Thus, the term operation $t^{\mathbf{A}}$ gives all the valuations of t in $\mathbf{A}$: $t^{\mathbf{A}}(\mathbf{a}) = h(t)$, where $h(x_i) = a_i$ for all $i \in [n]$. (As noted above, the arity of a term is not unique, so the arity of the induced term operation must be specified if necessary. The arity is usually clear from the context or does not matter.)

An identity is a pair (s, t) of terms, usually written as $s \approx t$. An identity $s \approx t$ is *trivial* if $s = t$. A groupoid $\mathbf{A} = (A, \circ)$ *satisfies* an identity $s \approx t$, in symbols, $\mathbf{A} \models s \approx t$, if $s^{\mathbf{A}} = t^{\mathbf{A}}$, or, equivalently, if $h(s) = h(t)$ for all assignments $h\colon X \to A$.

A *bracketing* of size n is a term in the language of groupoids obtained by inserting pairs of parentheses in the string $x_1 x_2 \cdots x_n$ appropriately. The number of distinct bracketings of size n equals the $(n-1)$-st Catalan number C_{n-1}. We denote by B_n the set of all bracketings of size n. A *bracketing identity* of size n is an identity $t \approx t'$ where $t, t' \in B_n$.

Let $\mathbf{A} = (A, \circ)$ be a groupoid. For each $n \in \mathbb{N}_+$, we define the equivalence relation $\sigma_n(\mathbf{A})$ on B_n by the rule that $(t, t') \in \sigma_n(\mathbf{A})$ if and only if $\mathbf{A}$ satisfies the identity $t \approx t'$. We call the sequence $(\sigma_n(\mathbf{A}))_{n \in \mathbb{N}_+}$ the *fine associative spectrum* of $\mathbf{A}$. The *associative spectrum* of $\mathbf{A}$ is the sequence $(s_n(\mathbf{A}))_{n \in \mathbb{N}_+}$, where $s_n(\mathbf{A}) := |B_n / \sigma_n(\mathbf{A})|$. Equivalently, $s_n(\mathbf{A})$ is the number of distinct term operations induced by the bracketings of size n on $\mathbf{A}$. We clearly have $1 \leq s_n(\mathbf{A}) \leq C_{n-1}$. If the operation is associative, then $s_n(\mathbf{A}) = 1$ for all $n \in \mathbb{N}_+$. At the other extreme, we have groupoids whose associative spectrum is *Catalan*, i.e., $s_n(\mathbf{A}) = C_{n-1}$ for $n \geq 2$; we call such groupoids *antiassociative.*[1]

The associative spectrum can be seen as a measure of how far the groupoid operation is from being associative. Intuitively, the faster the associative spectrum grows, the less associative the operation is considered. This notion was introduced by Csákány and Waldhauser [6], and it appears in the literature under different names, such as "subassociativity type" (Braitt, Silberger [4]), and "the number of $*$-equivalence classes of parenthesizations of $x_0 * x_1 * \cdots * x_n$" (Hein, Huang [8], Huang, Mickey, Xu [11]).

The *opposite* of a bracketing $t \in B_n$, denoted t^{op} is the bracketing obtained by writing t backwards and changing x_i to x_{n-i+1} for $i \in [n]$.

Fact 2.1. *A groupoid* $\mathbf{A}$ *satisfies* $t \approx t'$ *if and only if* $\mathbf{A}^{\mathrm{op}}$ *satisfies* $t^{\mathrm{op}} \approx t'^{\mathrm{op}}$.

Based on Fact 2.1, one can obtain the following result.

Lemma 2.2 (Csákány, Waldhauser [6, Statement 2.4])**.** *Isomorphic groupoids have the same associative spectrum. A groupoid and its opposite groupoid have the same associative spectrum.*

Since there are only one bracketing of size 1, namely x_1, and only one bracketing of size 2, namely $(x_1 x_2)$, it it obvious that $s_1(\mathbf{A}) = s_2(\mathbf{A}) = 1$ for every groupoid $\mathbf{A}$. Therefore, we may always assume that $n \geq 3$ when we consider the n-th component of an associative spectrum.

3. Binary trees and variants of Catalan numbers

3.1. **Binary trees.** A *tree* is a directed graph T that has a designated vertex u called the *root* and in which there is a unique walk from the root to any other vertex v. Hence a tree is acyclic, and the edges are directed away from the root. In this paper, we draw trees in such a way that the root is on the top and edges are directed downwards; with this convention there is no need to indicate the direction of edges. In a tree, the outneighbours of a vertex v are called its *children,* and v is called the *parent* of its children. The vertices reachable from v are called its *descendants,*

[1] This is not to be confused with the following property that is also often called *antiassociativity:* for all $a, b, c \in A$, $a \circ (b \circ c) \neq (a \circ b) \circ c$.

and v is an *ancestor* of any of its descendant. Two vertices are *siblings* if they have the same parent. A childless vertex is called a *leaf;* non-leaves are called *internal vertices.* We denote by $\mathrm{Int}(T)$ the set of all internal vertices of T. A subgraph of a tree induced by a vertex v and all its descendants is called the *subtree* rooted at v.

An *ordered tree* or *plane tree* is a tree in which a linear ordering is specified for the children of each vertex. We think of ordering the children from left to right, so that if v has outdegree k and its children are ordered as $u_0 < u_1 < \cdots < u_{k-1}$, then u_0 is the leftmost child and u_{k-1} is the rightmost child of v. Diagrams presenting plane trees shall be drawn in such a way that the children of a vertex are drawn left-to-right; such a drawing uniquely specifies the ordering of children.

A *binary tree* is a plane tree in which every internal vertex has exactly two children; the two children are referred to as the *left child* and the *right child.* We denote by T_n the set of all (isomorphism classes of) binary trees with n leaves. The subtree rooted at the left child (right child) of a vertex v is referred to as the *left* (*right*) *subtree* of v.

Let T be a plane tree. The *address* of a vertex v in T, denoted by $\alpha_T(v)$, is a word over $\mathbb{N}$ defined by the following recursion. The address of the root is the empty word ε. If v is an internal node with address w, and the children of v are $u_0 < u_1 < \cdots < u_{k-1}$, then the addresses of the child u_i is wi. Thus, the address of a vertex conveys the sequence of choices of children made along the unique path from the root to the given vertex.

The length of the unique path from the root to a vertex v in T is called the *depth* of v in T and is denoted by $d_T(v)$. In a binary tree T, we also define the *left depth* of a vertex v in T, denoted by $\delta_T(v)$, as the number of left steps on the unique path from the root of T to v, i.e., the number of 0's in $\alpha_T(v)$. The *right depth* of v in T is defined analogously and is denoted by $\rho_T(v)$.

The vertices of a plane tree T are totally ordered by the lexicographic ordering of their addresses (with respect to the natural ordering of $\mathbb{N}$): $v \leq v'$ if and only if $\alpha_T(v) \leq^{\mathrm{lex}} \alpha_T(v')$. This ordering is referred to as the *left-to-right order* of the vertices, and it corresponds to the so-called preorder traversal of the tree.

The addresses of two consecutive leaves of a binary tree are related in the following way.

Lemma 3.1. *Let T be a binary tree with leaves $1, 2, \dots, n$ in the left-to-right order. Then for all $i \in [n-1]$, $\alpha_T(i) = u01^k$ and $\alpha_T(i+1) = u10^\ell$ for some $k, \ell \in \mathbb{N}$, where u is the address of the deepest common ancestor of the leaves i and $i+1$.*

Proof. Obvious, as the leaves i and $i+1$ are the rightmost leaf of the left subtree and the leftmost leaf of the right subtree, respectively, of the deepest common ancestor of i and $i+1$. □

New binary trees can be built from given ones by joining two trees under a new root vertex. Let T_1 and T_2 be binary trees. We denote by $T_1 \wedge T_2$ the binary tree that is obtained by taking the disjoint union of T_1 and T_2, adding a new vertex u and designating it as the root of $T_1 \wedge T_2$, and setting the root of T_1 as the left child of u and the root of T_2 as the right child of u.

Another way of building new binary trees from given ones is adding new leaves. Let T be a binary tree, and assume its leaves are $1, 2, \dots, n$ in the left-to-right order. Now let $i \in [n]$, and let T_i^+ be the binary tree obtained by adding two new vertices p and q, which are designated as the left child and the right child of vertex

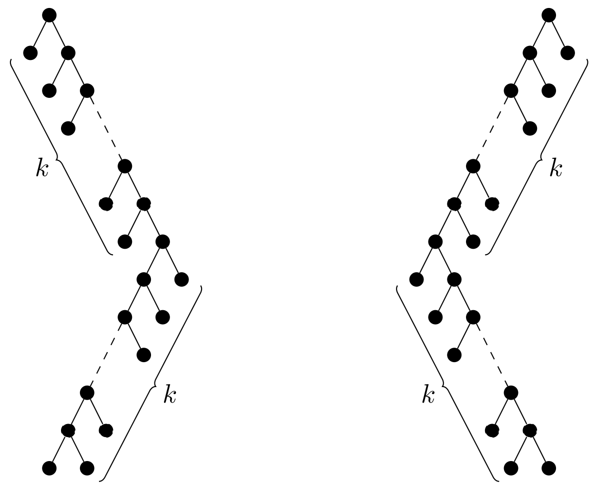


FIGURE 1. Two binary trees with the same depth sequence modulo k.

i, respectively. In this way we turned the leaf i of T into an internal vertex in T_i^+, and T_i^+ has $n+1$ leaves.

It is well known that binary trees with n leaves are in a one-to-one correspondence with bracketings of size n; hence the number of binary trees with n leaves is C_{n-1}. A canonical bijection between B_n and T_n is given by the restriction to B_n of the map τ from the set of all groupoid terms to the set of all binary trees defined recursively as follows. For any variable x_i, let $\tau(x_i)$ be the binary tree with one vertex. For a term $t = (t_1 \cdot t_2)$, let $\tau(t) := \tau(t_1) \wedge \tau(t_2)$. We often identify a bracketing $t \in B_n$ with $\tau(t)$, and we sometimes write $T(t)$ for $\tau(t)$.

If $T = \tau(t)$ for some bracketing $t \in B_n$, then the tree $\tau(t^{\mathrm{op}})$ is called the *opposite tree* of T and is denoted by T^{op}. The opposite tree of T can be thought of as obtained from T by reflection over a vertical line.

3.2. **Modular (left, right) depth sequences.** Let T be a binary tree with n leaves, and assume its leaves are $1, 2, \dots, n$ in the left-to-right order. The *depth sequence* of T is the tuple $d_T := (d_T(1), d_T(2), \dots, d_T(n))$. Similarly, the *left depth sequence* of T is the tuple $\delta_T := (\delta_T(1), \delta_T(2), \dots, \delta_T(n))$, and the *right depth sequence* of T is $\rho_T := (\rho_T(1), \rho_T(2), \dots, \rho_T(n))$. A binary tree is uniquely determined by its depth sequence, and it is also uniquely determined by its left (or right) depth sequence (see Csákány, Waldhauser [6, Statements 2.7, 2.8]).

We may also consider (left, right) depth sequences modulo some $k \in \mathbb{N}$. Let d_T^k, δ_T^k, ρ_T^k be the sequences obtained from d_T, δ_T, ρ_T, respectively, by taking componentwise remainders under division by k. These are called the (*left, right*) *depth sequences* of T *modulo* k, or *modular* (*left, right*) *depth sequences* of T. As the following example demonstrates, binary trees are not uniquely determined by their modular (left, right) depth sequences.

Example 3.2. For any $k \in \mathbb{N}_+$, the two binary trees with $2k+1$ leaves shown in Figure 1 have the same depth sequence modulo k, namely $(1, 2, \dots, k-1, 0, 0, 0, k-1, \dots, \dots, \dots, 2, 1)$. Similarly, the two binary trees with $k+2$ leaves shown in Figure 2

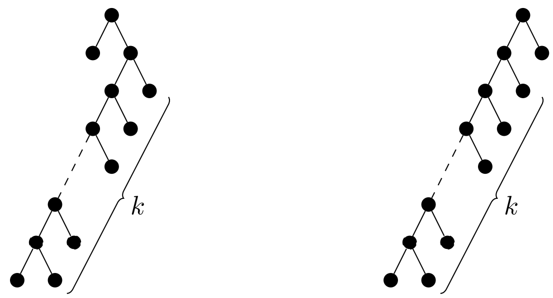


FIGURE 2. Two binary trees with the same left depth sequence modulo k.

have the same left depth sequence modulo k, namely $(1, 0, k-1, k-2, \dots, 1, 0)$, and their opposite trees have the same right depth sequence modulo k.

The (left, right) depth sequences of trees built with the constructions introduced earlier in this subsection can be described easily in terms of the (left, right) depth sequences of the given trees.

Lemma 3.3. *Let T and T' be binary trees with (left, right) depth sequences $d_T = (d_1, \dots, d_m)$, $\delta_T = (\delta_1, \dots, \delta_m)$, $\rho_T = (\rho_1, \dots, \rho_m)$, $d_{T'} = (d'_1, \dots, d'_n)$, $\delta_{T'} = (\delta'_1, \dots, \delta'_n)$, $\rho_{T'} = (\rho'_1, \dots, \rho'_n)$. Then the (left, right) depth sequences of $T \wedge T'$, T_i^+ ($i \in [m]$), and T^{op} are the following:*

$$\begin{aligned}
d_{T \wedge T'} &= (d_1 + 1, \dots, d_m + 1, d'_1 + 1, \dots, d'_n + 1),\\
\delta_{T \wedge T'} &= (\delta_1 + 1, \dots, \delta_m + 1, \delta'_1, \dots, \delta'_n),\\
\rho_{T \wedge T'} &= (\rho_1, \dots, \rho_m, \rho'_1 + 1, \dots, \rho'_n + 1);\\
d_{T_i^+} &= (d_1, \dots, d_{i-1}, d_i + 1, d_i + 1, d_{i+1}, \dots, d_m),\\
\delta_{T_i^+} &= (\delta_1, \dots, \delta_{i-1}, \delta_i + 1, \delta_i, \delta_{i+1}, \dots, \delta_m),\\
\rho_{T_i^+} &= (\rho_1, \dots, \rho_{i-1}, \rho_i, \rho_i + 1, \rho_{i+1}, \dots, \rho_m);\\
d_{T^{\mathrm{op}}} &= (d_m, \dots, d_1),\\
\delta_{T^{\mathrm{op}}} &= (\rho_m, \dots, \rho_1),\\
\rho_{T^{\mathrm{op}}} &= (\delta_m, \dots, \delta_1).
\end{aligned}$$

Proof. Straightforward verification. □

It is not obvious to the authors how to recognize whether a given n-tuple of natural numbers is the (left, right) depth sequence of some binary tree (modulo k). For the depth sequence modulo 2, Huang, Mickey, and Xu provided a rather simple necessary and sufficient condition [11, Lemma 6].

3.3. **Equivalence relations on binary trees based on modular (left, right) depth sequences.** In this subsection we are going to define several equivalence relations on the set T_n of binary trees with n leaves ($n \in \mathbb{N}_+$). Using the one-to-one correspondence between binary trees with n leaves and bracketings of n variables, we may equivalently view these as equivalence relations on bracketings: if $\sim$ is any one of the equivalence relations defined on binary trees and $t, t' \in B_n$, we let $t \sim t'$ if and only if $T(t) \sim T(t')$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	OEIS
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	A000012
2	1	1	2	5	10	21	42	85	170	341	682	1 365	2 730	5 461	10 922	A000975
3	1	1	2	5	14	42	129	398	1 223	3 752	11 510	35 305	108 217	331 434	1 014 304	
4	1	1	2	5	14	42	132	429	1 429	4 849	16 689	58 074	203 839	720 429	2 560 520	
5	1	1	2	5	14	42	132	429	1 430	4 862	16 795	58 773	207 906	742 203	2 670 389	
6	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 011	742 885	2 674 303	
7	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 439	
8	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 440	
C_{n-1}	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 440	A000108

TABLE 1. The number $T^{\mathrm{d}}_{k,n}$ of binary trees with n leaves up to k-depth-equivalence and Catalan numbers C_{n-1}.

Definition 3.4. For $k, \ell \in \mathbb{N}_+$ and $T, T' \in T_n$, we let

- $T \sim^{\mathrm{d}}_k T'$ if and only if $d^k_T = d^k_{T'}$, that is, $d_T(i) \equiv d_{T'}(i) \pmod{k}$ for all $i \in [n]$ (*k-depth-equivalence*);
- $T \sim^{\mathrm{L}}_k T'$ if and only if $\delta^k_T = \delta^k_{T'}$, that is, $\delta_T(i) \equiv \delta_{T'}(i) \pmod{k}$ for all $i \in [n]$ (*k-left-depth-equivalence*);
- $T \sim^{\mathrm{R}}_k T'$ if and only if $\rho^k_T = \rho^k_{T'}$, that is, $\rho_T(i) \equiv \rho_{T'}(i) \pmod{k}$ for all $i \in [n]$ (*k-right-depth-equivalence*);
- $T \sim^{\mathrm{LR}}_{k,\ell} T'$ if and only if $T \sim^{\mathrm{L}}_k T'$ and $T \sim^{\mathrm{R}}_\ell T'$ (*(k, ℓ)-depth-equivalence*).

We introduce the following notation for the number of equivalence classes of T_n with respect to the above equivalence relations:

$$T^{\mathrm{d}}_{k,n} := |T_n/{\sim^{\mathrm{d}}_k}|, \quad T^{\mathrm{L}}_{k,n} := |T_n/{\sim^{\mathrm{L}}_k}|, \quad T^{\mathrm{R}}_{k,n} := |T_n/{\sim^{\mathrm{R}}_k}|, \quad T^{\mathrm{LR}}_{k,\ell,n} := |T_n/{\sim^{\mathrm{LR}}_{k,\ell}}|.$$

It is clear that $T \sim^{\mathrm{L}}_k T'$ if and only if $T^{\mathrm{op}} \sim^{\mathrm{R}}_k T'^{\mathrm{op}}$, and consequently $T^{\mathrm{L}}_{k,n} = T^{\mathrm{R}}_{k,n}$.

Regarding the number of equivalence classes of the k-depth-equivalence relation $\sim^{\mathrm{d}}_k$, only a few particular cases are well understood. The 0-depth-equivalence relation $\sim^{\mathrm{d}}_0$ is just the equality relation, so the numbers $T^{\mathrm{d}}_{0,n}$ coincide with the Catalan numbers: $T^{\mathrm{d}}_{0,n} = C_{n-1}$ for all $n \geq 1$. The 1-depth-equivalence relation $\sim^{\mathrm{d}}_1$ is entirely trivial; all binary trees with n leaves are 1-depth-equivalent, so $T^{\mathrm{d}}_{1,n} = 1$ for all $n \geq 1$. The 2-depth-equivalence was investigated by Huang, Mickey and Xu [11], and the numbers $T^{\mathrm{d}}_{2,n}$ were shown to be given by the sequence A000975 in *The On-Line Encyclopedia of Integer Sequences* (OEIS) [17], which is known to have several characterizations, for example, for $n \geq 2$,

$$T^{\mathrm{d}}_{2,n} = \left\lfloor \frac{2^n}{3} \right\rfloor = \frac{2^{n+1} - 3 - (-1)^{n+1}}{6} = \begin{cases} \frac{2^n - 1}{3}, & \text{if } n \text{ is even}, \\ \frac{2^n - 2}{3}, & \text{if } n \text{ is odd}. \end{cases}$$

We are not aware of any results concerning moduli greater than 2. We have computed the values of $T^{\mathrm{d}}_{k,n}$ for small n and k with the help of the GAP computer algebra system [7] and present them in Table 1. Apart from the first two rows, these sequences do not seem to match any entry in the OEIS.

In contrast, the number $T^{\mathrm{L}}_{k,n}$ of $\sim^{\mathrm{L}}_k$-equivalence classes of T_n is well understood for any $k, n \in \mathbb{N}_+$; these numbers are given by the so-called k-modular Catalan numbers $C_{k,n}$ defined by Hein and Huang [8]: $T^{\mathrm{L}}_{k,n} = C_{k,n-1}$. Closed formulas for

$k \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	OEIS
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	A000012
2	1	1	2	4	8	16	32	64	128	256	512	1 024	2 048	4 096	8 192	A011782
3	1	1	2	5	13	35	96	267	750	2 123	6 046	17 303	49 721	143 365	414 584	A005773
4	1	1	2	5	14	41	124	384	1 210	3 865	12 482	40 677	133 572	441 468	1 467 296	A159772
5	1	1	2	5	14	42	131	420	1 375	4 576	15 431	52 603	180 957	627 340	2 189 430	A261588
6	1	1	2	5	14	42	132	428	1 420	4 796	16 432	56 966	199 444	704 146	2 504 000	A261589
7	1	1	2	5	14	42	132	429	1 429	4 851	16 718	58 331	205 632	731 272	2 620 176	A261590
8	1	1	2	5	14	42	132	429	1 430	4 861	16 784	58 695	207 452	739 840	2 658 936	A261591
9	1	1	2	5	14	42	132	429	1 430	4 862	16 795	58 773	207 907	742 220	2 670 564	A261592
10	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 785	207 998	742 780	2 673 624	
11	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 011	742 885	2 674 304	
12	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 899	2 674 424	
13	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 439	
14	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 440	
15	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 440	
C_{n-1}	1	1	2	5	14	42	132	429	1 430	4 862	16 796	58 786	208 012	742 900	2 674 440	A000108

TABLE 2. The number $T^{\mathrm{L}}_{k,n}$ of binary trees with n leaves up to k-left-depth-equivalence, i.e., modular Catalan number $C_{k,n-1}$.

modular Catalan numbers are known [8, Theorem 1.1]:

$$C_{k,n} = \sum_{\substack{\lambda \subseteq (k-1)^n \\ |\lambda| < n}} \frac{n - |\lambda|}{n} m_\lambda(1^n) = \sum_{0 \leq j \leq (n-1)/k} \frac{(-1)^j}{n} \binom{n}{j} \binom{2n - jk}{n+1}.$$

(For explanation of the symbols used in the first summation formula, please refer to [8].) In particular, $C_{2,n} = 2^{n-2}$ for $n \geq 2$. The numbers $T^{\mathrm{L}}_{k,n} = C_{k,n-1}$ for $k, n \leq 15$ are evaluated in Table 2.

As for the number $T^{\mathrm{LR}}_{k,\ell,n}$ of $\sim^{\mathrm{LR}}_{k,\ell}$-equivalence classes of T_n, Hein and Huang [9, 10, Section 1, last paragraph] conjectured, based on computational evidence, that $T^{\mathrm{LR}}_{k,\ell,n} = T^{\mathrm{L}}_{k+\ell-1,n}$, for all $k, \ell, n \geq 1$. We have verified this with the help of a computer for $k, \ell, n \leq 14$.

Next we define more general equivalence relations on binary trees that are also based on left and right depth sequences. As we shall see in the next section, these relations are of key importance in describing associative spectra of linear quasigroups.

Definition 3.5. For $a, b, m \in \mathbb{N}$, define the equivalence relation $\sim^{\mathrm{lin}}_{a,b,m}$ on T_n by the following rule. Assume that T and T' are binary trees in T_n and their leaves are $1, 2, \dots, n$ in the left-to-right order. Set

$$T \sim^{\mathrm{lin}}_{a,b,m} T' :\Longleftrightarrow \forall i \in [n] \colon a\delta_T(i) + b\rho_T(i) \equiv a\delta_{T'}(i) + b\rho_{T'}(i) \pmod{m}.$$

Let $T^{\mathrm{lin}}_{a,b,m,n} := |T_n/{\sim^{\mathrm{lin}}_{a,b,m}}|$.

We have enumerated the numbers $T^{\mathrm{lin}}_{a,b,m,n}$ for small values of the parameters a, b, m, n and present them in Appendix 5. It remains an open problem to determine these numbers for arbitrary a, b, m, n. The following lemma shows some relationships between numbers of this form, as well as to the other variants of Catalan numbers of Definition 3.4.

Lemma 3.6. *Let $a, b, m \in \mathbb{N}$.*

(i) $T^{\mathrm{lin}}_{a,b,m,n} = T^{\mathrm{lin}}_{b,a,m,n}$.

(ii) *For any $\ell \in \mathbb{N}_+$, we have $\sim^{\mathrm{lin}}_{a,b,m} = \sim^{\mathrm{lin}}_{\ell a,\ell b,\ell m}$; consequently, $T^{\mathrm{lin}}_{a,b,m,n} = T^{\mathrm{lin}}_{\ell a,\ell b,\ell m,n}$.*

(iii) *If ℓ is a unit modulo m, then $\sim^{\mathrm{lin}}_{a,b,m} = \sim^{\mathrm{lin}}_{\ell a,\ell b,m}$; consequently, $T^{\mathrm{lin}}_{a,b,m,n} = T^{\mathrm{lin}}_{\ell a,\ell b,m,n}$.*

(iv) *If $\gcd(a,b) = 1$ and $m = ab$, then $\sim^{\mathrm{lin}}_{a,b,m} = \sim^{\mathrm{LR}}_{b,a}$; consequently, $T^{\mathrm{lin}}_{a,b,m,n} = T^{\mathrm{LR}}_{b,a,n} = T^{\mathrm{LR}}_{a,b,n}$.*

Proof. (i) Since for all $t, t' \in B_n$,

$$\begin{aligned} t \sim^{\mathrm{lin}}_{a,b,m} t' &\iff \forall i \in [n] \colon a\delta_t(x_i) + b\rho_t(x_i) \equiv a\delta_{t'}(x_i) + b\rho_{t'}(x_i) \pmod{m} \\ &\iff \forall i \in [n] \colon a\rho_{t^{\mathrm{op}}}(x_i) + b\delta_{t^{\mathrm{op}}}(x_i) \equiv a\rho_{t'^{\mathrm{op}}}(x_i) + b\delta_{t'^{\mathrm{op}}}(x_i) \pmod{m} \\ &\iff t^{\mathrm{op}} \sim^{\mathrm{lin}}_{b,a,m} t'^{\mathrm{op}}, \end{aligned}$$

we conclude that the map $t \mapsto t^{\mathrm{op}}$ induces a bijection between $B_n/{\sim^{\mathrm{lin}}_{a,b,m}}$ and $B_n/{\sim^{\mathrm{lin}}_{b,a,m}}$ for each $n \in \mathbb{N}_n$.

(ii) Clear because the congruences $ax + by \equiv 0 \pmod{m}$ and $\ell ax + \ell by \equiv 0 \pmod{\ell m}$ are equivalent.

(iii) Since ℓ is a unit modulo m, the congruences $ax + by \equiv 0 \pmod{m}$ and $\ell ax + \ell by \equiv 0 \pmod{m}$ are equivalent.

(iv) Since $\gcd(a,b) = 1$, it follows from the Chinese remainder theorem that $ax + by \equiv 0 \pmod{ab}$ is equivalent to $ax + by \equiv 0 \pmod{a}$ and $ax + by \equiv 0 \pmod{b}$, which in turn is equivalent to $ax \equiv 0 \pmod{b}$ and $by \equiv 0 \pmod{a}$. From $\gcd(a,b) = 1$, it follows that a is a unit modulo b and b is a unit modulo a; hence the last pair of congruences is equivalent to $x \equiv 0 \pmod{b}$ and $y \equiv 0 \pmod{a}$. We conclude that $t \sim^{\mathrm{lin}}_{a,b,m} t'$ if and only if $t \sim^{\mathrm{LR}}_{b,a} t'$, as claimed. □

3.4. **Equivalence of binary trees modulo a group.**

Definition 3.7. Let $\mathbf{G} = (G, \cdot)$ be a group with neutral element 1. For a family $(\gamma_i)_{i \in I}$ of elements of G and a word $w \in I^*$, define the group element γ_w by the following recursion: $\gamma_\varepsilon := 1$, and if $w = iw'$ for some $i \in I$ and $w' \in I^*$, then $\gamma_w := \gamma_i \cdot \gamma_{w'}$. (Compare this with the map φ_w defined in Subsection 4.2.)

Let $a, b \in G$, and let T and T' be binary trees with n leaves. Let $\gamma_0 := a$ and $\gamma_1 := b$. We say that T and T' are *(a,b)-equivalent modulo* $\mathbf{G}$*,* and we write $T \sim^{\mathbf{G}}_{a,b} T'$, if for all $i \in [n]$, $\gamma_{\alpha_T(x_i)} = \gamma_{\alpha_{T'}(x_i)}$. We denote by $T^{\mathbf{G}}_{a,b,n}$ the number of $\sim^{\mathbf{G}}_{a,b}$-equivalence classes of binary trees with n leaves.

Example 3.8. The various equivalence relations on binary trees that we have seen in the previous subsection are special instances of (a,b)-equivalence modulo some group $\mathbf{G}$.

(i) Let $\mathbf{G} = (\mathbb{Z}_k, +)$ for $k \in \mathbb{N}$, and consider $\sim^{\mathbf{G}}_{a,b}$. With $a = 1$, $b = 1$, we get k-depth-equivalence $\sim^{\mathrm{d}}_k$; with $a = 1$, $b = 0$, we get k-left-depth-equivalence $\sim^{\mathrm{L}}_k$; and with $a = 0$, $b = 1$, we get k-right-depth-equivalence $\sim^{\mathrm{R}}_k$. With arbitrary $a, b \in \mathbb{N}$, we get the equivalence relation $\sim^{\mathrm{lin}}_{a,b,k}$.

(ii) For $k, \ell \in \mathbb{N}$, taking $\mathbf{G} = (\mathbb{Z}_k, +) \times (\mathbb{Z}_\ell, +)$, $a = (1,0)$, $b = (0,1)$, we get (k,ℓ)-depth-equivalence $\sim^{\mathrm{LR}}_{k,\ell}$.

4. Linear quasigroups

4.1. **Affine quasigroups.** A quasigroup $\mathbf{A} = (A, \circ)$ is *affine* over a loop $(A, +)$ if there exist automorphisms $\varphi, \psi \in \mathrm{Aut}(A, +)$, and a constant $c \in A$ such that $x \circ y = (\varphi(x) + \psi(y)) + c$. If $c = 0$ in the above, then $\mathbf{A}$ is *linear* over $(A, +)$. The quintuple $(A, +, \varphi, \psi, c)$ is called an *arithmetic form* of $\mathbf{A}$. It is well known (see [12]) that

- an affine quasigroup with arithmetic form $(A, +, \varphi, \psi, c)$ is idempotent if and only if $c = 0$ and $\varphi + \psi = \mathrm{id}_A$ (pointwise addition of functions on the left side);
- an affine quasigroup with arithmetic form $(A, +, \varphi, \psi, c)$ is medial (i.e., it satisfies the identity $(xy)(uv) \approx (xu)(yv)$) if and only if $(A, +)$ is an abelian group and $\varphi\psi = \psi\varphi$ (proved independently by Bruck [5], Murdoch [16], Toyoda [18]).

4.2. **Bracketings over linear quasigroups.** Let $\varphi_i \colon A \to A$ $(i \in I)$ be a family of maps. We define, for each string $w \in I^*$, the map $\varphi_w \colon A \to A$ by the following recursion: $\varphi_\varepsilon := \mathrm{id}_A$, and if $w = iw'$ for some $i \in I$ and $w' \in I^*$, then $\varphi_w := \varphi_i \circ \varphi_{w'}$.

Proposition 4.1. *Let $\mathbf{A} = (A, \circ)$ be a linear quasigroup over a group $(A, +)$ with arithmetic form $(A, +, \varphi_0, \varphi_1, 0)$. Let $t, t' \in B_n$.*

(i) $t^{\mathbf{A}}(a_1, \dots, a_n) = \varphi_{\alpha_t(x_1)}(a_1) + \varphi_{\alpha_t(x_2)}(a_2) + \dots + \varphi_{\alpha_t(x_n)}(a_n)$.

(ii) $\mathbf{A}$ *satisfies $t \approx t'$ if and only if for all $i \in [n]$, $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$.*

Proof. (i) We proceed by induction on n. The claim holds for $n = 1$ because in this case we have $t = x_1$ and $t^{\mathbf{A}}(a_1) = \mathrm{id}_A(a_1) = \varphi_\varepsilon(a_1) = \varphi_{\alpha_t(x_1)}(a_1)$.

Assume that the claim holds for $n \leq k$ for some $k \geq 1$, and let $t \in B_{k+1}$. Then $t = (t_1 \cdot t_2)$ for some subterms t_1 and t_2. By the induction hypothesis, we have

$$t_1^{\mathbf{A}}(\mathbf{a}) = \varphi_{\alpha_{t_1}(x_1)}(a_1) + \varphi_{\alpha_{t_1}(x_2)}(a_2) + \dots + \varphi_{\alpha_{t_1}(x_\ell)}(a_\ell),$$
$$t_2^{\mathbf{A}}(\mathbf{a}) = \varphi_{\alpha_{t_2}(x_{\ell+1})}(a_{\ell+1}) + \varphi_{\alpha_{t_2}(x_{\ell+2})}(a_{\ell+2}) + \dots + \varphi_{\alpha_{t_2}(x_{k+1})}(a_{k+1}).$$

Using the fact that φ_0 and φ_1 are automorphisms of $(A, +)$, it follows that

$$\begin{aligned} t^{\mathbf{A}}(\mathbf{a}) &= \varphi_0(t_1^{\mathbf{A}}(\mathbf{a})) + \varphi_1(t_2^{\mathbf{A}}(\mathbf{a})) \\ &= \varphi_0\varphi_{\alpha_{t_1}(x_1)}(a_1) + \dots + \varphi_0\varphi_{\alpha_{t_1}(x_\ell)}(a_\ell) + \\ &\quad \varphi_1\varphi_{\alpha_{t_2}(x_{\ell+1})}(a_{\ell+1}) + \dots + \varphi_1\varphi_{\alpha_{t_2}(x_{k+1})}(a_{k+1}) \\ &= \varphi_{\alpha_t(x_1)}(a_1) + \dots + \varphi_{\alpha_t(x_{k+1})}(a_{k+1}). \end{aligned}$$

(ii) Assume first that $\mathbf{A}$ satisfies $t \approx t'$. By applying part (i), by assigning the neutral element 0 of $(A, +)$ to all variables but x_i, and by observing that any automorphism of $(A, +)$ maps 0 to itself, we get

$$t^{\mathbf{A}}(0, \dots, 0, a_i, 0, \dots, 0) = \varphi_{\alpha_t(x_i)}(a_i),$$
$$t'^{\mathbf{A}}(0, \dots, 0, a_i, 0, \dots, 0) = \varphi_{\alpha_{t'}(x_i)}(a_i),$$

which implies $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$ for all $i \in [n]$.

Assume now that $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$ for all $i \in [n]$. Then we have, by part (i), that

$$\begin{aligned} t^{\mathbf{A}}(a_1, \dots, a_n) &= \varphi_{\alpha_t(x_1)}(a_1) + \dots + \varphi_{\alpha_t(x_n)}(a_n) \\ &= \varphi_{\alpha_{t'}(x_1)}(a_1) + \dots + \varphi_{\alpha_{t'}(x_n)}(a_n) = t'^{\mathbf{A}}(a_1, \dots, a_n), \end{aligned}$$

that is, $t^{\mathbf{A}} = t'^{\mathbf{A}}$, so $\mathbf{A}$ satisfies the identity $t \approx t'$. □

4.3. **Special cases of linear quasigroups.**

Proposition 4.2. *Let $\mathbf{A} = (A, \circ)$ be a linear quasigroup over a group $(A, +)$ with arithmetic form $(A, +, \varphi_0, \varphi_1, 0)$. Let $t, t' \in B_n$.*

- (i) *If $\varphi_0 = \varphi_1$ and φ_0 has order k, then $\mathbf{A}$ satisfies $t \approx t'$ if and only if $t \sim_k^{\mathrm{d}} t'$. Consequently, $s_n(\mathbf{A}) = T_{k,n}^{\mathrm{d}}$.*
- (ii) *If $\varphi_1 = \mathrm{id}_A$ and φ_0 has order k, then $\mathbf{A}$ satisfies $t \approx t'$ if and only if $t \sim_k^{\mathrm{L}} t'$. Consequently, $s_n(\mathbf{A}) = T_{k,n}^{\mathrm{L}} = C_{k,n-1}$.*
- (iii) *If $\varphi_0 = \mathrm{id}_A$ and φ_1 has order k, then $\mathbf{A}$ satisfies $t \approx t'$ if and only if $t \sim_k^{\mathrm{R}} t'$. Consequently, $s_n(\mathbf{A}) = T_{k,n}^{\mathrm{R}} = C_{k,n-1}$.*

Proof. (i) Since $\varphi_0 = \varphi_1$, we have $\varphi_{\alpha_t(x_i)} = \varphi_0^{d_t(x_i)}$. Since φ_0 has order k, it follows that $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$ if and only if $d_t(x_i) \equiv d_{t'}(x_i) \pmod{k}$. By Proposition 4.1, $\mathbf{A}$ satisfies $t \approx t'$ if and only if $t \sim_k^{\mathrm{d}} t'$. The last claim is clear because $T_{k,n}^{\mathrm{d}} = |B_n/{\sim_k^{\mathrm{d}}}|$.

(ii) Since $\varphi_1 = \mathrm{id}_A$, we have $\varphi_{\alpha_t(x_i)} = \varphi_0^{\delta_t(x_i)}$. Since φ_0 has order k, it follows that $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$ if and only if $\delta_t(x_i) \equiv \delta_{t'}(x_i) \pmod{k}$. By Proposition 4.1, $\mathbf{A}$ satisfies $t \approx t'$ if and only if $t \sim_k^{\mathrm{L}} t'$. The last claim is clear because $C_{k,n-1} = T_{k,n}^{\mathrm{L}} = |B_n/{\sim_k^{\mathrm{L}}}|$.

(iii) The proof is similar to part (ii), and it also follows from Lemma 2.2 by using the fact that the affine quasigroup with arithmetic form $(A, +, \varphi, \psi, c)$ is the opposite groupoid of the affine quasigroup with arithmetic form $(A, +, \psi, \varphi, c)$. □

Proposition 4.3. *Let $\mathbf{A} = (A, \circ)$ be a linear quasigroup over a group $(A, +)$ with arithmetic form $(A, +, \varphi_0, \varphi_1, 0)$, and assume that φ_0 and φ_1 have orders k_0 and k_1, respectively, and $\varphi_0\varphi_1 = \varphi_1\varphi_0$. Let $t, t' \in B_n$.*

- (i) *If $t \sim_{k_0,k_1}^{\mathrm{LR}} t'$, then $\mathbf{A}$ satisfies $t \approx t'$. Consequently, $\sigma_n(\mathbf{A})$ is a coarsening of $\sim_{k_0,k_1}^{\mathrm{LR}}$ and hence $s_n(\mathbf{A}) \leq T_{k_0,k_1,n}^{\mathrm{LR}}$.*
- (ii) *If for all $p, q, r, s \in \mathbb{N}$, $\varphi_0^p\varphi_1^q = \varphi_0^r\varphi_1^s$ implies $p \equiv r \pmod{k_0}$ and $q \equiv s \pmod{k_1}$, then $\mathbf{A}$ satisfies $t \approx t'$ if and only if $t \sim_{k_0,k_1}^{\mathrm{LR}} t'$. Consequently, $s_n(\mathbf{A}) = T_{k_0,k_1,n}^{\mathrm{LR}}$.*

Proof. (i) Assume $t \sim_{k_0,k_1}^{\mathrm{LR}} t'$. Then $\delta_t(x_i) \equiv \delta_{t'}(x_i) \pmod{k_0}$ and $\rho_t(x_i) \equiv \rho_{t'}(x_i) \pmod{k_1}$ for all $i \in [n]$. Since $\varphi_0\varphi_1 = \varphi_1\varphi_0$ and φ_0 and φ_1 have orders k_0 and k_1, respectively, we have $\varphi_{\alpha_t(x_i)} = \varphi_0^{\delta_t(x_i)}\varphi_1^{\rho_t(x_i)} = \varphi_0^{\delta_{t'}(x_i)}\varphi_1^{\rho_{t'}(x_i)} = \varphi_{\alpha_{t'}(x_i)}$ for all $i \in [n]$. By Proposition 4.1, $\mathbf{A}$ satisfies $t \approx t'$.

(ii) By part (i), it suffices to show that $\mathbf{A} \models t \approx t'$ implies $t \sim_{k_0,k_1}^{\mathrm{LR}} t'$. So, assume that $\mathbf{A} \models t \approx t'$. Then for all $i \in [n]$, $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$, i.e., $\varphi_0^{\delta_t(x_i)}\varphi_1^{\rho_t(x_i)} = \varphi_0^{\delta_{t'}(x_i)}\varphi_1^{\rho_{t'}(x_i)}$ because $\varphi_0\varphi_1 = \varphi_1\varphi_0$. By our hypothesis, this implies that for all $i \in [n]$, $\delta_t(x_i) \equiv \delta_{t'}(x_i) \pmod{k_0}$ and $\rho_t(x_i) \equiv \rho_{t'}(x_i) \pmod{k_1}$, in other words, $t \sim_{k_0,k_1}^{\mathrm{LR}} t'$. □

Proposition 4.4. *Let $\mathbf{A} = (A, \circ)$ be a linear quasigroup over a group $(A, +)$ with arithmetic form $(A, +, \varphi_0, \varphi_1, 0)$, and assume that $\varphi_0 = \pi^a$ and $\varphi_1 = \pi^b$ for some permutation π of A and $a, b \in \mathbb{N}$. Assume that π has order ℓ. Let $t, t' \in B_n$. Then $\mathbf{A} \models t \approx t'$ if and only if $t \sim_{a,b,\ell}^{\mathrm{lin}} t'$. Consequently, $s_n(\mathbf{A}) = T_{a,b,\ell,n}^{\mathrm{lin}}$.*

Proof. We have $\varphi_{\alpha_t(x_i)} = (\pi^a)^{\delta_t(x_i)}(\pi^b)^{\rho_t(x_i)} = \pi^{a\delta_t(x_i)+b\rho_t(x_i)}$ and, similarly, $\varphi_{\alpha_{t'}(x_i)} = \pi^{a\delta_{t'}(x_i)+b\rho_{t'}(x_i)}$. Since π has order ℓ, it follows that $\varphi_{\alpha_t(x_i)} = \varphi_{\alpha_{t'}(x_i)}$ if and only if $a\delta_t(x_i) + b\rho_t(x_i) \equiv a\delta_{t'}(x_i) + b\rho_{t'}(x_i) \pmod{\ell}$. The claim then follows from Proposition 4.1. □

Remark 4.5. The condition of Proposition 4.4 is equivalent to the condition that $(x, y) = (\delta_t(x_i) - \delta_{t'}(x_i), \rho_t(x_i) - \rho_{t'}(x_i))$ is a solution of the congruence $ax + by \equiv 0 \pmod{\ell}$. It is well known that such a congruence has $\gamma\ell$ solutions, where $\gamma := \gcd(a, b, \ell)$. A method for determining the solutions is described by Lehmer [14, p. 155].

4.4. **Associative spectra of linear quasigroups.**

Definition 4.6. For a group $\mathbf{G} = (G, \cdot)$ and $g_0, g_1 \in G$, let $\Lambda_{\mathbf{G}}(g_0, g_1)$ denote the following set of pairs of integers:

$$\Lambda_{\mathbf{G}}(g_0, g_1) = \{(r, s) \in \mathbb{Z} \times \mathbb{Z} : g_1 g_0^r g_1^{-1} = g_0 g_1^s g_0^{-1}\}.$$

Lemma 4.7. *For any group $\mathbf{G}$ and $g_0, g_1 \in G$, the set $\Lambda_{\mathbf{G}}(g_0, g_1)$ is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.*

Proof. The defining condition $g_1 g_0^r g_1^{-1} = g_0 g_1^s g_0^{-1}$ of $\Lambda_{\mathbf{G}}(g_0, g_1)$ is equivalent to $\phi(g_0^r) = g_1^s$, where $\phi(x) = g_0^{-1} g_1 x g_1^{-1} g_0$ is the conjugation by $g_1^{-1} g_0$. If $(r, s), (r', s') \in \Lambda_{\mathbf{G}}(g_0, g_1)$, then, using the fact that ϕ is an automorphism of $\mathbf{G}$, we have

$$\phi(g_0^{r+r'}) = \phi(g_0^r g_0^{r'}) = \phi(g_0^r)\phi(g_0^{r'}) = g_1^s g_1^{s'} = g_1^{s+s'},$$

thus $(r + r', s + s') \in \Lambda_{\mathbf{G}}(g_0, g_1)$. Similarly, $(r, s) \in \Lambda_{\mathbf{G}}(g_0, g_1)$ implies $(-r, -s) \in \Lambda_{\mathbf{G}}(g_0, g_1)$:

$$\phi(g_0^{-r}) = \phi((g_0^r)^{-1}) = \phi(g_0^r)^{-1} = (g_1^s)^{-1} = g_1^{-s}. \qquad \square$$

Lemma 4.8. *Let $\mathbf{G}$ be a group, let $g_0, g_1 \in G$, and let T, T' be binary trees with leaves $1, 2, \dots, n$ (in the left-to-right order). Then $T \sim^{\mathbf{G}}_{g_0,g_1} T'$ holds if and only if $(\delta_T(i) - \delta_{T'}(i), \rho_{T'}(i) - \rho_T(i)) \in \Lambda_{\mathbf{G}}(g_0, g_1)$ for all $i \in [n]$.*

Proof. First let us make some preliminary observations that we will use in the proof. To simplify notation, we let $\delta_i = \delta_T(i)$, $\rho_i = \rho_T(i)$, $\delta'_i = \delta_{T'}(i)$ and $\rho'_i = \rho_{T'}(i)$ for $i \in [n]$. For a leaf $i \in [n-1]$, let z and z' be the deepest common ancestors of i and $i+1$ in T and T', respectively. Setting $u = \alpha_T(z)$ and $v = \alpha_{T'}(z')$, we have

$$\alpha_T(i) = u01^p,\ \alpha_T(i+1) = u10^q,\ \alpha_{T'}(i) = v01^{p'},\ \alpha_{T'}(i+1) = v10^{q'} \tag{1}$$

for some $p, q, p', q' \in \mathbb{N}$. This implies the following relationships among the depths:

$$\delta_i = \delta_T(z) + 1, \quad \delta'_i = \delta_{T'}(z') + 1, \quad \delta_{i+1} = \delta_T(z) + q, \quad \delta'_{i+1} = \delta_{T'}(z') + q',$$
$$\rho_i = \rho_T(z) + p, \quad \rho'_i = \rho_{T'}(z') + p', \quad \rho_{i+1} = \rho_T(z) + 1, \quad \rho'_{i+1} = \rho_{T'}(z') + 1.$$

We can thus conclude that

$$(\delta_i - \delta'_i) - (\delta_{i+1} - \delta'_{i+1}) = q' - q \text{ and } (\rho'_i - \rho_i) - (\rho'_{i+1} - \rho_{i+1}) = p' - p. \tag{2}$$

Let $\gamma_0 = g_0$ and $\gamma_1 = g_1$, and let us simply write $\gamma_i = \gamma_{\alpha_T(i)}$, $\gamma'_i = \gamma_{\alpha_{T'}(i)}$ (see Definition 3.7). We can compute these elements of $\mathbf{G}$ with the help of (1):

$$\gamma_i = \gamma_u g_0 g_1^p,\ \gamma_{i+1} = \gamma_u g_1 g_0^q,\ \gamma'_i = \gamma_v g_0 g_1^{p'},\ \gamma'_{i+1} = \gamma_v g_1 g_0^{q'}.$$

Therefore, we have

$$\gamma'_i \gamma_i^{-1} = \gamma_v g_0 g_1^{p'-p} g_0^{-1} \gamma_u^{-1} \text{ and } \gamma'_{i+1} \gamma_{i+1}^{-1} = \gamma_v g_1 g_0^{q'-q} g_1^{-1} \gamma_u^{-1}. \tag{3}$$

Now we are ready to begin the proof. Assume first that $T \sim^{\mathbf{G}}_{g_0,g_1} T'$, i.e., $\gamma_i = \gamma'_i$ for all $i \in [n]$. We are going to prove by induction on i that $(\delta_i - \delta'_i, \rho'_i - \rho_i) \in \Lambda_{\mathbf{G}}(g_0, g_1)$. The base case $i = 1$ is straightforward: $(\delta_1 - \delta'_1, \rho'_1 - \rho_1) = (\delta_1 - \delta'_1, 0) \in \Lambda_{\mathbf{G}}(g_0, g_1)$ is equivalent to $g_0^{\delta_i - \delta'_i} = 1$, and this is certainly true, as $g_0^{\delta_1} = \gamma_1 = \gamma'_1 = g_0^{\delta'_1}$. Now, for the induction step, let us assume that $(\delta_i - \delta'_i, \rho'_i - \rho_i) \in \Lambda_{\mathbf{G}}(g_0, g_1)$ holds for some $i \in [n-1]$. Since $\Lambda_{\mathbf{G}}(g_0, g_1)$ is a group, in order to prove that $(\delta_{i+1} - \delta'_{i+1}, \rho'_{i+1} - \rho_{i+1}) \in \Lambda_{\mathbf{G}}(g_0, g_1)$, it suffices to verify that $(q' - q, p' - p) \in \Lambda_{\mathbf{G}}(g_0, g_1)$, according to (2). We know that $\gamma_i = \gamma'_i$ and $\gamma_{i+1} = \gamma'_{i+1}$, and this implies that $\gamma'_i \gamma_i^{-1} = \gamma'_{i+1} \gamma_{i+1}^{-1} = 1$. From (3) it follows then that $g_0 g_1^{p'-p} g_0^{-1} = g_1 g_0^{q'-q} g_1^{-1}$, and this shows that we indeed have $(q' - q, p' - p) \in \Lambda_{\mathbf{G}}(g_0, g_1)$.

For the converse, assume that $(\delta_i - \delta'_i, \rho_i - \rho'_i) \in \Lambda_{\mathbf{G}}(g_0, g_1)$ for all $i \in [n]$. We are going to prove by induction on i that $\gamma_i = \gamma'_i$. The base case $\gamma_1 = \gamma'_1$ is equivalent to $g_0^{\delta_1} = g_0^{\delta'_1}$, and this follows immediately, as $(\delta_1 - \delta'_1, \rho'_1 - \rho_1) = (\delta_1 - \delta'_1, 0) \in \Lambda_{\mathbf{G}}(g_0, g_1)$. For the induction step, let us assume that $\gamma_i = \gamma'_i$ for some $i \in [n-1]$. We assumed that $(\delta_i - \delta'_i, \rho'_i - \rho_i)$ and $(\delta_{i+1} - \delta'_{i+1}, \rho'_{i+1} - \rho_{i+1})$ belong to the group $\Lambda_{\mathbf{G}}(g_0, g_1)$, hence $(q' - q, p' - p) \in \Lambda_{\mathbf{G}}(g_0, g_1)$ by (2). This fact together with (3) and the induction hypothesis $\gamma_i = \gamma'_i$ allow us to deduce that $\gamma_{i+1} = \gamma'_{i+1}$:

$$1 = \gamma'_i \gamma_i^{-1} = \gamma_v g_0 g_1^{p'-p} g_0^{-1} \gamma_u^{-1} = \gamma_v g_1 g_0^{q'-q} g_1^{-1} \gamma_u^{-1} = \gamma'_{i+1} \gamma_{i+1}^{-1}. \tag{4}$$ □

Remark 4.9. Propositions 4.3 and 4.4 follow as special cases of Lemma 4.8.

Theorem 4.10. *The fine associative spectrum of a linear quasigroup over a group is of the form $\sim^{\mathrm{L}}_k \cap \sim^{\mathrm{lin}}_{a,b,m}$ for suitable integers k, a, b, m.*

Proof. Let $\mathbf{A} = (A, \circ)$ be a linear quasigroup over a group $(A, +)$ with arithmetic form $(A, +, \varphi_0, \varphi_1, 0)$. Proposition 4.1 implies that the fine spectrum of $\mathbf{A}$ is $\sim^{\mathbf{G}}_{\varphi_0,\varphi_1}$, where $\mathbf{G}$ is the automorphism group of $(A, +)$. Let us consider the group $\Lambda := \Lambda_{\mathbf{G}}(\varphi_0, \varphi_1) \leq \mathbb{Z} \times \mathbb{Z}$. Every subgroup of $\mathbb{Z}^2$ can be generated by (at most) two elements, hence there exists a matrix $M \in \mathbb{Z}^{2\times 2}$ such that Λ consists of all linear combinations of the column vectors of M with integer coefficients. As every integer matrix has an Hermite normal form, there exists a unimodular matrix $U \in \mathbb{Z}^{2\times 2}$ so that $H := MU$ is a lower triangular matrix. Let $H = \begin{pmatrix} u & 0 \\ v & w \end{pmatrix}$; then $(r, s) \in \Lambda$ if and only if there exist $x, y \in \mathbb{Z}$ such that $r = ux$ and $s = vx + wy$. Assuming $u \neq 0$, the latter equality is equivalent to $us = uvx + uwy$, and taking $r = ux$ into account, this yields $uwy = us - vr$. Thus we have $(r, s) \in \Lambda$ if and only if $u \mid r$ and $uw \mid us - vr$.

Now Lemma 4.8 shows that $T \sim^{\mathbf{G}}_{\varphi_0,\varphi_1} T'$ holds for binary trees T and T' if and only if $u \mid \delta_T(i) - \delta_{T'}(i)$ and $uw \mid u(\rho_{T'}(i) - \rho_T(i)) - v(\delta_T(i) - \delta_{T'}(i))$ for all $i \in [n]$. These divisibilities are equivalent to the congruences $\delta_T(i) \equiv \delta_{T'}(i) \pmod{u}$ and $v\delta_T(i) + u\rho_T(i) \equiv v\delta_{T'}(i) + u\rho_{T'}(i) \pmod{uw}$. Thus the fine associative spectrum of $\mathbf{A}$ is $\sim^{\mathbf{G}}_{\varphi_0,\varphi_1} = \sim^{\mathrm{L}}_u \cap \sim^{\mathrm{lin}}_{v,u,uw}$, and this proves the theorem in the case $u \neq 0$.

If $u = 0$, then we have $r = 0$ for all $(r, s) \in \Lambda$, hence Lemma 4.8 implies that $T \sim^{\mathbf{G}}_{\varphi_0,\varphi_1} T'$ can hold only if T and T' have the same left depth sequence. This means that $\mathbf{A}$ is antiassociative, i.e., its fine spectrum is trivial, and it can be written, e.g., as $\sim^{\mathrm{L}}_0 \cap \sim^{\mathrm{lin}}_{0,0,0}$ (note that the modulo 0 congruence is just the equality relation). □

Remark 4.11. The result in the previous theorem seems a bit asymmetric. This is due to the fact that we worked with the column space of the matrix M. The row space would have given an equivalence relation of the form $\sim_k^{\mathrm{R}} \cap \sim_{a,b,m}^{\mathrm{lin}}$. On the other hand, we can see from the proof that $\gcd(v,w) \mid s$, thus we can write $\sim_{\varphi_0,\varphi_1}^{\mathbf{G}} = \sim_u^{\mathrm{L}} \cap \sim_{v,u,uw}^{\mathrm{lin}} \cap \sim_{\gcd(v,w)}^{\mathrm{R}}$. However, it is perhaps not worth introducing a fifth parameter just to obtain a symmetric form.

5. Numerical data

Table 3 shows the number of $\sim_{a,b,m}^{\mathrm{lin}}$-equivalence classes of T_n, for $a,b,m \leq 14$. Note that, by Lemma 3.6, the triples (a,b,m) and (a',b',m') yield the same sequence if

- $a' = \ell a$, $b' = \ell b$, $c' = \ell c$ for some $\ell \in \mathbb{N}_+$;
- $a' = b$, $b' = a$, $m' = m$; or
- $a' = \ell a$, $b' = \ell b$, $m' = m$ for some unit ℓ modulo m.

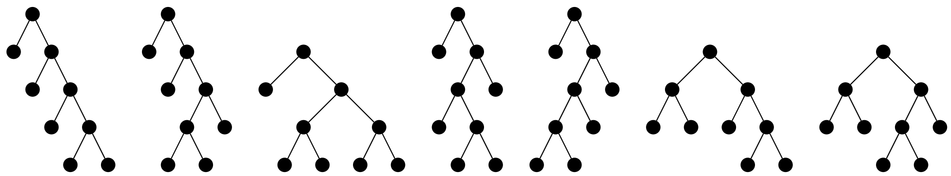
Therefore we list in the table only those triples (a,b,m) for which $\gcd(a,b,m) = 1$, $a \leq b$, and a and b are the smallest possible with respect to multiplication by units. The table entries are arranged in the lexicographical order of the first 14 terms of the sequence $(T_{a,b,m,n}^{\mathrm{lin}})_{n \in \mathbb{N}_+}$.

a	b	m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	2	2	1	1	2	4	8	16	32	64	128	256	512	1024	2048	4096	$T^{\mathrm{L}}_{2,n}$
1	1	2	1	1	2	5	10	21	42	85	170	341	682	1365	2730	5461	$T^{\mathrm{d}}_{2,n}$
1	3	3	1	1	2	5	13	35	96	267	750	2123	6046	17303	49721	143365	$T^{\mathrm{L}}_{3,n}$
1	4	4	1	1	2	5	14	41	124	384	1210	3865	12482	40677	133572	441468	$T^{\mathrm{L}}_{4,n}$
2	3	6	1	1	2	5	14	41	124	384	1210	3865	12482	40677	133572	441468	?
1	2	3	1	1	2	5	14	42	128	390	1185	3586	10862	32929	99883	303000	?
1	1	3	1	1	2	5	14	42	129	398	1223	3752	11510	35305	108217	331434	$T^{\mathrm{d}}_{3,n}$
1	2	4	1	1	2	5	14	42	131	420	1374	4561	15306	51793	176404	603990	?
1	5	5	1	1	2	5	14	42	131	420	1375	4576	15431	52603	180957	627340	$T^{\mathrm{L}}_{5,n}$
1	6	6	1	1	2	5	14	42	132	428	1420	4796	16432	56966	199444	704146	$T^{\mathrm{L}}_{6,n}$
2	5	10	1	1	2	5	14	42	132	428	1420	4796	16432	56966	199444	704146	?
3	4	12	1	1	2	5	14	42	132	428	1420	4796	16432	56966	199444	704146	?
1	3	4	1	1	2	5	14	42	132	429	1425	4807	16402	56472	195860	683420	?
1	1	4	1	1	2	5	14	42	132	429	1429	4849	16689	58074	203839	720429	$T^{\mathrm{d}}_{4,n}$
1	3	6	1	1	2	5	14	42	132	429	1429	4851	16718	58331	205631	731257	?
1	7	7	1	1	2	5	14	42	132	429	1429	4851	16718	58331	205632	731272	$T^{\mathrm{L}}_{7,n}$
1	2	6	1	1	2	5	14	42	132	429	1430	4861	16784	58695	207450	739810	?
1	4	6	1	1	2	5	14	42	132	429	1430	4861	16784	58695	207452	739839	?
1	8	8	1	1	2	5	14	42	132	429	1430	4861	16784	58695	207452	739840	$T^{\mathrm{L}}_{8,n}$
2	7	14	1	1	2	5	14	42	132	429	1430	4861	16784	58695	207452	739840	?
1	4	5	1	1	2	5	14	42	132	429	1430	4862	16790	58708	207382	738815	?
1	1	5	1	1	2	5	14	42	132	429	1430	4862	16795	58773	207906	742203	$T^{\mathrm{d}}_{5,n}$
1	2	5	1	1	2	5	14	42	132	429	1430	4862	16795	58773	207907	742219	?
1	4	8	1	1	2	5	14	42	132	429	1430	4862	16795	58773	207907	742220	?
1	9	9	1	1	2	5	14	42	132	429	1430	4862	16795	58773	207907	742220	$T^{\mathrm{L}}_{9,n}$
2	3	12	1	1	2	5	14	42	132	429	1430	4862	16795	58773	207907	742220	?
1	10	10	1	1	2	5	14	42	132	429	1430	4862	16796	58785	207998	742780	$T^{\mathrm{L}}_{10,n}$
1	5	6	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208005	742795	?
1	1	6	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	$T^{\mathrm{d}}_{6,n}$
1	2	8	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	?
1	3	9	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	?
1	5	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	?
1	6	8	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	?
1	6	9	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	?
1	11	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208011	742885	$T^{\mathrm{L}}_{11,n}$
1	12	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742899	$T^{\mathrm{L}}_{12,n}$
1	1	7	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{7,n}$
1	1	8	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{8,n}$
1	1	9	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{9,n}$
1	1	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{10,n}$
1	1	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{11,n}$
1	1	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{12,n}$
1	1	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{13,n}$
1	1	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{d}}_{14,n}$
1	2	7	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	2	9	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	2	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	2	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	2	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	2	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	2	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	3	7	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	3	8	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	3	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?

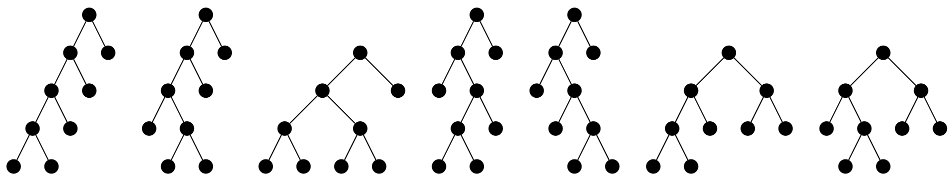
a	b	m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
1	3	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	3	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	3	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	3	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	4	9	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	4	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	4	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	4	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	4	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	5	8	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	5	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	5	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	5	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	6	7	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	6	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	6	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	6	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	6	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	7	8	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	7	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	7	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	7	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	8	9	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	8	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	8	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	8	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	9	10	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	9	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	9	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	10	11	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	10	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	10	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	11	12	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	12	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	12	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	13	13	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{L}}_{13,n}$
1	13	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	?
1	14	14	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	$T^{\mathrm{L}}_{14,n}$

TABLE 3. $T^{\mathrm{lin}}_{a,b,m,n}$

Below we list all binary trees with five leaves, and we indicate their left and right depth sequences. Denoting a tree by its left depth sequence, it is easy to verify that the only nontrivial $\sim^{\mathrm{LR}}_{3,1}$-equivalence class is $\{13210, 43210\}$, the only nontrivial $\sim^{\mathrm{LR}}_{1,3}$-equivalence class is $\{11110, 22210\}$, and the only nontrivial $\sim^{\mathrm{LR}}_{2,2}$-equivalence class is $\{11210, 33210\}$. Moreover, the only nontrivial $\sim^{\mathrm{d}}_{2}$-equivalence classes are $\{11210, 33210\}$, $\{12110, 32210\}$, $\{12210, 32110\}$, and $\{21110, 23210\}$.



11110 11210 12110 12210 13210 21110 21210
01234 01233 01223 01232 01222 01123 01122



43210 33210 32210 23210 22210 32110 22110
01111 01211 01121 01221 01231 01112 01212

References

[1] J. Berman, S. Burris, A computer study of 3-element groupoids, in: A. Ursini, P. Aglianò (eds.), *Logic and Algebra (Pontignano, 1994),* Lecture Notes in Pure and Appl. Math., 180, Marcel Dekker, Inc., New York, 1996. pp. 379–429.

[2] M. Braitt, D. Hobby, D. Silberger, Completely dissociative groupoids, *Math. Bohem.* **137** (2012) 79–97.

[3] M. Braitt, D. Hobby, D. Silberger, Antiassociative groupoids, *Math. Bohem.* **142** (2017) 27–46.

[4] M. S. Braitt, D. Silberger, Subassociative groupoids, *Quasigroups Related Systems* **14** (2006) 11–26.

[5] R. H. Bruck, Some results in the theory of quasigroups. *Trans. Amer. Math. Soc.* **55** (1944) 19–52.

[6] B. Csákány, T. Waldhauser, Associative spectra of binary operations, *Mult.-Valued Log.* **5** (2000) 175–200.

[7] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.10.2, 2019. (`https://www.gap-system.org`)

[8] N. Hein, J. Huang, Modular Catalan numbers, *European J. Combin.* **61** (2017) 197–218.

[9] N. Hein, J. Huang, Variations of the Catalan numbers from some nonassociative binary operations, *Sém. Lothar. Combin.* **80B** (2018), Art. 31, 12 pp.

[10] N. Hein, J. Huang, Variations of the Catalan numbers from some nonassociative binary operations, *Discrete Math.* **345** (2022) 112711 18 pp.

[11] J. Huang, M. Mickey, J. Xu, The nonassociativity of the double minus operation, *J. Integer Seq.* **20** (2017) Art. 17.10.3, 11 pp.

[12] P. Jedlička, D. Stanovský, P. Vojtěchovský, Distributive and trimedial quasigroups of order 243, *Discrete Math.* **340** (2017) 404–415.

[13] A. Kotzig, C. Reischer, Associativity index of finite quasigroups, *Glas. Mat. Ser. III* **18 (38)** (1983) 243–253.

[14] D. N. Lehmer, Certain theorems in the theory of quadratic residues, *Amer. Math. Monthly* **20** (1913) 151–157.

[15] S. Liebscher, T. Waldhauser, On associative spectra of operations, *Acta Sci. Math. (Szeged)* **75** (2009) 433–456.

[16] D. C. Murdoch, Structure of abelian quasi-groups, *Trans. Amer. Math. Soc.* **49** (1941) 392–409.

[17] N. J. A. Sloane (ed.), *The On-Line Encyclopedia of Integer Sequences,* published electronically at `https://oeis.org`, 2019.

[18] K. Toyoda, On axioms of linear functions, *Proc. Imp. Acad. Tokyo* **17** (1941) 221–227.

[19] D. L. Walker, *Power graphs of quasigroups,* master's thesis, University of South Florida, 2019. https://scholarcommons.usf.edu/etd/7984