# Reflection-closed varieties of multisorted algebras and minor identities 

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#### Abstract

The notion of reflection is considered in the setting of multisorted algebras. The Galois connection induced by the satisfaction relation between multisorted algebras and minor identities provides a characterization of reflection-closed varieties: a variety of multisorted algebras is reflection-closed if and only if it is definable by minor identities. Minorequational theories of multisorted algebras are described by explicit closure conditions. It is also observed that nontrivial varieties of multisorted algebras of a non-composable type are reflection-closed.


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## 1. Introduction

Motivated by considerations of the complexity of constraint satisfaction problems, Barto et al. [2] introduced an algebraic construction called reflection. Given an algebra $\mathbf{A}=\left(A, F^{\mathbf{A}}\right)$ of type $\tau$, a set $B$, and maps $h_{1}: B \rightarrow A$ and $h_{2}: A \rightarrow B$, we can define an algebra $\mathbf{B}=\left(B, F^{\mathbf{B}}\right)$ of type $\tau$ in which the operations are given by the rule

$$
\begin{equation*}
f^{\mathbf{B}}\left(x_{1}, \ldots, x_{n}\right):=h_{2}\left(f^{\mathbf{A}}\left(h_{1}\left(x_{1}\right), \ldots, h_{1}\left(x_{n}\right)\right)\right) . \tag{1.1}
\end{equation*}
$$

The algebra $\mathbf{B}$ is called a reflection of $\mathbf{A}$. Reflections are a common generalization of subalgebras and homomorphic images. It was shown in [2, Corollary 5.4] that the classes of algebras closed under reflections and products are precisely the classes defined by height-1 identities.

[^0]Multisorted (or heterogeneous) algebras generalize the notion of an algebra so as to include functions that take arguments and values from possibly different sets. Much of the general theory of usual one-sorted (also called homogeneous) algebras applies to multisorted algebras, and the basics of the theory of multisorted algebras were established as early as in the 1960's and 1970's. In particular, subalgebras, morphisms, congruences, direct products, and free algebras were defined in the setting of multisorted algebras in the papers by Higgins [7] and Birkhoff and Lipson [3]. Furthermore, Higgins [7] defined varieties of multisorted algebras and proved Birkhoff's HSP theorem for multisorted algebras. Further considerations on varieties are included, e.g., in the paper by Taylor [9].

The defining equality (1.1) of reflections allows an immediate generalization from algebras to multisorted algebras in which the carrier comprises two sets $A$ and $B$ and the operations are functions $f: A^{n} \rightarrow B$ of several arguments from $A$ to $B$ ("2-algebras"; see Example 2.13 (5)). With a little modification of the definition, the notion of reflection can be further generalized to arbitrary multisorted algebras (see Section 4).

In this paper, we consider reflections of multisorted algebras and ask for a characterization of reflection-closed varieties. As it turns out, the right notion for such a characterization are the so-called minor identities (also known as height-1 identities or primitive identities), i.e., identities of a special form, where all terms have exactly one occurrence of a function symbol. We thus set out to investigate the Galois connection Mod-mId induced by the relation of satisfaction between multisorted algebras and minor identities. Analogously to the first Birkhoff theorem, the Galois closures of this Galois connection are precisely the reflection-closed varieties of multisorted algebras, i.e., $\operatorname{Mod} \operatorname{mId} \mathcal{K}=\operatorname{RP} \mathcal{K}$. (For usual one-sorted algebras, this was proved by Barto et al. [2].) We also characterize by explicit closure conditions the minor-equational theories of multisorted algebras, i.e., the closed sets of minor identities of the Galois connection Mod-mId. (For usual one-sorted algebras, this was done by Čupona and Markovski [5].)

We also discuss how reflection-closed varieties and usual varieties of multisorted algebras are related to each other. These notions can be quite different in general, but for varieties of multisorted algebras of a so-called non-composable type, the only varieties that are not reflection-closed are in a certain sense trivial.

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## 2. Multisorted algebras

We will start with recalling the definitions of basic concepts in the theory of multisorted sets and multisorted algebras. We will mainly follow the notation and terminology used in the book by Wechler [10].

Definition 2.1. We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_{+}$the set of positive integers. For $n \in \mathbb{N}$, let $[n]:=\{1, \ldots, n\}$. Note that $[0]=\emptyset$.

Definition 2.2. We write tuples $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ interchangeably as words $a_{1} a_{2} \ldots a_{n}$. The set of all words over a set $S$ is denoted by $W(S)$. The empty word is denoted by $\varepsilon$. The length of a word $w \in W(S)$ is the number of letters in $w$ and it is denoted by $|w|$. Thus, $\left|w_{1} w_{2} \ldots w_{n}\right|=n$. For $s \in S$, the number of occurrences of $s$ in $w$ is denoted by $|w|_{s}$.

Definition 2.3. Let $S$ be a set of elements called sorts. An $S$-indexed family of sets is called an $S$-sorted set. Given $S$-sorted sets $A=\left(A_{s}\right)_{s \in S}$ and $B=$ $\left(B_{s}\right)_{s \in S}$, we say that $A$ is an ( $S$-sorted) subset of $B$ and we write $A \subseteq B$ if $A_{s} \subseteq B_{s}$ for all $s \in S$. The union and intersection of $S$-sorted sets $A$ and $B$ are defined componentwise: $A \cup B:=\left(A_{s} \cup B_{s}\right)_{s \in S}$ and $A \cap B:=\left(A_{s} \cap B_{s}\right)_{s \in S}$. For any subset $S^{\prime} \subseteq S$, we denote by $\left.A\right|_{S^{\prime}}$ the $S$-sorted subset of $A$ given by

$$
\left(\left.A\right|_{S^{\prime}}\right)_{s}:= \begin{cases}A_{s}, & \text { if } s \in S^{\prime} \\ \emptyset, & \text { if } s \notin S^{\prime}\end{cases}
$$

When we make statements such as "let $A$ be an $S$-sorted set", it is understood that the member of the family $A$ indexed by $s \in S$ is denoted by $A_{s}$.

Definition 2.4. Let $A$ be an $S$-sorted set. If $A_{s} \neq \emptyset$, then we say that sort $s$ is essential in $A$; otherwise sort $s$ is inessential in $A$. Let $S_{A}:=\left\{s \in S \mid A_{s} \neq \emptyset\right\}$ be the set of essential sorts in $A$. It follows immediately from the definitions that $\left.A\right|_{S_{A}}=A$ and $S_{\left.A\right|_{S^{\prime}}} \subseteq S^{\prime}$ for any $S^{\prime} \subseteq S$.

Definition 2.5. Let $A$ and $B$ be $S$-sorted sets. An $S$-sorted mapping $f$ from $A$ to $B$, denoted by $f: A \rightarrow B$, is a family $\left(f_{s}\right)_{s \in S}$ of maps $f_{s}: A_{s} \rightarrow B_{s}$. If $x \in A_{s}$ and there is no risk of confusion about the sort $s$, we may write $f(x)$ instead of $f_{s}(x)$.

Definition 2.6. For an $S$-sorted set $A=\left(A_{s}\right)_{s \in S}$ and $w=w_{1} w_{2} \ldots w_{n} \in W(S)$, let $A_{w}:=A_{w_{1}} \times A_{w_{2}} \times \ldots \times A_{w_{n}}$. Note that $A_{\varepsilon}=\{\emptyset\}$.

Definition 2.7. A pair $(w, s) \in W(S) \times S$ is called a declaration over $S$. Let $A$ be an $S$-sorted set. A declaration $(w, s)$ with $w=w_{1} \ldots w_{n}$ is reasonable in $A$ if $A_{s}=\emptyset$ implies $A_{w_{i}}=\emptyset$ for some $i$, or, equivalently, if $A_{w} \neq \emptyset$ implies $A_{s} \neq \emptyset$. Note that the declaration $(\varepsilon, s)$ is reasonable in $A$ if and only if $A_{s} \neq \emptyset$.

An $S$-sorted operation on $A$ is any function $f: A_{w} \rightarrow A_{s}$ for some declaration $(w, s)$ that is reasonable in $A$. Note that it is possible that $A_{w}=\emptyset$, in which case $f$ is just the empty function $\emptyset \rightarrow A_{s}$. The word $w$ is called the arity of $f$ and the element $s$ is called the (output) sort of $f$. The elements of $S$ occurring in the word $w$ are called the input sorts of $f$. We denote the declaration, arity, sort, and the set of input sorts of $f$ by $\operatorname{dec}(f), \operatorname{ar}(f)$, $\operatorname{sort}(f)$, and $\operatorname{inp}(f)$, respectively. If $|w|=n$, then we also say that $f$ has numerical arity $n$, or that $f$ is $n$-ary.

Definition 2.8. A (multisorted similarity) type is a triple $\tau=(S, \Sigma, \mathrm{dec})$, where $S$ is a set of sorts, $\Sigma$ is a set of function symbols, and dec is a mapping dec:
$\Sigma \rightarrow W(S) \times S$. If $f \in \Sigma$ and $\operatorname{dec}(f)=(w, s)$, we say that $f$ has arity $w$ and sort $s$. Using the same notation as for functions, we denote the arity, sort and the set of input sorts of a function symbol $f$ by $\operatorname{ar}(f)$, sort $(f)$, and $\operatorname{inp}(f)$, respectively. For $w \in W(S), s \in S$, we write $\Sigma_{(w, s)}:=\{f \in \Sigma \mid \operatorname{dec}(f)=$ $(w, s)\}, \Sigma_{s}:=\{f \in \Sigma \mid \operatorname{sort}(f)=s\}$.

A (multisorted) algebra of type $\tau$ is a system $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$, where $A=$ $\left(A_{s}\right)_{s \in S}$ is an $S$-sorted set, called the carrier (or universe) of $\mathbf{A}$, and $\Sigma^{\mathbf{A}}=$ $\left(f^{\mathbf{A}}\right)_{f \in \Sigma}$ is a family of $S$-sorted operations on $A$, each $f^{\mathbf{A}}$ of declaration $\operatorname{dec}(f)$. It is implicit in the definition that $\operatorname{dec}(f)$ is reasonable in $A$ for every $f \in \Sigma$. Denote by $\operatorname{Alg}(\tau)$ the class of all multisorted algebras of type $\tau$.

Remark 2.9. One can find in the literature different definitions of multisorted algebras that differ in whether or not the sets $A_{s}$ in the carrier $\left(A_{s}\right)_{s \in S}$ of an algebra may be empty. Following the approach taken by Higgins [7], we allow carriers with empty components.

Definition 2.10. Let $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ and $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ be multisorted algebras of type $\tau=(S, \Sigma$, dec $)$. We say that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ if $B \subseteq A$ and for every $f \in \Sigma_{(w, s)}$, the operation $f^{\mathbf{B}}$ equals the restriction of $f^{\mathbf{A}}$ to $B_{w}$. Running the risk of being a bit sloppy, we may designate subalgebras of $\mathbf{A}$ simply by their carrier sets and make statements such as " $B$ is a subalgebra of $\mathbf{A}$ " when we mean that $B$ is the carrier of a subalgebra of $\mathbf{A}$.

It is possible that some components $B_{s}$ of the carrier of $\mathbf{B}$ are empty. However, if $\Sigma^{\mathbf{A}}$ contains a nullary operation which selects an element $a \in A_{s}$, then we require that $a \in B_{s}$.

If $C$ is an $S$-sorted subset of $A$, then the subalgebra of $\mathbf{A}$ generated by $C$, denoted by $\langle C\rangle_{\mathbf{A}}$, is the smallest subalgebra $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ of $\mathbf{A}$ such that $C \subseteq B$.

Definition 2.11. Let $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ and $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ be algebras of type $\tau=$ $(S, \Sigma, \mathrm{dec})$. A homomorphism of $\mathbf{A}$ to $\mathbf{B}$ is an $S$-sorted mapping $\varphi: A \rightarrow B$ such that for every $f \in \Sigma_{(w, s)}$ with $w=w_{1} \ldots w_{n}$, it holds that

$$
f^{\mathbf{B}}\left(\varphi_{w_{1}}\left(a_{1}\right), \ldots, \varphi_{w_{n}}\left(a_{n}\right)\right)=\varphi_{s}\left(f^{\mathbf{A}}\left(a_{1}, \ldots, a_{n}\right)\right)
$$

for all $\left(a_{1}, \ldots, a_{n}\right) \in A_{w}$. If every $\varphi_{s}$ is a surjective map onto $B_{s}$, then $\mathbf{B}$ is referred to as a homomorphic image of $\mathbf{A}$.

Definition 2.12. Let $\Gamma$ be an index set of a family of multisorted algebras $\mathbf{A}_{\gamma}=\left(\left(A_{\gamma, s}\right)_{s \in S},\left(f^{\mathbf{A}_{\gamma}}\right)_{f \in \Sigma}\right)$ of type $\tau=(S, \Sigma, \operatorname{dec}), \gamma \in \Gamma$. The direct product $\prod_{\gamma \in \Gamma} \mathbf{A}_{\gamma}$ is the algebra $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ of type $\tau$, where $B_{s}=\prod_{\gamma \in \Gamma} A_{\gamma, s}$ and

$$
f^{\mathbf{B}}\left(\left(a_{\gamma, 1}\right)_{\gamma \in \Gamma}, \ldots,\left(a_{\gamma, n}\right)_{\gamma \in \Gamma}\right)=\left(f^{\mathbf{A}_{\gamma}}\left(a_{\gamma, 1}, \ldots, a_{\gamma, n}\right)\right)_{\gamma \in \Gamma} .
$$

If $\mathbf{A}_{\gamma}=\mathbf{A}$ for all $\gamma \in \Gamma$, then we speak of the $\Gamma$-th direct power of $\mathbf{A}$, and we write $\mathbf{A}^{\Gamma}$ for $\prod_{\gamma \in \Gamma} \mathbf{A}$.

Note that the empty product $\prod_{\gamma \in \emptyset} \mathbf{A}_{\gamma}$ is the algebra $\left(B_{s}, \Sigma^{\mathbf{B}}\right)$ where $B_{s}=\{\emptyset\}$ for all $s \in S$. We will denote the empty product by $\Pi \emptyset$.

Example 2.13. Examples of multisorted algebras include the following.
(1) If the set $S$ of sorts is a singleton, then $S$-sorted sets, mappings, operations, algebras, etc., correspond to the usual ones. Every algebra in the usual sense can be viewed as a multisorted algebra of type $\tau=(S, \Sigma$, dec $)$ where $|S|=1$. Such algebras are called one-sorted (or homogeneous).
(2) Given a multisorted similarity type $\tau=(S, \Sigma$, dec), we can construct the canonical trivial algebra $\mathbf{S}=\left(\tilde{S}, \Sigma^{\mathbf{S}}\right)$ of type $\tau$, in which the carrier $\tilde{S}=$ $\left(\tilde{S}_{s}\right)_{s \in S}$ consists of one-element sets only, $\tilde{S}_{s}:=\{s\}$ for every $s \in S$, and for any $f \in \Sigma_{(w, s)}$, the operation $f^{\text {S }}$ is trivially defined as the constant $\operatorname{map} w \mapsto s$.
(3) Let $\tau=(S, \Sigma$, dec $)$ be a multisorted similarity type, and let $Y=\left(Y_{s}\right)_{s \in S}$ be an $S$-sorted set in which the components are pairwise disjoint and also disjoint from the function symbols $\Sigma$. The elements of $Y$ are referred to as variables. We often encounter the $S$-sorted standard set of variables, namely, $X=\left(X_{s}\right)_{s \in S}$ where $X_{s}=\left\{x_{i}^{s} \mid i \in \mathbb{N}\right\}$.

The $S$-sorted set $T_{\tau}(Y)=\left(T_{\tau}^{s}(Y)\right)_{s \in S}$ of terms of type $\tau$ over $Y$ is defined as follows. Each set $T_{\tau}^{s}(Y)$ of terms of type $\tau$ over $Y$ of sort $s$ is the least set of words over $\Sigma \cup Y$ such that $Y_{s} \subseteq T_{\tau}^{s}(Y)$ and for all function symbols $f \in \Sigma_{(w, s)}$ and for all $\left(t_{1}, \ldots, t_{n}\right) \in\left(T_{\tau}(Y)\right)_{w}$, the word $f t_{1} \ldots t_{n}$ belongs to $T_{\tau}^{s}(Y)$. For better readability, we may add some punctuation marks and write $f\left(t_{1}, \ldots, t_{n}\right)$ instead of $f t_{1} \ldots t_{n}$.

The terms of type $\tau$ over $Y$ carry a multisorted algebra $\mathbf{T}_{\tau}(Y)=$ $\left(T_{\tau}(Y), \Sigma^{\mathbf{T}_{\tau}(Y)}\right)$ of type $\tau$ in which the operations are defined as follows. For each $f \in \Sigma_{(w, s)}$, let $f^{\mathbf{T}_{\tau}(Y)}\left(t_{1}, \ldots, t_{n}\right):=f t_{1} \ldots t_{n}$ for all $\left(t_{1}, \ldots, t_{n}\right) \in\left(T_{\tau}(Y)\right)_{w}$. The algebra $\mathbf{T}_{\tau}(Y)$ is called the term algebra of type $\tau$ over $Y$.
(4) An operation on a set $A$ (an ordinary set, not an $S$-sorted set) is a map $f: A^{n} \rightarrow A$ for some $n \in \mathbb{N}_{+}$, called the arity of $f$. The $i$-th $n$-ary projection on $A$ is the operation $\mathrm{pr}_{i}^{n}: A^{n} \rightarrow A,\left(a_{1}, \ldots, a_{n}\right) \mapsto a_{i}$. The composition of $f: A^{n} \rightarrow A$ with $g_{1}, \ldots, g_{n}: A^{m} \rightarrow A$ is the operation $f\left(g_{1}, \ldots, g_{n}\right): A^{m} \rightarrow A$ given by the rule

$$
f\left(g_{1}, \ldots, g_{n}\right)(\mathbf{a}):=f\left(g_{1}(\mathbf{a}), \ldots, g_{n}(\mathbf{a})\right),
$$

for all $\mathbf{a} \in A^{m}$. The set of all $n$-ary operations on $A$ is denoted by $\mathcal{O}_{A}^{(n)}$, and $\mathcal{O}_{A}:=\bigcup \mathcal{O}_{A}^{(n)}$. A clone on a set $A$ is a set $\mathcal{C} \subseteq \mathcal{O}_{A}$ of operations on $A$ that contains all projections and is closed under composition.

Clones on $A$ are sometimes viewed as multisorted algebras. Namely, let

$$
\begin{aligned}
S & :=\mathbb{N}_{+}, \\
\Sigma & :=\left\{C_{n, m} \mid n, m \in \mathbb{N}_{+}\right\} \cup\left\{e_{n, i} \mid n, i \in \mathbb{N}_{+}, 1 \leq i \leq n\right\}, \\
\operatorname{dec}\left(C_{n, m}\right) & :=(n \underbrace{m \ldots m}_{n}, m), \\
\operatorname{dec}\left(e_{n, i}\right) & :=(\varepsilon, n),
\end{aligned}
$$

and define the algebra $\mathbf{F}=\left(F, \Sigma^{\mathbf{F}}\right)$ of type $\tau=(S, \Sigma$, dec $)$, where $F=$ $\left(F_{n}\right)_{n \in \mathbb{N}_{+}}$with $F_{n}:=\mathcal{O}_{A}^{(n)}$, and

$$
\begin{aligned}
C_{n, m}^{\mathbf{F}}\left(f, g_{1}, \ldots, g_{n}\right) & :=f\left(g_{1}, \ldots, g_{n}\right) \\
e_{n, i}^{\mathbf{F}} & :=\operatorname{pr}_{i}^{n}
\end{aligned}
$$

The subalgebras of $\mathbf{F}$ are then in one-to-one correspondence with the clones on $A$ in an obvious way.
(5) A 2-algebra is a multisorted algebra of type $\tau=(S, \Sigma$, dec), where $S=$ $\{1,2\}$ and for every $f \in \Sigma, \operatorname{dec}(f)=(\underbrace{1 \ldots 1}_{n}, 2)$ for some $n \in \mathbb{N}$. In other words, the carrier of a 2-algebra comprises two sets $A$ and $B$, and the operations are functions $f: A^{n} \rightarrow B$ of several arguments from $A$ to $B$.

Let us make a simple but very useful observation about the subalgebras of the canonical trivial algebra $\mathbf{S}=\left(\tilde{S}, \Sigma^{\mathbf{S}}\right)$ of type $\tau=(S, \Sigma$, dec) (see Example $2.13(2)$ ). We first introduce the shorthand $\tilde{S}^{\prime}:=\left.\tilde{S}\right|_{S^{\prime}}$, for any subset $S^{\prime} \subseteq S$ (for notation, see Definition 2.3), i.e., $\tilde{S}^{\prime}$ is the $S$-sorted set with $\tilde{S}_{s}^{\prime}=\{s\}$ if $s \in S^{\prime}$ and $\tilde{S}_{s}^{\prime}=\emptyset$ if $s \notin S^{\prime}$. Obviously, for subsets $S^{\prime}$ and $S^{\prime \prime}$ of $S$, the set inclusion $S^{\prime} \subseteq S^{\prime \prime}$ holds if and only if $\tilde{S}^{\prime} \subseteq \tilde{S}^{\prime \prime}$ holds. In the sequel, we will often slightly abuse the notation and write $\left\langle S^{\prime}\right\rangle_{\mathbf{S}}$ to mean the unique set $S^{\prime \prime} \subseteq S$ such that $\left\langle\tilde{S}^{\prime}\right\rangle_{\mathbf{S}}=\tilde{S}^{\prime \prime}$. We will keep the formally correct notation in the following lemma and its proof.

Lemma 2.14. For a multisorted algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ of type $\tau=(S, \Sigma$, dec $)$, $\widetilde{S_{A}}$ is a subalgebra of the canonical trivial algebra $\mathbf{S}=\left(\tilde{S}, \Sigma^{\mathbf{S}}\right)$ of type $\tau$, i.e., $\left\langle\widetilde{S_{A}}\right\rangle_{\mathbf{S}}=\widetilde{S_{A}}$.

Proof. Since $\mathbf{A}$ is an algebra, the declaration of every $f \in \Sigma$ is reasonable in $A$. Clearly $S_{A}=S_{\widetilde{S_{A}}}$, so the declaration of every $f \in \Sigma$ is reasonable in $\widetilde{S_{A}}$, too. Moreover, for each $f \in \Sigma_{(w, s)}$, the uniquely determined operation $\left(\widetilde{S_{A}}\right)_{w} \rightarrow\left(\widetilde{S_{A}}\right)_{s}$ coincides with the restriction of $f^{\mathbf{S}}$ to $\widetilde{S_{A}}$. Therefore $\widetilde{S_{A}}$ is a subalgebra of $\mathbf{S}$.

## 3. Minor terms and minor identities

As we have seen in Example 2.13(3), terms can be defined in the multisorted setting in the expected way: the output sorts of the terms $t_{1}, \ldots, t_{n}$ must match with the input sorts of the function symbol $f$ when a complex term $f\left(t_{1}, \ldots, t_{n}\right)$ is formed. However, the notion of identity (or equation) requires a bit of care. It is not sufficient to define an identity simply as a pair of terms. One also has to specify the variables that are to be valuated when one decides whether an identity is satisfied by an algebra. This sounds superfluous, and it is indeed so in the case of one-sorted algebras, but for multisorted algebras this makes a difference. Namely, an identity would be trivially satisfied by an algebra $\mathbf{A}=(A, F)$ if there is a variable of sort $s$ to which a value must be assigned but the set $A_{s}$ is empty. If there is no such variable, then the identity
may or may not be satisfied by $\mathbf{A}$, depending on whether the two terms of the identity get the same value in all assignments of values to variables. For further discussion and examples on this, see Wechler [10, Section 4.1.1].

As explained above, for a reasonable definition of an identity in the multisorted setting, it is necessary to specify the variables to which values are assigned. What really matters here are the sorts of such variables. For this reason, we have chosen to indicate only the sorts of the variables that are valuated, not the variables themselves. Consequently, our definition of an identity is slightly, but not in any essential way different from what is commonly seen in the literature (e.g., Adámek, Rosický, Vitale [1], Manca, Salibra [8, Definition 1.9], Wechler [10, Section 4.1.1]).

Definition 3.1. Let $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ be an algebra of type $\tau=(S, \Sigma$, dec $)$ and let $Y$ be an $S$-sorted set of variables. A valuation of $Y$ in $A$ is an $S$-sorted mapping $\beta: Y \rightarrow A$. (Note that valuations $\beta: Y \rightarrow A$ exist if and only if $S_{Y} \subseteq S_{A}$.) The map $\beta$ admits a unique homomorphic extension $\beta^{\#}: \mathbf{T}_{\tau}(Y) \rightarrow \mathbf{A}$ (see Example 2.13(3)). For a term $t \in T_{\tau}(Y)$, we call $\beta^{\#}(t)$ the value of $t$ in $\mathbf{A}$ under $\beta$.

Definition 3.2. Let $\tau=(S, \Sigma$, dec) be a multisorted similarity type, and let $Y$ be an $S$-sorted set of variables. An identity of sort $s$ of type $\tau$ over variables $Y$ is a triple $\left(S^{\prime}, t_{1}, t_{2}\right)$, where $S^{\prime} \subseteq S$ and $t_{1}, t_{2} \in T_{\tau}^{s}\left(\left.Y\right|_{S^{\prime}}\right)$. We will use a more suggestive notation for identities and write $t_{1} \approx_{S^{\prime}} t_{2}$ for $\left(S^{\prime}, t_{1}, t_{2}\right)$. We say that $t_{1} \approx_{S^{\prime}} t_{2}$ is valuated on sorts $S^{\prime}$. We denote the set of all identities of sort $s$ of type $\tau$ over $Y$ by $I D_{\tau}^{s}(Y)$, and we set $I D_{\tau}(Y):=\bigcup_{s \in S} I D_{\tau}^{s}(Y)$.

An algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ of type $\tau$ is said to satisfy the identity $t_{1} \approx_{S^{\prime}} t_{2}$ if $\beta^{\#}\left(t_{1}\right)=\beta^{\#}\left(t_{2}\right)$ for all valuation maps $\beta:\left.Y\right|_{S^{\prime}} \rightarrow A$. In this case we also write $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$. Note that $\mathbf{A}$ satisfies $t_{1} \approx_{S^{\prime}} t_{2}$ vacuously if $Y_{s} \neq \emptyset$ and $A_{s}=\emptyset$ for some $s \in S^{\prime}$.

Remark 3.3. In the literature (e.g., $[1,8,10]$ ), identities are often written as $\forall Y^{\prime}\left(t_{1}=t_{2}\right)$, where $Y^{\prime}$ is a subset of the $S$-sorted set $Y$ of variables and $t_{1}$ and $t_{2}$ are terms. Using this notation, an identity $t_{1} \approx_{S^{\prime}} t_{2}$ (according to our Definition 3.2, where a set $S^{\prime}$ is given instead of a set of variables) would be written as $\left.\forall Y\right|_{S^{\prime}}\left(t_{1}=t_{2}\right)$.

Lemma 3.4. Let $\mathbf{A} \in \operatorname{Alg}(\tau)$, and let $t_{1} \approx_{S^{\prime}} t_{2} \in I D_{\tau}^{s}(Y)$. If $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$, then $\mathbf{A} \models t_{1} \approx_{S^{\prime \prime}} t_{2}$ for all $S^{\prime \prime}$ with $S^{\prime} \subseteq S^{\prime \prime} \subseteq S$.

Proof. For every valuation $\beta:\left.Y\right|_{S^{\prime \prime}} \rightarrow A$, we have

$$
\beta^{\#}\left(t_{1}\right)=\left(\left.\beta\right|_{S^{\prime}}\right)^{\#}\left(t_{1}\right)=\left(\left.\beta\right|_{S^{\prime}}\right)^{\#}\left(t_{2}\right)=\beta^{\#}\left(t_{2}\right)
$$

Thus, if there exists a valuation $\beta:\left.Y\right|_{S^{\prime \prime}} \rightarrow A$, then $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$ implies that $\mathbf{A} \models t_{1} \approx_{S^{\prime \prime}} t_{2}$. If there is no such valuation, then $\mathbf{A} \models t_{1} \approx_{S^{\prime \prime}} t_{2}$ vacuously.

Remark 3.5. A valuation $\beta:\left.Y\right|_{S^{\prime \prime}} \rightarrow A$ exists if and only if $Y_{s} \neq \emptyset \Longrightarrow A_{s} \neq$ $\emptyset$ for every $s \in S^{\prime \prime}$. If this is the case, then the converse of Lemma 3.4 is also true (i.e., $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$ if and only if $\mathbf{A} \models t_{1} \approx_{S^{\prime \prime}} t_{2}$ ).

Definition 3.6. Terms containing exactly one function symbol are called minor terms. We denote by $M T_{\tau}^{s}(Y)$ the set of all minor terms of sort $s$ of type $\tau=(S, \Sigma, \operatorname{dec})$ over $Y$, and we set $M T_{\tau}(Y):=\bigcup_{s \in S} M T_{\tau}^{s}(Y)$.

In other words, a general minor term $t \in M T_{\tau}^{s}(Y)$ is of the form $f(\sigma(1), \ldots, \sigma(n))$, where $f \in \Sigma$ with $\operatorname{dec}(f)=(w, s), w=w_{1} \ldots w_{n}$, and $\sigma:[n] \rightarrow Y$ is a map respecting the sorts, i.e., satisfying $\sigma(i) \in Y_{w_{i}}$ for all $i \in[n]$. We denote this term by $f_{\sigma}$. The value of $f_{\sigma}$ in $\mathbf{A}$ under a valuation $\beta: Y \rightarrow A$ is $\beta^{\#}\left(f_{\sigma}\right)=f^{\mathbf{A}}(\beta(\sigma(1)), \ldots, \beta(\sigma(n)))=f^{\mathbf{A}}(\beta \circ \sigma)$.

Note that constants $f \in \Sigma$ are also minor terms, corresponding to the case $n=0$. Recall that $[0]=\emptyset$, so $f_{\sigma}=f$ for every $\sigma:[0] \rightarrow Y$ and for any valuation $\beta: Y \rightarrow A$ we have $\beta^{\#}(f)=f^{\mathbf{A}}$.

An identity $t_{1} \approx_{S^{\prime}} t_{2}$ is called a minor identity if both $t_{1}$ and $t_{2}$ are minor terms. Minor identities are also known as height-1 identities (see Barto et al. [2]) or primitive identities (see Čupona, Markovski [4,5] and Čupona et al. [6]). We denote by $\operatorname{MID}_{\tau}^{s}(Y)$ the set of all minor identities of sort of type $\tau$ over $Y$, and we set $M I D_{\tau}(Y):=\bigcup_{s \in S} M I D_{\tau}^{s}(Y)$.

Definition 3.7. The satisfaction relation induces a Galois connection between multisorted algebras and identities of type $\tau$. For a class $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$ of algebras of type $\tau$ and for a set $\mathcal{J} \subseteq I D_{\tau}(Y)$ of identities of type $\tau$, let

$$
\begin{aligned}
\operatorname{Id}_{Y} \mathcal{K} & :=\left\{t_{1} \approx_{S^{\prime}} t_{2} \in I D_{\tau}(Y) \mid \forall \mathbf{A} \in \mathcal{K}: \mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}\right\}, \\
\operatorname{mId}_{Y} \mathcal{K} & :=\left\{t_{1} \approx_{S^{\prime}} t_{2} \in M I D_{\tau}(Y) \mid \forall \mathbf{A} \in \mathcal{K}: \mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}\right\}, \\
\operatorname{Mod} \mathcal{J} & :=\left\{\mathbf{A} \in \operatorname{Alg}(\tau) \mid \forall t_{1} \approx_{S^{\prime}} t_{2} \in \mathcal{J}: \mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}\right\} .
\end{aligned}
$$

When $Y$ is the standard set of variables $X$, then we write $\operatorname{Id} \mathcal{K}$ and $\operatorname{mId} \mathcal{K}$ for $\operatorname{Id}_{X} \mathcal{K}$ and $\operatorname{mId}_{X} \mathcal{K}$, respectively.

By Birkhoff's theorem for (multisorted) algebras, $\operatorname{Mod} \mathcal{J}$ is a variety for any set $\mathcal{J} \subseteq I D_{\tau}(Y)$ of identities. We call a variety $\mathcal{V}$ a minor variety if $\mathcal{V}=\operatorname{Mod} \mathcal{J}$ for some set $\mathcal{J} \subseteq \operatorname{MID}_{\tau}(Y)$ of minor identities. Minor varieties of one-sorted algebras were investigated by Čupona and Markovski [4,5] and Čupona et al. [6].

Example 3.8. As a concrete example of minor varieties, we present here the minor varieties of groupoids (one-sorted algebras with a single binary operation), which were determined by Čupona et al. [6]. It is easy to verify that every minor identity in the language of groupoids is equivalent to one of the following:

$$
\begin{equation*}
x y \approx x y, \quad x y \approx y x, \quad x x \approx y y, \quad x y \approx x z, \quad x y \approx z y, \quad x y \approx z u \tag{3.1}
\end{equation*}
$$

where we have written identities as is usual in the classical framework and the binary operation as juxtaposition. Therefore, there are six varieties defined by a single minor identity:


Figure 1. Minor varieties of groupoids

$$
\begin{array}{rlrl}
\mathcal{G}:=\operatorname{Mod}\{x y \approx x y\} & & \text { (all groupoids), } \\
\mathcal{C} & :=\operatorname{Mod}\{x y \approx y x\} & & \text { (commutative groupoids), } \\
\mathcal{U} & :=\operatorname{Mod}\{x x \approx y y\} & & \text { (unipotent groupoids), } \\
\mathcal{L}:=\operatorname{Mod}\{x y \approx x z\} & & \text { (left unars), } \\
\mathcal{R} & :=\operatorname{Mod}\{x y \approx z y\} & & \text { (right unars), } \\
\mathcal{K}:=\operatorname{Mod}\{x y \approx z u\} & & \text { (constant groupoids). }
\end{array}
$$

The only new variety that can be formed as the intersection of any of the above is $\mathcal{C U}:=\operatorname{Mod}\{x y \approx y x, x x \approx y y\}$ (commutative unipotent groupoids). The lattice of minor varieties of groupoids is represented by the Hasse diagram shown in Figure 1.

Example 3.9. As another example, we determine the minor variety generated by the variety of groups. Recall that a group is a one-sorted algebra $\left(A ; \cdot,{ }^{-1}, e\right)$ of type $(2,1,0)$ satisfying the identities

$$
x \cdot(y \cdot z) \approx(x \cdot y) \cdot z, \quad e \cdot x \approx x, \quad x \cdot e \approx x, \quad x \cdot x^{-1} \approx e, \quad x^{-1} \cdot x \approx e
$$

Every minor identity in the language of groups is equivalent to one of the groupoid identities listed in (3.1) or one of the following:

$$
\begin{align*}
& x y \approx z^{-1}, \quad x y \approx x^{-1}, \quad x y \approx y^{-1}, \quad x x \approx y^{-1}, \quad x x \approx x^{-1}, \quad x y \approx e \\
& x x \approx e, \quad x^{-1} \approx y^{-1}, \quad x^{-1} \approx x^{-1}, \quad x^{-1} \approx e, \quad e \approx e \tag{3.2}
\end{align*}
$$

As trivial identities, $x^{-1} \approx x^{-1}$ and $e \approx e$ are equivalent to $x y \approx x y$. It is an easy exercise to find, for each one of the nontrivial identities listed in (3.1) and (3.2), an example of a group that does not satisfy that identity. Hence the only minor identities satisfied by the variety of groups are the trivial ones, and we conclude that the minor variety generated by the variety of groups is the variety of all algebras of type $(2,1,0)$.

Example 3.10. Our last example involves multisorted algebras that are not one-sorted, and it aims at illustrating the role of empty components in carriers of algebras, as well as the importance of specifying the sorts on which identities are valuated. Consider the algebraic similarity type $\tau=(S, \Sigma$, dec) with $S=$ $\{s, t\}, \Sigma=\{\cdot, *\}$ and $\operatorname{dec}(\cdot)=(s s, s), \operatorname{dec}(*)=(s t, t)$. Algebras of type $\tau$ satisfying the identity $x *(y * u) \approx_{S}(x \cdot y) * u$ are called groupoid actions. Groupoid actions that additionally satisfy the identity $x \cdot(y \cdot z) \approx_{\{s\}}(x \cdot y) \cdot z$ are called semigroup actions. (Note that the defining identities of groupoid actions and semigroup actions are not minor identities.)

Let us determine the minor varieties of algebras of type $\tau$. To this end, we introduce some notation. For a variety $\mathcal{V}$ of groupoids, as in Example 3.8, let us denote by $\mathcal{V}^{*}$ the set of all algebras $\mathbf{A}$ of type $\tau$ such that $\left(A_{s}, \cdot\right)$ is in $\mathcal{V}$. Let $\mathcal{T}$ be the class of all algebras $\mathbf{A}$ of type $\tau$ such that $A_{t}=\emptyset$.

Let $\mathcal{J}$ be a set of identities in the language of groupoids (usual onesorted), and let $\mathcal{J}^{\prime} \subseteq \mathcal{J}$. Define

$$
\mathcal{I}:=\left\{t_{1} \approx_{\{s\}} t_{2} \mid t_{1} \approx t_{2} \in \mathcal{J}^{\prime}\right\} \cup\left\{t_{1} \approx_{\{s, t\}} t_{2} \mid t_{1} \approx t_{2} \in \mathcal{J} \backslash \mathcal{J}^{\prime}\right\}
$$

An algebra A of type $\tau$ satisfies the set $\mathcal{I}$ of identities if and only if $\left(A_{s}, \cdot\right) \mid=$ $t_{1} \approx t_{2}$ for every $t_{1} \approx t_{2} \in \mathcal{J}^{\prime}$ and $\left(A_{s}, \cdot\right) \models t_{1} \approx t_{2}$ or $A_{t}=\emptyset$ for every $t_{1} \approx t_{2} \in \mathcal{J} \backslash \mathcal{J}^{\prime}$. This condition is equivalent to the following: $\left(A_{s}, \cdot\right) \models t_{1} \approx t_{2}$ for every $t_{1} \approx t_{2} \in \mathcal{J}$, or $A_{t}=\emptyset$ and $\left(A_{s}, \cdot\right) \vDash t_{1} \approx t_{2}$ for every $t_{1} \approx t_{2} \in \mathcal{J}^{\prime}$. In other words, $\mathbf{A} \models \mathcal{I}$ if and only if $\mathbf{A} \in \mathcal{V}^{*} \cup\left(\mathcal{V}^{\prime *} \cap \mathcal{T}\right)$, where $\mathcal{V}:=\operatorname{Mod} \mathcal{J}$ and $\mathcal{V}^{\prime}:=\operatorname{Mod} \mathcal{J}^{\prime}$ are varieties of groupoids.

Consequently, the varieties of algebras of type $\tau$ defined by identities of sort $s$ are of the form $\left\lfloor\mathcal{V}, \mathcal{V}^{\prime}\right\rceil:=\mathcal{V}^{*} \cup\left(\mathcal{V}^{\prime *} \cap \mathcal{T}\right)$, where $\mathcal{V}, \mathcal{V}^{\prime}$ are varieties of groupoids such that $\mathcal{V} \subseteq \mathcal{V}^{\prime}$. We can deduce from Figure 1 that there are 20 varieties of type $\tau$ that are defined by minor identities of sort $s$, and they are shown in Figure 2.

It is easy to verify that every minor identity of type $\tau$ of sort $t$ is equivalent to one of the following:

$$
x * u \approx_{S} x * u, \quad x * u \approx_{S} x * v, \quad x * u \approx_{S} y * u, \quad x * u \approx_{S} y * v
$$

Therefore, there are four varieties defined by a single minor identity of sort $t$ :

$$
\begin{aligned}
\mathcal{M} & :=\operatorname{Mod}\left(x * u \approx_{S} x * u\right), & \mathcal{N}:=\operatorname{Mod}\left(x * u \approx_{S} x * v\right), \\
\mathcal{O} & :=\operatorname{Mod}\left(x * u \approx_{S} y * u\right), & \mathcal{P}:=\operatorname{Mod}\left(x * u \approx_{S} y * v\right) .
\end{aligned}
$$

Intersections of these four varieties do not yield any new varieties. These varieties are shown in Figure 3.

We conclude that the minor varieties of type $\tau$ are of the form $\mathcal{X} \cap \mathcal{Y}$, where $\mathcal{X}$ is a variety defined by minor identities of sort $s$ (see, Figure 2) and $\mathcal{Y}$ is a variety defined by minor identities of sort $t$ (see, Figure 3). Consequently, the total number of minor varieties of type $\tau$ is $20 \cdot 4=80$, and the lattice of minor varieties is isomorphic to the direct product of the lattices shown in Figures 2 and 3.


Figure 2. Minor varieties of the groupoid action type defined by identities of sort $s$


Figure 3. Minor varieties of the groupoid action type defined by identities of sort $t$

## 4. Reflections

We are now going to generalize the notion of reflection (see Barto et al. [2]) to the multisorted setting.

Definition 4.1. Let $A$ and $B$ be $S$-sorted sets. A reflection of $A$ into $B$ is a pair $\left(h, h^{\prime}\right)$ of $S_{B}$-sorted mappings $h=\left(h_{s}\right)_{s \in S_{B}}, h^{\prime}=\left(h_{s}^{\prime}\right)_{s \in S_{B}}, h_{s}: B_{s} \rightarrow A_{s}$,
$h_{s}^{\prime}: A_{s} \rightarrow B_{s}$. Note that reflections of $A$ into $B$ exist if and only if $S_{B} \subseteq S_{A}$. For, if $S_{B} \subseteq S_{A}$, then $A_{s}$ and $B_{s}$ are nonempty for all $s \in S_{B}$ and there clearly exist maps $h_{s}: B_{s} \rightarrow A_{s}$ and $h_{s}^{\prime}: A_{s} \rightarrow B_{s}$. If $S_{B} \nsubseteq S_{A}$, then there is $s \in S_{B} \backslash S_{A}$, whence $A_{s}=\emptyset$ and $B_{s} \neq \emptyset$, so there is no map $h_{s}: B_{s} \rightarrow A_{s}$.

Assume that $A$ and $B$ are $S$-sorted sets with $S_{B} \subseteq S_{A}$ and $\left(h, h^{\prime}\right)$ is a reflection of $A$ into $B$. If $(w, s) \in W(S) \times S$ is a declaration that is reasonable in both $A$ and $B$ and $f: A_{w} \rightarrow A_{s}$, then we can define the $\left(h, h^{\prime}\right)$-reflection of $f$ to be the map $f_{\left(h, h^{\prime}\right)}: B_{w} \rightarrow B_{s}$ given by the rule

$$
f_{\left(h, h^{\prime}\right)}\left(b_{1}, \ldots, b_{n}\right)=h_{s}^{\prime}\left(f\left(h_{w_{1}}\left(b_{1}\right), \ldots, h_{w_{n}}\left(b_{n}\right)\right)\right)
$$

for all $\left(b_{1}, \ldots, b_{n}\right) \in B_{w}$, which we write simply as $f_{\left(h, h^{\prime}\right)}(\mathbf{b})=h_{s}^{\prime}\left(f\left(h_{w}(\mathbf{b})\right)\right)$ for all $\mathbf{b} \in B_{w}$. This is illustrated by the commutative diagram shown below. Note that if $B_{w_{i}}=\emptyset$ for some $i \in\{1, \ldots, n\}$, then $f_{\left(h, h^{\prime}\right)}=\emptyset$.


Let $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ and $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ be algebras of type $\tau=(S, \Sigma$, dec $)$. If $\left(h, h^{\prime}\right)$ is a reflection of $A$ into $B$ and for all $f \in \Sigma$ it holds that $f^{\mathbf{B}}=\left(f^{\mathbf{A}}\right)_{\left(h, h^{\prime}\right)}$, then $\mathbf{B}$ is called the $\left(h, h^{\prime}\right)$-reflection of $\mathbf{A}$. (Note that for every $f \in \Sigma$, the declaration $\operatorname{dec}(f)$ of $f^{\mathbf{A}}$ and $f^{\mathbf{B}}$ is reasonable in both $A$ and $B$, because $\mathbf{A}$ and $\mathbf{B}$ are algebras.) We say that $\mathbf{B}$ is a reflection of $\mathbf{A}$ if $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of $\mathbf{A}$ for some reflection $\left(h, h^{\prime}\right)$ of $A$ into $B$.

For a class $\mathcal{K}$ of multisorted algebras of type $\tau$, let $\mathrm{R} \mathcal{K}, \mathrm{H} \mathcal{K}, \mathrm{SK}$ and PK denote the classes of all reflections, homomorphic images, subalgebras and products of algebras of $\mathcal{K}$, respectively.

Lemma 4.2 (cf. [2, Lemma 4.4]). Let $\mathcal{K}$ be a class of multisorted algebras of type $\tau$. Then the following statements hold.
(1) $\mathrm{HK} \subseteq \mathrm{RK}$ and $\mathrm{SK} \subseteq \mathrm{RK}$.
(2) $R R \mathcal{K} \subseteq R \mathcal{K}$.
(3) $\mathrm{PRK} \subseteq \mathrm{RP} \mathcal{K}$.

Proof. (1) Assume that $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right) \in \mathbf{H K}$. Then there exists an algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right) \in \mathcal{K}$ such that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$. Let $\varphi$ be a homomorphism of $\mathbf{A}$ to $\mathbf{B}$, with each $\varphi_{s}: A_{s} \rightarrow B_{s}$ surjective. Then there exist mappings $h_{s}: B_{s} \rightarrow A_{s}$ such that $\varphi_{s} \circ h_{s}=\operatorname{id}_{B_{s}}$. Then for each $f \in \Sigma_{(w, s)}$ with $w=w_{1} \ldots w_{n}$ and for all $\left(b_{1}, \ldots, b_{n}\right) \in B_{w}$, we have

$$
\begin{aligned}
f^{\mathbf{B}}\left(b_{1}, \ldots, b_{n}\right) & =f^{\mathbf{B}}\left(\varphi_{w_{1}}\left(h_{w_{1}}\left(b_{1}\right)\right), \ldots, \varphi_{w_{n}}\left(h_{w_{n}}\left(b_{n}\right)\right)\right) \\
& =\varphi_{s}\left(f^{\mathbf{A}}\left(h_{w_{1}}\left(b_{1}\right), \ldots, h_{w_{n}}\left(b_{n}\right)\right)\right) .
\end{aligned}
$$

We clearly have $S_{B}=S_{A}$ because the homomorphism $\varphi$ is surjective. Setting $h=\left(h_{s}\right)_{s \in S_{B}}$ and $h^{\prime}=\left(\varphi_{s}\right)_{s \in S_{B}}$, we conclude that $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of A. Thus $\mathrm{H} \mathcal{K} \subseteq \mathrm{R} \mathcal{K}$.

Assume then that $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right) \in \mathbf{S} \mathcal{K}$. Then there exists an algebra $\mathbf{A}=$ $\left(A, \Sigma^{\mathbf{A}}\right) \in \mathcal{K}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Then clearly $S_{B} \subseteq S_{A}$. Let $h=$ $\left(h_{s}\right)_{s \in S_{B}}$ and $h^{\prime}=\left(h_{s}^{\prime}\right)_{s \in S_{B}}$ where each $h_{s}: B_{s} \rightarrow A_{s}$ is the inclusion map of $B_{s}$ into $A_{s}$ and each $h_{s}^{\prime}: A_{s} \rightarrow B_{s}$ is an arbitrary extension of the identity map on $B_{s}$. Then for each $f \in \Sigma_{(w, s)}$ with $w=w_{1} \ldots w_{n}$, and for all $\left(b_{1}, \ldots, b_{n}\right) \in$ $B_{w}$, we clearly have $f^{\mathbf{B}}\left(b_{1}, \ldots, b_{n}\right)=h_{s}^{\prime}\left(f^{\mathbf{A}}\left(h_{w_{1}}\left(b_{1}\right), \ldots, h_{w_{n}}\left(b_{n}\right)\right)\right.$, so $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of $\mathbf{A}$. Thus $\mathrm{SK} \subseteq \mathrm{R} \mathcal{K}$.
(2) Assume that $\mathbf{C} \in R R \mathcal{K}$. Then there exist algebras $\mathbf{B} \in \mathrm{R} \mathcal{K}$ and $\mathbf{A} \in \mathcal{K}$ such that $\mathbf{C}$ is a reflection of $\mathbf{B}$, witnessed by $\left(h, h^{\prime}\right)=\left(\left(h_{s}\right)_{s \in S_{C}},\left(h_{s}^{\prime}\right)_{s \in S_{C}}\right)$ where $h_{s}: C_{s} \rightarrow B_{s}$ and $h_{s}^{\prime}: B_{s} \rightarrow C_{s}$, and $\mathbf{B}$ is a reflection of $\mathbf{A}$, witnessed by $\left(k, k^{\prime}\right)=\left(\left(k_{s}\right)_{s \in S_{B}},\left(k_{s}^{\prime}\right)_{s \in S_{B}}\right)$ where $k_{s}: B_{s} \rightarrow A_{s}$ and $k_{s}^{\prime}: A_{s} \rightarrow B_{s}$. Then $S_{C} \subseteq S_{B} \subseteq S_{A}$, so we can define a reflection $\left(\ell, \ell^{\prime}\right)$ of $A$ into $C$ using the $S_{C^{-}}$ sorted maps $\ell=\left(\ell_{s}\right)_{s \in S_{C}}$ where each $\ell_{s}: C_{s} \rightarrow A_{s}$ is given by $\ell_{s}:=k_{s} \circ h_{s}$ and $\ell^{\prime}=\left(\ell_{s}^{\prime}\right)_{s \in S_{C}}$ where each $\ell_{s}^{\prime}: A_{s} \rightarrow C_{s}$ is given by $\ell_{s}^{\prime}:=h_{s}^{\prime} \circ k_{s}^{\prime}$. Furthermore, for every $f \in \Sigma_{(w, s)}$ with $w=w_{1} \ldots w_{n}$, we have

$$
\begin{aligned}
f^{\mathbf{C}}\left(c_{1}, \ldots, c_{n}\right) & =h_{s}^{\prime}\left(f^{\mathbf{B}}\left(h_{w_{1}}\left(c_{1}\right), \ldots, h_{w_{n}}\left(c_{n}\right)\right)\right) \\
& =h_{s}^{\prime}\left(k_{s}^{\prime}\left(f^{\mathbf{A}}\left(k_{w_{1}}\left(h_{w_{1}}\left(c_{1}\right)\right), \ldots, k_{w_{n}}\left(h_{w_{n}}\left(c_{n}\right)\right)\right)\right)\right) \\
& =\ell_{s}^{\prime}\left(f^{\mathbf{A}}\left(\ell_{w_{1}}\left(c_{1}\right), \ldots, \ell_{w_{n}}\left(c_{n}\right)\right)\right)
\end{aligned}
$$

for all $\left(c_{1}, \ldots, c_{n}\right) \in C_{w}$. We conclude that $\mathbf{C}$ is a reflection of $\mathbf{A}$. Thus RRK $\subseteq$ RK.
(3) Assume that $\mathbf{C} \in \mathrm{PRK}$. Then $\mathbf{C}=\prod_{\gamma \in \Gamma} \mathbf{B}_{\gamma}$ for some algebras $\mathbf{B}_{\gamma}=$ $\left(B_{\gamma}, \Sigma^{\mathbf{B}_{\gamma}}\right)$, and each $\mathbf{B}_{\gamma}$ is a reflection of some $\mathbf{A}_{\gamma}=\left(A_{\gamma}, \Sigma^{\mathbf{A}_{\gamma}}\right) \in \mathcal{K}$, witnessed by $\left(\left(h_{\gamma, s}\right)_{s \in S_{B_{\gamma}}},\left(h_{\gamma, s}^{\prime}\right)_{s \in S_{B_{\gamma}}}\right)$, where $h_{\gamma, s}: B_{\gamma, s} \rightarrow A_{\gamma, s}, h_{\gamma, s}^{\prime}: A_{\gamma, s} \rightarrow B_{\gamma, s}$.

Observe that if $C_{s}=\prod_{\gamma \in \Gamma} B_{\gamma, s} \neq \emptyset$, then $B_{\gamma, s} \neq \emptyset$ for every $\gamma \in \Gamma$. Therefore $S_{C} \subseteq S_{B_{\gamma}} \subseteq S_{A_{\gamma}}$ for every $\gamma \in \Gamma$. Define the $S_{C}$-sorted maps $h=\left(h_{s}\right)_{s \in S_{C}}$ and $h^{\prime}=\left(h_{s}^{\prime}\right)_{s \in S_{C}}$, where $h_{s}: \prod_{\gamma \in \Gamma} B_{\gamma, s} \rightarrow \prod_{\gamma \in \Gamma} A_{\gamma, s}$ and $h_{s}^{\prime}: \prod_{\gamma \in \Gamma} A_{\gamma, s} \rightarrow \prod_{\gamma \in \Gamma} B_{\gamma, s}$ are defined componentwise in terms of the $h_{\gamma, s}$ and $h_{\gamma, s}^{\prime}$ as $h_{s}\left(\left(b_{\gamma}\right)_{\gamma \in \Gamma}\right)=\left(h_{\gamma, s}\left(b_{\gamma}\right)\right)_{\gamma \in \Gamma}$ and $h_{s}^{\prime}\left(\left(a_{\gamma}\right)_{\gamma \in \Gamma}\right)=\left(h_{\gamma, s}^{\prime}\left(a_{\gamma}\right)\right)_{\gamma \in \Gamma}$. Then, for every operation $f \in \Sigma_{(w, s)}$ with $w=w_{1} \ldots w_{n}$ and for all tuples $\left(\left(b_{1, \gamma}\right)_{\gamma \in \Gamma}, \ldots,\left(b_{n, \gamma}\right)_{\gamma \in \Gamma}\right) \in\left(\prod_{\gamma \in \Gamma} B_{\gamma}\right)_{w}$, we have

$$
\begin{aligned}
& f^{\Pi \mathbf{B}_{\gamma}}\left(\left(b_{1, \gamma}\right)_{\gamma \in \Gamma}, \ldots,\left(b_{n, \gamma}\right)_{\gamma \in \Gamma}\right)=\left(f^{\mathbf{B}_{\gamma}}\left(b_{1, \gamma}, \ldots, b_{n, \gamma}\right)\right)_{\gamma \in \Gamma} \\
& \quad=\left(h_{\gamma, s}^{\prime}\left(f^{\mathbf{A}_{\gamma}}\left(h_{\gamma, w_{1}}\left(b_{1, \gamma}\right), \ldots, h_{\gamma, w_{n}}\left(b_{n, \gamma}\right)\right)\right)\right)_{\gamma \in \Gamma} \\
& \quad=h_{s}^{\prime}\left(\left(f^{\mathbf{A}_{\gamma}}\left(h_{\gamma, w_{1}}\left(b_{1, \gamma}\right), \ldots, h_{\gamma, w_{n}}\left(b_{n, \gamma}\right)\right)\right)_{\gamma \in \Gamma}\right) \\
& \quad=h_{s}^{\prime}\left(f^{\Pi \mathbf{A}_{\gamma}}\left(\left(h_{\gamma, w_{1}}\left(b_{1, \gamma}\right)\right)_{\gamma \in \Gamma}, \ldots,\left(h_{\gamma, w_{n}}\left(b_{n, \gamma}\right)\right)_{\gamma \in \Gamma}\right)\right) \\
& \quad=h_{s}^{\prime}\left(f^{\Pi \mathbf{A}_{\gamma}}\left(h_{w_{1}}\left(\left(b_{1, \gamma}\right)_{\gamma \in \Gamma}\right), \ldots, h_{w_{n}}\left(\left(b_{n, \gamma}\right)_{\gamma \in \Gamma}\right)\right)\right) .
\end{aligned}
$$

This shows that the algebra $\mathbf{C}=\prod_{\gamma \in \Gamma} \mathbf{B}_{\gamma}$ is the $\left(h, h^{\prime}\right)$-reflection of the product $\prod_{\gamma \in \Gamma} \mathbf{A}_{\gamma}$. Thus $\mathbf{C} \in \operatorname{RPK}$, so $\mathrm{PRK} \subseteq \mathrm{RP} \mathcal{K}$.

Remark 4.3. Note that the converse of the inclusion of Lemma 4.2(3), namely $R P \mathcal{K} \subseteq P R \mathcal{K}$, does not hold in general. For example, take $\mathcal{K}:=\emptyset$. Then $\mathrm{P} \emptyset=\{\Pi \emptyset\}$. Since for any $S$-sorted set $A=\left(A_{s}\right)_{s \in S}$, an algebra with carrier $A$ can be obtained as a reflection of $\Pi \emptyset$ (the proof of this assertion is essentially included in the proof of Theorem 6.3, implication $(1) \Longrightarrow(3))$, it follows that $R P \mathcal{K}$ contains algebras with arbitrary carrier sets. On the other hand, $R \emptyset=\emptyset$, whence $P R \emptyset=\left\{\prod \emptyset\right\}$. Thus RP $\emptyset \nsubseteq P R \emptyset$.

In order to give also a nonempty counterexample, let $\mathbf{A}=\left(\{0,1\} ; f^{\mathbf{A}}\right)$ and $\mathbf{B}=\left(\{a, b, c\} ; f^{\mathbf{B}}\right)$ where $f^{\mathbf{A}}$ and $f^{\mathbf{B}}$ are the identity functions on the corresponding sets. We define maps $h$ and $h^{\prime}$ as follows:

$$
\begin{aligned}
h: B \rightarrow A^{2}, & a \mapsto(0,0), b \mapsto(0,1), c \mapsto(1,0) ; \\
h^{\prime}: A^{2} \rightarrow B, & (0,0) \mapsto a,(0,1) \mapsto b,(1,0) \mapsto c,(1,1) \mapsto c .
\end{aligned}
$$

Then $\mathbf{B}$ is the $\left(h, h^{\prime}\right)$-reflection of $\mathbf{A}^{2}$, hence $\mathbf{B} \in \operatorname{RP}\{\mathbf{A}\}$. On the other hand, if $\mathbf{B}$ were in $\operatorname{PR}\{\mathbf{A}\}$, then $\mathbf{B}$ would be a reflection of $\mathbf{A}$, since, having a prime number of elements, it cannot be a proper product. However, B cannot be a reflection of $\mathbf{A}$, because the range of $f^{\mathbf{B}}$ is larger than the range of $f^{\mathbf{A}}$. We can conclude that $\mathbf{B} \notin \operatorname{PR}\{\mathbf{A}\}$, thus $\operatorname{RP}\{\mathbf{A}\} \nsubseteq \operatorname{PR}\{\mathbf{A}\}$.

The following proposition shows that it would, in principle, be sufficient to consider multisorted algebras with carriers in which the same set is associated to every essential sort (i.e., $A_{s}=A_{t}$ for all $s, t \in S_{A}$ ). Every $S$-sorted algebra is reflection-equivalent to such an algebra with a single carrier set. This comes, however, at the cost of the carrier sets becoming possibly much larger than in the given algebra.

Proposition 4.4. Let $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ be an algebra of type $\tau=(S, \Sigma$, dec $)$. Then there exists an algebra $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ of type $\tau$ such that $S_{A}=S_{B}, B_{i}=B_{j}$ for all $i, j \in S_{B}$, and $\mathbf{A}$ and $\mathbf{B}$ are reflections of each other.

Proof. Let $C$ be a set of cardinality greater than or equal to the cardinality of each of the sets $A_{i}, i \in S$ (for example, we may choose $C:=\bigcup_{i \in S} A_{i}$ ). For $i \in S$, let $B_{i}:=C$ if $i \in S_{A}$ and let $B_{i}:=\emptyset$ if $i \notin S_{A}$. Then clearly $S_{A}=S_{B}$. For each $i \in S_{A}$, let $h_{i}^{\prime}: A_{i} \rightarrow C$ be an injection, and let $h_{i}: C \rightarrow A_{i}$ be a pseudoinverse of $h_{i}^{\prime}$, i.e., a map such that $h_{i}\left(h_{i}^{\prime}(a)\right)=a$ for all $a \in A_{i}$. Such maps $h_{i}^{\prime}$ and $h_{i}$ exist because $\left|A_{i}\right| \leq|C|$. Let $h:=\left(h_{s}\right)_{s \in S_{B}}, h^{\prime}:=\left(h_{s}^{\prime}\right)_{s \in S_{B}}$. Let $\mathbf{B}:=\left(B, \Sigma^{\mathbf{B}}\right)$, with $f^{\mathbf{B}}:=\left(f^{\mathbf{A}}\right)_{\left(h, h^{\prime}\right)}$ for each $f \in \Sigma$ (for notation, see Definition 4.1). Then $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of $\mathbf{A}$ by definition. Furthermore, for each $f \in \Sigma$, say, of declaration $\left(w_{1} \ldots w_{n}, s\right)$, it holds that

$$
\begin{aligned}
\left(f^{\mathbf{B}}\right)_{\left(h^{\prime}, h\right)}\left(a_{1}, \ldots, a_{n}\right) & =h_{s}\left(h_{s}^{\prime}\left(f^{\mathbf{A}}\left(h_{w_{1}}\left(h_{w_{1}}^{\prime}\left(a_{1}\right)\right), \ldots, h_{w_{n}}\left(h_{w_{n}}^{\prime}\left(a_{n}\right)\right)\right)\right)\right) \\
& =f^{\mathbf{A}}\left(a_{1}, \ldots, a_{n}\right)
\end{aligned}
$$

that is, $f^{\mathbf{A}}=\left(f^{\mathbf{B}}\right)_{\left(h^{\prime}, h\right)}$ for every $f \in \Sigma$. In other words, $\mathbf{A}$ is an $\left(h^{\prime}, h\right)-$ reflection of $\mathbf{B}$.

## 5. The Galois connection mId-Mod

It is known from the classical Birkhoff theorem for (multisorted) algebras that HSP-closed classes are equational classes. By Lemma 4.2, RP-closed classes are also HSP-closed and therefore must be characterizable by identities. In this section we prove that the "right" kind of identities for this setting are the minor identities: $\operatorname{ModmId} \mathcal{K}=\operatorname{RP} \mathcal{K}$ for every class $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$ of multisorted algebras. For the proof we need the following technical lemma, which essentially states that (under some reasonable assumptions) the validity of an identity does not change if we rename the variables and extend the set of variables.

For a term $t$, denote by $\operatorname{var}(t)$ the $S$-sorted set of variables occurring in $t$, i.e., $\operatorname{var}(t)=\left(v_{s}\right)_{s \in S}$ where $v_{s}$ is the set of variables of sort $s$ occurring in $t$.

Lemma 5.1. Let $\tau=(S, \Sigma$, dec) be a multisorted similarity type, and let $Y$ be an $S$-sorted set of variables. Let $\mu:=t_{1} \approx_{S^{\prime}} t_{2} \in M I D_{\tau}^{s}(Y)$, and assume that $S^{\prime} \subseteq S_{Y}$. Let $Y^{\prime}:=\operatorname{var}\left(t_{1}\right) \cup \operatorname{var}\left(t_{2}\right)$. Let $Z$ be an $S$-sorted set of variables such that $S^{\prime} \subseteq S_{Z}$ and there exists an injective $S$-sorted map $\delta: Y^{\prime} \rightarrow Z$. Let $t_{1}^{\prime}$ and $t_{2}^{\prime}$ be terms in $M T_{\tau}^{s}(Z)$ that are obtained from $t_{1}$ and $t_{2}$, respectively, by replacing each occurrence of a variable symbol $y \in Y^{\prime}$ by $\delta(y)$, and let $\mu^{\prime}:=t_{1}^{\prime} \approx_{S^{\prime}} t_{2}^{\prime} \in \operatorname{MID}_{\tau}^{s}(Z)$. Then for every algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ of type $\tau$, it holds that $\mathbf{A} \models \mu$ if and only if $\mathbf{A} \models \mu^{\prime}$.

Proof. Assume first that $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$. We want to show that $\mathbf{A} \models t_{1}^{\prime} \approx_{S^{\prime}} t_{2}^{\prime}$. If $S^{\prime} \nsubseteq S_{A}$, then $\mathbf{A} \models t_{1}^{\prime} \approx_{S^{\prime}} t_{2}^{\prime}$ holds vacuously (note that $S^{\prime} \subseteq S_{Z}$ ), so we may assume that $S^{\prime} \subseteq S_{A}$. Let $\beta:\left.Z\right|_{S^{\prime}} \rightarrow A$ be a valuation map, and define $\gamma:\left.Y\right|_{S^{\prime}} \rightarrow A$ by the rule

$$
\gamma_{s}(x)= \begin{cases}\beta_{s}\left(\delta_{s}(x)\right), & \text { if } x \in Y_{s}^{\prime} \\ a_{s}, & \text { if } x \in Y_{s} \backslash Y_{s}^{\prime}\end{cases}
$$

where $a_{s}$ is an arbitrary fixed element of $A_{s}$. It is clear that $\beta^{\#}\left(t_{1}^{\prime}\right)=\gamma^{\#}\left(t_{1}\right)$ and $\beta^{\#}\left(t_{2}^{\prime}\right)=\gamma^{\#}\left(t_{2}\right)$. Since $\mathbf{A}=t_{1} \approx_{S^{\prime}} t_{2}$, we have $\gamma^{\#}\left(t_{1}\right)=\gamma^{\#}\left(t_{2}\right)$. Consequently, $\beta^{\#}\left(t_{1}^{\prime}\right)=\beta^{\#}\left(t_{2}^{\prime}\right)$, and we conclude that $\mathbf{A} \models t_{1}^{\prime} \approx_{S^{\prime}} t_{2}^{\prime}$.

The proof of the converse implication is very similar. Assume that $\mathbf{A}=$ $t_{1}^{\prime} \approx_{S^{\prime}} t_{2}^{\prime}$. We want to show that $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$. We may assume that $S^{\prime} \subseteq$ $S_{A}$, for otherwise $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$ holds vacuously (note that $S^{\prime} \subseteq S_{Y}$ ). Let $\gamma:\left.Y\right|_{S^{\prime}} \rightarrow A$ be a valuation map, and define $\beta:\left.Z\right|_{S^{\prime}} \rightarrow A$ by the rule

$$
\beta_{s}(x)= \begin{cases}\gamma(y), & \text { if } x=\delta_{s}(y) \text { for } y \in Y_{s}^{\prime} \\ a_{s}, & \text { otherwise }\end{cases}
$$

where $a_{s}$ is an arbitrary fixed element of $A_{s}$. Again, it is clear that $\beta^{\#}\left(t_{1}^{\prime}\right)=$ $\gamma^{\#}\left(t_{1}\right)$ and $\beta^{\#}\left(t_{2}\right)=\gamma^{\#}\left(t_{2}\right)$. Since $\mathbf{A} \models t_{1}^{\prime} \approx_{S^{\prime}} t_{2}^{\prime}$, we have $\beta^{\#}\left(t_{1}^{\prime}\right)=\beta^{\#}\left(t_{2}^{\prime}\right)$. Consequently, $\gamma^{\#}\left(t_{1}\right)=\gamma^{\#}\left(t_{2}\right)$, and we conclude that $\mathbf{A} \models t_{1} \approx_{S^{\prime}} t_{2}$.

Theorem 5.2. Let $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$. Then $\operatorname{ModmId} \mathcal{K}=\operatorname{RPK}$.
Proof. For any set $\mathcal{J}$ of minor identities, the inclusion $\mathrm{P}(\operatorname{Mod} \mathcal{J}) \subseteq \operatorname{Mod} \mathcal{J}$ holds by the classical (multisorted) Birkhoff theorem. In order to show that $\mathrm{R}(\operatorname{Mod} \mathcal{J}) \subseteq \operatorname{Mod} \mathcal{J}$, let $\mathbf{B} \in \mathrm{R}(\operatorname{Mod} \mathcal{J})$; then $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of
some $\mathbf{A} \in \operatorname{Mod} \mathcal{J}$ for some $h: B \rightarrow A$ and $h^{\prime}: A \rightarrow B$. We need to show that $\mathbf{B} \models f_{\sigma} \approx_{S^{\prime}} g_{\pi}$ for every $f_{\sigma} \approx_{S^{\prime}} g_{\pi} \in \mathcal{J}$. For every $\beta:\left.X\right|_{S^{\prime}} \rightarrow B$, we have

$$
\begin{aligned}
\beta^{\#}\left(f_{\sigma}\right) & =f^{\mathbf{B}}(\beta \circ \sigma)=h^{\prime}\left(f^{\mathbf{A}}(h \circ \beta \circ \sigma)\right)=h^{\prime}\left(f_{\sigma}^{\mathbf{A}}(h \circ \beta)\right) \\
& =h^{\prime}\left(g_{\pi}^{\mathbf{A}}(h \circ \beta)\right)=h^{\prime}\left(g^{\mathbf{A}}(h \circ \beta \circ \pi)\right)=g^{\mathbf{B}}(\beta \circ \pi)=\beta^{\#}\left(g_{\pi}\right),
\end{aligned}
$$

where the fourth equality holds because $\mathbf{A} \vDash f_{\sigma} \quad \approx_{S^{\prime}} \quad g_{\pi}$, whence $f_{\sigma}^{\mathbf{A}}(h \circ \beta)=(h \circ \beta)^{\#}\left(f_{\sigma}\right)=(h \circ \beta)^{\#}\left(g_{\pi}\right)=g_{\pi}^{\mathbf{A}}(h \circ \beta)$. We have proved the inclusion $\mathrm{RPK} \subseteq \operatorname{Mod} \operatorname{mId} \mathcal{K}$.

It remains to show $\operatorname{Mod} \operatorname{mId} \mathcal{K} \subseteq \operatorname{RPK}$. Assume that $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ is an algebra of type $\tau=(S, \Sigma$, dec) satisfying every minor identity that holds in $\mathcal{K}$. We want to show that $\mathbf{B} \in \mathrm{RP} \mathcal{K}$.

Let $Y=\left(Y_{s}\right)_{s \in S}$ be the $S$-sorted set of variables with $Y_{s}:=B_{s} \times\{s\}$ for all $s \in S$ (i.e., we take the variable symbols to be the disjoint union of the sets $B_{s}$ ), and let

$$
\mathcal{N}:=\left\{t_{1} \approx_{S_{Y}} t_{2} \in M I D_{\tau}(Y) \mid \mathcal{K} \notin t_{1} \approx_{S_{Y}} t_{2}\right\}
$$

be the set of minor identities over $Y$ valuated on the set $S_{Y}$ that do not hold in $\mathcal{K}$.

We first consider the case $\mathcal{N} \neq \emptyset$. Then for each $\nu \in \mathcal{N}$, say $\nu=f_{\sigma} \approx_{S_{Y}}$ $g_{\pi}$ with $f \in \Sigma_{(w, s)}, \sigma:[n] \rightarrow Y$ with $|w|=n, g \in \Sigma_{(u, s)}, \pi:[m] \rightarrow Y$ with $|u|=m$, there exists a counterexample $\mathbf{A}_{\nu}=\left(A_{\nu}, \Sigma^{\mathbf{A}_{\nu}}\right) \in \mathcal{K}$ that does not satisfy $\nu$. This means that there exists a valuation map $\beta_{\nu}:\left.Y\right|_{S_{Y}} \rightarrow A_{\nu}$ such that $f^{\mathbf{A}_{\nu}}\left(\beta_{\nu} \circ \sigma\right) \neq g^{\mathbf{A}_{\nu}}\left(\beta_{\nu} \circ \pi\right)$; hence $S_{Y} \subseteq S_{A_{\nu}}$. Now let $\mathbf{P}:=\prod_{\nu \in \mathcal{N}} \mathbf{A}_{\nu}$ be the product of all the counterexamples. Then $\mathbf{P}=\left(P, \Sigma^{\mathbf{P}}\right)$ and $S_{P}=$ $\bigcap_{\nu \in \mathcal{N}} S_{A_{\nu}} \supseteq S_{Y}$. Note that $\mathbf{P} \in \mathrm{PK}$.

For every $y \in Y_{s}$ with $s \in S_{Y}$, the tuple $\bar{y}:=\left(\beta_{\nu}(y)\right)_{\nu \in \mathcal{N}}$ is an element of $P_{s}$. Let $h=\left(h_{s}\right)_{s \in S_{Y}}$ where each $h_{s}: B_{s} \rightarrow P_{s}$ is the map $b \mapsto \overline{(b, s)}$ (note that $\left.(b, s) \in Y_{s}\right)$. For each $s \in S_{Y}$, let $Z_{s}:=\left\{f^{\mathbf{P}}\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right) \mid f \in\right.$ $\left.\Sigma_{(w, s)},\left(y_{1}, \ldots, y_{n}\right) \in Y_{w}\right\} \subseteq P_{s}$. Now we shall define maps $h_{s}^{\prime}: P_{s} \rightarrow B_{s}$, for $s \in S_{B}$ and we set $h^{\prime}=\left(h_{s}^{\prime}\right)_{s \in S_{B}}$ such that $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of $\mathbf{P}$. For any $z \in P_{s} \backslash Z_{s}$, the value $h_{s}^{\prime}(z)$ can be chosen arbitrarily in $B_{s}$. For an element $f^{\mathbf{P}}\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right) \in Z_{s}$, where $y_{i}:=\left(b_{i}, w_{i}\right)$ with $b_{i} \in B_{w_{i}}(i=1, \ldots, n)$, define $h_{s}^{\prime}\left(f^{\mathbf{P}}\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right)\right):=f^{\mathbf{B}}\left(b_{1}, \ldots, b_{n}\right)$ according to the reflection property (cf. Definition 4.1).

We have to verify that $h_{s}^{\prime}$ is well defined. Suppose, to the contrary, that $f^{\mathbf{P}}\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right)=g^{\mathbf{P}}\left(\bar{z}_{1}, \ldots, \bar{z}_{m}\right)$ but $f^{\mathbf{B}}\left(b_{1}, \ldots, b_{n}\right) \neq g^{\mathbf{B}}\left(c_{1}, \ldots, c_{m}\right)$ for some $f \in \Sigma_{(w, s)}, g \in \Sigma_{(u, s)},\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right) \in P_{w},\left(\bar{z}_{1}, \ldots, \bar{z}_{m}\right) \in P_{u}$, where $y_{i}:=$ $\left(b_{i}, w_{i}\right)$ for $b_{i} \in B_{w_{i}}(i=1, \ldots, n)$ and $z_{i}:=\left(c_{i}, u_{i}\right)$ for $c_{i} \in B_{u_{i}}(i=1, \ldots, m)$. From the latter it follows that $\mathbf{B}$ does not satisfy the minor identity $\mu:=$ $f_{\sigma} \approx_{S_{Y}} g_{\pi} \in \operatorname{MID}_{\tau}^{s}(Y)$, where $\sigma:[n] \rightarrow Y, i \mapsto y_{i}$ and $\pi:[m] \rightarrow Y, i \mapsto z_{i}$. Write $Y^{\prime}:=\operatorname{var}\left(f_{\sigma}\right) \cup \operatorname{var}\left(g_{\pi}\right)$, and let $\delta: Y^{\prime} \rightarrow X$ be an injective map to the set $X$ of standard variables. Let $\mu^{\prime}:=f_{\sigma}^{\prime} \approx_{S_{Y}} g_{\pi}^{\prime} \in M I D_{\tau}^{s}(X)$, where $f_{\sigma}^{\prime}$ and $g_{\pi}^{\prime}$ are the minor terms in $M T_{\tau}^{s}(X)$ that are obtained from $f_{\sigma}$ and $g_{\pi}$ by replacing each occurrence of a variable symbol $y \in Y^{\prime}$ by $\delta(y)$. Since $\mathbf{B} \not \vDash \mu$, it follows from Lemma 5.1 that $\mathbf{B} \not \vDash \mu^{\prime}$. Since $\mathbf{B} \in \operatorname{ModmId} \mathcal{K}$, this implies $\mathcal{K} \not \vDash \mu^{\prime}$, whence $\mathcal{K} \not \vDash \mu$ by Lemma 5.1. Therefore $\mu \in \mathcal{N}$. Then, by the definition of $\mathbf{A}_{\mu}$ and
$\beta_{\mu}$, we have $f^{\mathbf{A}_{\mu}}\left(\beta_{\mu}\left(y_{1}\right), \ldots \beta_{\mu}\left(y_{n}\right)\right) \neq g^{\mathbf{A}_{\mu}}\left(\beta_{\mu}\left(z_{1}\right), \ldots, \beta_{\mu}\left(z_{m}\right)\right)$. This means that the $\mu$-th coordinates of the tuples $f^{\mathbf{P}}\left(\bar{y}_{1}, \ldots, \bar{y}_{n}\right)$ and $g^{\mathbf{P}}\left(\bar{z}_{1}, \ldots, \bar{z}_{m}\right)$ are different, contradicting our assumption. We conclude that $\mathbf{B} \in \mathrm{RP} \subseteq R P \mathcal{K}$.

Finally, we must consider the case $\mathcal{N}=\emptyset$, i.e., $\mathcal{K}$ satisfies every minor identity $f_{\sigma} \approx_{S_{Y}} g_{\pi}$ where $f_{\sigma}, g_{\pi} \in M T_{\tau}^{s}(Y)$ for some $s \in S$. Let $f \in \Sigma_{(w, s)}$, $g \in \Sigma_{(u, s)},\left(a_{1}, \ldots, a_{n}\right) \in B_{w},\left(b_{1}, \ldots, b_{m}\right) \in B_{u}$, and define $\sigma:[n] \rightarrow Y$, $\sigma(i)=\left(a_{i}, w_{i}\right)$ and $\pi:[m] \rightarrow Y, \pi(i)=\left(b_{i}, u_{i}\right)$. Let $\beta:\left.Y\right|_{S_{Y}} \rightarrow B, \beta((x, s))=$ $x$. Since $\mathbf{B} \models f_{\sigma} \approx_{S_{Y}} g_{\pi}$, we have $\beta^{\#}\left(f_{\sigma}\right)=\beta^{\#}\left(g_{\pi}\right)$. Therefore,
$f^{\mathbf{B}}\left(a_{1}, \ldots, a_{n}\right)=f^{\mathbf{B}}(\beta \circ \sigma)=\beta^{\#}\left(f_{\sigma}\right)=\beta^{\#}\left(g_{\pi}\right)=g^{\mathbf{B}}(\beta \circ \pi)=g^{\mathbf{B}}\left(b_{1}, \ldots, b_{n}\right)$.
Since the choice of $f, g$ and the $a_{i}$ and $b_{i}$ was arbitrary, it follows that there exist constants $c_{s} \in B_{s}\left(s \in S_{B}\right)$ such that every function $f^{\mathbf{B}}$ of sort $s$ is constant $c_{s}$. Let $\mathbf{D}=\left(D, \Sigma^{\mathbf{D}}\right):=\Pi \emptyset$ be the empty product of algebras in $\mathcal{K}$. Then $\mathbf{D} \in \mathrm{PK}$. As noted in Definition $2.12, D_{s}$ equals the singleton $\{\emptyset\}$ for every $s \in S$. Define $h=\left(h_{s}\right)_{s \in S_{B}}$ and $h^{\prime}=\left(h_{s}^{\prime}\right)_{s \in S_{B}}$ where $h_{s}: B_{s} \rightarrow D_{s}$, $b \mapsto \emptyset$ for all $b \in B_{s}$ and $h_{s}^{\prime}: D_{s} \rightarrow B_{s}, \emptyset \mapsto c_{s}$. Then for each $f \in \Sigma_{(w, s)}$ with $w=w_{1} \ldots w_{n}$, we have $f^{\mathbf{B}}\left(a_{1}, \ldots, a_{n}\right)=c_{s}=h_{s}^{\prime}\left(f^{\mathbf{D}}\left(h_{w_{1}}\left(a_{1}\right), \ldots, h_{w_{n}}\left(a_{n}\right)\right)\right)$. Therefore $\mathbf{B}$ is an $\left(h, h^{\prime}\right)$-reflection of $\mathbf{D}$. Thus $\mathbf{B} \in \mathrm{RPK}$.

Theorem 5.2 characterizes the closed classes of algebras corresponding to the Galois connection mId-Mod as the classes that are closed under reflections and direct products. Next we describe the Galois closed classes of minor identities in terms of closure conditions, which are analogous to the classical characterization of equational theories as fully invariant congruences of free algebras. In order to state the result, we make use of the canonical trivial algebra $\mathbf{S}$ of type $\tau$ defined in Example 2.13(2). Recall also Lemma 2.14 and the notational shorthand $\left\langle S^{\prime}\right\rangle_{\mathbf{S}}$ involving subalgebras of $\mathbf{S}$ introduced in the paragraph preceding Lemma 2.14.

Theorem 5.3. Let $\mathcal{J} \subseteq M I D_{\tau}(X)$ be a set of minor identities of type $\tau=$ $(S, \Sigma, \operatorname{dec})$ over $X$. Then $\mathcal{J}=\operatorname{mId} \operatorname{Mod} \mathcal{J}$ if and only if $\mathcal{J}$ satisfies the following conditions:
(1) For every $S^{\prime} \subseteq S$ and $s \in S$, the set

$$
\mathcal{J}_{s}^{\left(S^{\prime}\right)}:=\left\{\left(f_{\sigma}, g_{\pi}\right) \mid f_{\sigma} \approx_{S^{\prime}} g_{\pi} \in \mathcal{J}, \operatorname{sort}(f)=\operatorname{sort}(g)=s\right\}
$$

is an equivalence relation on $M T_{\tau}^{s}\left(\left.X\right|_{S^{\prime}}\right)$.
(2) If $t_{1} \approx_{S^{\prime}} t_{2} \in \mathcal{J}$ and $S^{\prime} \subseteq S^{\prime \prime}$, then $t_{1} \approx_{S^{\prime \prime}} t_{2} \in \mathcal{J}$ ("sort expansion").
(3) If $t_{1} \approx_{S^{\prime}} t_{2} \in M I D_{\tau}(X)$ and $t_{1} \approx_{\left\langle S^{\prime}\right\rangle_{\mathrm{S}}} t_{2} \in \mathcal{J}$, then $t_{1} \approx_{S^{\prime}} t_{2} \in \mathcal{J}$ ("sort contraction").
(4) If $f_{\sigma} \approx_{S^{\prime}} g_{\pi} \in \mathcal{J}$, then $f_{\lambda \circ \sigma} \approx_{S^{\prime}} g_{\lambda \circ \pi} \in \mathcal{J}$ for all $\lambda:\left.\left.X\right|_{S^{\prime}} \rightarrow X\right|_{S^{\prime}}$ ("closure under minors").

Proof. We will prove the equivalent statement that a set $\mathcal{J} \subseteq M I D_{\tau}(X)$ of minor identities is of the form $\mathcal{J}=\operatorname{mId} \mathcal{K}$ for some set $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$ of algebras if and only if $\mathcal{J}$ satisfies conditions (1)-(4).

Assume first that $\mathcal{J}=\operatorname{mId} \mathcal{K}$ for some $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$. It is easy to verify that condition (1) holds. Condition (2) holds by Lemma 3.4.

Let $t_{1} \approx_{S^{\prime}} t_{2} \in \operatorname{MID}_{\tau}(X)$ and assume that $t_{1} \approx_{\left\langle S^{\prime}\right\rangle_{\mathrm{s}}} t_{2} \in \mathcal{J}$. Suppose, to the contrary, that $t_{1} \approx_{S^{\prime}} t_{2} \notin \mathcal{J}$. Then there exists an algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right) \in \mathcal{K}$ such that $\mathbf{A} \not \vDash t_{1} \approx_{S^{\prime}} t_{2}$, i.e., there exists a valuation $\beta:\left.X\right|_{S^{\prime}} \rightarrow A$ such that $\beta^{\#}\left(t_{1}\right) \neq \beta^{\#}\left(t_{2}\right)$. This is possible only if $S^{\prime} \subseteq S_{A}$, which implies $\left\langle S^{\prime}\right\rangle_{\mathbf{S}} \subseteq\left\langle S_{A}\right\rangle_{\mathbf{S}}=S_{A}$ by Lemma 2.14. Consequently, there exist maps $\left.X\right|_{\left\langle S^{\prime}\right\rangle_{\mathrm{s}}} \rightarrow A$, and for any extension $\gamma:\left.X\right|_{\left\langle S^{\prime}\right\rangle_{\mathrm{s}}} \rightarrow A$ of $\beta$, it holds that $\gamma^{\#}\left(t_{1}\right)=\beta^{\#}\left(t_{1}\right) \neq \beta^{\#}\left(t_{2}\right)=\gamma^{\#}\left(t_{2}\right)$. Therefore $\mathbf{A} \notin t_{1} \approx_{\left\langle S^{\prime}\right\rangle_{\mathbf{S}}} t_{2}$, so $t_{1} \approx_{\left\langle S^{\prime}\right\rangle_{\mathrm{S}}} t_{2} \notin \mathcal{J}$, a contradiction. We conclude that condition (3) holds.

Let then $\left(f_{\sigma}, g_{\pi}\right) \in \mathcal{J}_{s}^{\left(S^{\prime}\right)}$ and $\lambda:\left.\left.X\right|_{S^{\prime}} \rightarrow X\right|_{S^{\prime}}$. Then for every valuation $\operatorname{map} \beta:\left.X\right|_{S^{\prime}} \rightarrow A$, we have $\beta^{\#}\left(f_{\lambda \circ \sigma}\right)=f(\beta \circ \lambda \circ \sigma)=(\beta \circ \lambda)^{\#}\left(f_{\sigma}\right)=$ $(\beta \circ \lambda)^{\#}\left(g_{\pi}\right)=g(\beta \circ \lambda \circ \pi)=\beta^{\#}\left(g_{\lambda \circ \pi}\right)$. Consequently, $\left(f_{\lambda \circ \sigma}, g_{\lambda \circ \pi}\right) \in \mathcal{J}_{s}^{\left(S^{\prime}\right)}$, that is, condition (4) holds.

For the converse implication, assume that $\mathcal{J}$ satisfies conditions (1)-(4). For each $\delta=f_{\sigma} \approx_{S^{\prime}} g_{\pi} \in M I D_{\tau}(X) \backslash \mathcal{J}$, we will construct an algebra $\mathbf{F}_{\delta} \in$ $\operatorname{Alg}(\tau)$ such that $\mathbf{F}_{\delta} \neq \mathcal{J}$ but $\mathbf{F}_{\delta} \not \vDash \delta$. Taking $\mathcal{K}$ to be the set of all such "separating" algebras $\mathbf{F}_{\delta}$, for every $\delta \in \operatorname{MID}_{\tau}(X) \backslash \mathcal{J}$, we have $\mathcal{J}=\operatorname{mId} \mathcal{K}$.

Let $\delta=f_{\sigma} \approx_{S^{\prime}} g_{\pi} \in M I D_{\tau}(X) \backslash \mathcal{J}$. Let $S^{\prime \prime}:=\left\langle S^{\prime}\right\rangle_{\mathbf{S}}$. Define the algebra $\mathbf{F}_{\delta}=\left(F, \Sigma^{\mathbf{F}_{\delta}}\right)$ of type $\tau$ as follows. Let $q:=\operatorname{sort}(f)=\operatorname{sort}(g)$. For $s \in S$, let

$$
F_{s}:= \begin{cases}\emptyset, & \text { if } s \in S \backslash S^{\prime \prime} \\ X_{s}, & \text { if } s \in S^{\prime \prime} \backslash\{q\} \\ X_{q} \cup M T_{\tau}^{q}\left(\left.X\right|_{S^{\prime \prime}}\right) / \mathcal{J}_{q}^{\left(S^{\prime \prime}\right)}, & \text { if } s=q\end{cases}
$$

Note that the quotient $M T_{\tau}^{q}\left(\left.X\right|_{S^{\prime \prime}}\right) / \mathcal{J}_{q}^{\left(S^{\prime \prime}\right)}$ appearing in the definition of $F_{q}$ is a well-defined object, because $\mathcal{J}_{q}^{\left(S^{\prime \prime}\right)}$ is an equivalence relation on $M T_{\tau}^{q}\left(\left.X\right|_{S^{\prime \prime}}\right)$ by condition (1). We will denote the $\mathcal{J}_{q}^{\left(S^{\prime \prime}\right)}$-equivalence class of a term $t \in$ $M T_{\tau}^{q}\left(\left.X\right|_{S^{\prime \prime}}\right)$ by $[t]$. For $d \in \Sigma_{(w, s)}$, the operation $d^{\mathbf{F}_{\delta}}: F_{w} \rightarrow F_{s}$ is defined by the following rules (for notation, see Definition 2.7). If $\operatorname{inp}(d) \nsubseteq S^{\prime \prime}$, then $d^{\mathbf{F}_{\delta}}=\emptyset$. If $\operatorname{inp}(d) \subseteq S^{\prime \prime}$ and $s \neq q$, then $d^{\mathbf{F}_{\delta}}(\alpha)=x_{1}^{s}$ for all $\alpha \in F_{w}$. If $\operatorname{inp}(d) \subseteq S^{\prime \prime}$ and $s=q$, then $d^{\mathbf{F}_{\delta}}(\alpha)=\left[d_{\varphi \circ \alpha}\right]$, where $\varphi: F \rightarrow X$ is given by $x_{i}^{s} \mapsto x_{i}^{s}$ for any $\left.x_{i}^{s} \in X\right|_{S^{\prime \prime}}$ and $[t] \mapsto x_{1}^{q}$ for any $[t] \in M T_{\tau}^{q}\left(\left.X\right|_{S^{\prime \prime}}\right) / \mathcal{J}_{q}^{\left(S^{\prime \prime}\right)}$. Note that $S_{F}=S^{\prime \prime}=\left\langle S^{\prime}\right\rangle_{\mathbf{S}}$, from which it follows by Lemma 2.14 that the declaration of every $d \in \Sigma$ is reasonable in $F$, so the algebra $\mathbf{F}_{\delta}$ is well defined.

We show first that $\mathbf{F}_{\delta} \not \vDash \delta$. Let $\beta:\left.X\right|_{S^{\prime}} \rightarrow F$ be the inclusion map $x \mapsto x$. Then $\beta^{\#}\left(f_{\sigma}\right)=f^{\mathbf{F}_{\delta}}(\beta \circ \sigma)=\left[f_{\varphi \circ \beta \circ \sigma}\right]=\left[f_{\sigma}\right]$ and $\beta^{\#}\left(g_{\pi}\right)=g^{\mathbf{F}_{\delta}}(\beta \circ \pi)=$ $\left[g_{\varphi \circ \beta \circ \pi}\right]=\left[g_{\pi}\right]$. Since $f_{\sigma} \approx_{S^{\prime \prime}} g_{\pi} \notin \mathcal{J}$ by condition (3), we have $\left[f_{\sigma}\right] \neq\left[g_{\pi}\right]$, and we conclude that $\mathbf{F}_{\delta} \not \models \delta$.

Finally we show that $\mathbf{F}_{\delta} \models \mathcal{J}$. Let $d_{\rho} \approx_{T} d_{\rho^{\prime}}^{\prime} \in \mathcal{J}$. If sort $(d) \neq q$, then $\mathbf{F}_{\delta}$ obviously satisfies the identity $d_{\rho} \approx_{T} d_{\rho^{\prime}}^{\prime}$. Assume that $\operatorname{sort}(d)=\operatorname{sort}\left(d^{\prime}\right)=q$. If $T \nsubseteq S^{\prime \prime}=S_{F}$, then $\mathbf{F}_{\delta} \models d_{\rho} \approx_{T} d_{\rho^{\prime}}^{\prime}$ holds vacuously. Thus we may assume that $T \subseteq S^{\prime \prime}$. Let $\beta:\left.X\right|_{T} \rightarrow F$. By condition (2) we have $d_{\rho} \approx_{S^{\prime \prime}} d_{\rho^{\prime}}^{\prime} \in \mathcal{J}$, and by condition (4) we have $d_{\varphi \circ \beta \circ \rho} \approx_{S^{\prime \prime}} d_{\varphi \circ \beta \circ \rho^{\prime}}^{\prime} \in \mathcal{J}$. Then

$$
\beta^{\#}\left(d_{\rho}\right)=d^{\mathbf{F}_{\delta}}(\beta \circ \rho)=\left[d_{\varphi \circ \beta \circ \rho}\right]=\left[d_{\varphi \circ \beta \circ \rho^{\prime}}^{\prime}\right]=d^{\prime} \mathbf{F}_{\delta}\left(\beta \circ \rho^{\prime}\right)=\beta^{\#}\left(d_{\rho^{\prime}}^{\prime}\right)
$$

Thus $\mathbf{F}_{\delta}$ satisfies $d_{\rho} \approx_{T} d_{\rho^{\prime}}^{\prime}$. We conclude that $\mathbf{F}_{\delta} \models \mathcal{J}$.

Remark 5.4. Theorem 5.2 was proved in the case of usual one-sorted algebras by Barto et al. [2, Corollary 5.4]. As for Theorem 5.3, sort expansion and sort contraction play no role when $|S|=1$, and the theorem reduces to the description of closed sets of minor identities given by Čupona and Markovski [5, Theorem 2.1].

## 6. How are HSP and RP related?

It is clear from Lemma 4.2 that every RP-closed class is also HSP-closed. The converse is not true, and we would like to describe which HSP-closed classes are not RP-closed. For similarity types of a special form that does not admit compositions of terms, we can provide a complete description: the HSP-closed classes that are not RP-closed are somewhat "trivial" in this case. For arbitrary similarity types, a characterization eludes us.

Definition 6.1. A multisorted similarity type $\tau=(S, \Sigma$, dec) is non-composable if $S$ can be partitioned into two subsets $I$ and $O$ such that for every $f \in \Sigma$, it holds that $\operatorname{inp}(f) \subseteq I$ and $\operatorname{sort}(f) \in O$. A type is composable if it is not non-composable.

Examples of non-composable similarity types include all types of 2-algebras, as introduced in Example 2.13(5).

Definition 6.2. The height of a term $t$, denoted by $h(t)$, is defined inductively as follows:
(1) Variable symbols have height 0 , i.e., $h(x)=0$ for all $x \in Y_{s}, s \in S$.
(2) If $t=f$, where $f \in \Sigma$ and $\operatorname{dec}(f)=(\varepsilon, s)$ (constant), then $h(t)=1$.
(3) If $t=f\left(t_{1}, \ldots, t_{n}\right)$, where $f \in \Sigma, \operatorname{dec}(f)=\left(w_{1} \ldots w_{n}, s\right), n \geq 1$, and $t_{1}, \ldots, t_{n}$ are terms, then $h(t)=\max \left(h\left(t_{1}\right), \ldots, h\left(t_{n}\right)\right)+1$.
Theorem 6.3. Let $\tau=(S, \Sigma, \mathrm{dec})$ be a non-composable similarity type, and let $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$ be an HSP-closed class of algebras. Then the following are equivalent.
(1) $\mathcal{K}$ is R -closed.
(2) For all $s \in S, \mathcal{K} \notin x_{1}^{s} \approx_{S} x_{2}^{s}$.
(3) For all $s \in S$, there exists $\mathbf{A} \in \mathcal{K}$ such that $S_{A}=S$ and $\left|A_{s}\right| \geq 2$.

Proof. (1) $\Longrightarrow$ (3) Assume that $\mathcal{K}$ is R-closed. Since HSPK $=\mathcal{K}$, we have $\mathbf{P}=\left(P, \Sigma^{\mathbf{P}}\right):=\Pi \emptyset \in \mathcal{K}$. Let $A=\left(A_{s}\right)_{s \in S}$ be an $S$-sorted set with $\left|A_{s}\right| \geq 2$ for all $s \in S$, let $h: A \rightarrow P, h^{\prime}: P \rightarrow A$ be arbitrary maps, and let $\mathbf{A}$ be the ( $h, h^{\prime}$ )-reflection of $\mathbf{P}$; hence $\mathbf{A} \in \mathrm{RK} \subseteq \mathcal{K}$. The required condition is then satisfied by $\mathbf{A}$ for every $s \in S$.
$(3) \Longrightarrow(2)$ An algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ with $S_{A}=S$ and $\left|A_{s}\right| \geq 2$ clearly does not satisfy the identity $x_{1}^{s} \approx_{S} x_{2}^{s}$. Since $\mathcal{K}$ contains such an algebra for every $s \in S$, it follows that $\mathcal{K} \not \vDash x_{1}^{s} \approx_{S} x_{2}^{s}$ for all $s \in S$.
(2) $\Longrightarrow(1)$ Assume that $\mathcal{K} \not \models x_{1}^{s} \approx_{S} x_{2}^{s}$ for all $s \in S$. Let $\mathcal{J}:=\operatorname{Id} \mathcal{K}$; since $\mathcal{K}$ is HSP-closed, we have $\mathcal{K}=\operatorname{Mod} \mathcal{J}$. We need to show that every identity in $\mathcal{J}$ is satisfied by all reflections of every algebra in $\mathcal{K}$. Let $\mu:=t_{1} \approx_{S^{\prime}} t_{2} \in \mathcal{J}$.

If $t_{1}=t_{2}$, then $\mu$ is satisfied by every algebra in $\operatorname{Alg}(\tau)$. Therefore we may assume that $t_{1} \neq t_{2}$. Since $\tau$ is non-composable, the terms $t_{1}$ and $t_{2}$ have height at most 1. Consider the different possibilities. If $h\left(t_{1}\right)=h\left(t_{2}\right)=0$, then $\mu=x_{i}^{s} \approx_{S^{\prime}} x_{j}^{s}$ with $i \neq j$. It is clear that then $\mathcal{K} \models x_{1}^{s} \approx_{S^{\prime}} x_{2}^{s}$, from which it follows by Lemma 3.4 that $\mathcal{K} \models x_{1}^{s} \approx_{S} x_{2}^{s}$. This contradicts our assumption and shows that this case is impossible. If $h\left(t_{1}\right)=h\left(t_{2}\right)=1$, then $\mu$ is a minor identity, and $\mathrm{RK} \models \mu$ holds by Theorem 5.2. Finally, if $h\left(t_{1}\right) \neq h\left(t_{2}\right)$, say $h\left(t_{1}\right)=1$ and $h\left(t_{2}\right)=0$, then $\mu=f_{\sigma} \approx_{S^{\prime}} x_{i}^{s}$ for some $f \in \Sigma_{(w, s)}$. Note that $s \notin \operatorname{inp}(f)$, because $\tau$ is non-composable. Then in fact $f_{\sigma} \approx_{S^{\prime}} x_{j}^{s} \in \mathcal{J}$ for every $j \in \mathbb{N}$. By symmetry and transitivity, we get $x_{1}^{s} \approx_{S^{\prime}} x_{2}^{s} \in \mathcal{J}$. As above, this leads to a contradiction and shows that this last case is impossible. We conclude that $\mathrm{RK} \subseteq \mathcal{K}$.

Remark 6.4. Note that the proofs of the implications $(1) \Longrightarrow(3) \Longrightarrow(2)$ of Theorem 6.3 did not rely on the assumption that $\tau$ is non-composable, and it is also easy to see that (2) and (3) are actually equivalent for every type $\tau$ (whether it is composable or not). Hence the crucial part is $(3) \Longrightarrow(1)$ (or, equivalently, $(2) \Longrightarrow(1))$, and we will prove in the next proposition that this implication actually characterizes non-composable types.

Proposition 6.5. Let $\tau$ be a similarity type. If for every HSP-closed class $\mathcal{K} \subseteq$ $\operatorname{Alg}(\tau)$, the conditions (1)-(3) of Theorem 6.3 are equivalent, then $\tau$ is noncomposable.

Proof. We prove the contrapositive. Assume that $\tau$ is composable. Then there exist $w=w_{1} \ldots w_{n}, u=u_{1} \ldots u_{m} \in W(S), s \in S$ and $i \in[n]$ such that $\Sigma_{(w, s)} \neq \emptyset$ and $\Sigma_{\left(u, w_{i}\right)} \neq \emptyset$. Without loss of generality, we may assume that $i=1$. Let $f \in \Sigma_{(w, s)}$ and $g \in \Sigma_{\left(u, w_{1}\right)}$, let $S^{\prime}:=\operatorname{inp}(f) \cup \operatorname{inp}(g)$, and let

$$
\mu:=f\left(g\left(y_{1}, \ldots, y_{m}\right), z_{2}, \ldots, z_{n}\right) \approx_{S^{\prime}} f\left(g\left(y_{1}^{\prime}, \ldots, y_{m}^{\prime}\right), z_{2}^{\prime}, \ldots, z_{n}^{\prime}\right)
$$

where $y_{1}, \ldots, y_{m}, z_{2}, \ldots, z_{n}, y_{1}^{\prime}, \ldots, y_{m}^{\prime}, z_{2}^{\prime}, \ldots, z_{n}^{\prime}$ are pairwise distinct variables with $y_{i}, y_{i}^{\prime} \in X_{u_{i}}, z_{i}, z_{i}^{\prime} \in X_{w_{i}}$, and let $\mathcal{K}:=\operatorname{Mod} \mu$.

Define an $S$-sorted algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$ of type $\tau$ as follows. The carrier is $A=\left(A_{s}\right)_{s \in S}$ with $A_{s}:=\{0,1,2\}$ for all $s \in S$. Define $f^{\mathbf{A}}: A_{w} \rightarrow A_{s}$ and $g^{\mathbf{A}}: A_{u} \rightarrow A_{w_{1}}$ by the rules

$$
\begin{array}{ll}
f^{\mathbf{A}}\left(a_{1}, \ldots, a_{n}\right):=\psi\left(a_{1}\right), & \\
\text { where } \psi: 0 \mapsto 0,1 \mapsto 0,2 \mapsto 2, \\
g^{\mathbf{A}}\left(a_{1}, \ldots, a_{m}\right):=\varphi\left(a_{1}\right), & \\
\text { where } \varphi: 0 \mapsto 0,1 \mapsto 1,2 \mapsto 0 .
\end{array}
$$

The other operations in $\Sigma^{\mathbf{A}}$ can be defined in an arbitrary way. Since $\operatorname{Im} g=$ $\{0,1\}$, we have

$$
f^{\mathbf{A}}\left(g^{\mathbf{A}}\left(a_{1}, \ldots, a_{m}\right), c_{2}, \ldots, c_{n}\right)=\psi\left(g^{\mathbf{A}}\left(a_{1}, \ldots, a_{m}\right)\right)=0
$$

for all $a_{1}, \ldots, a_{m}, c_{2}, \ldots, c_{n} \in\{0,1,2\}$. Hence $\mathcal{A} \models \mu$, i.e., $\mathbf{A} \in \mathcal{K}$. Thus condition (3) of Theorem 6.3 is satisfied with $\mathbf{A}$ for every $s \in S$.

Let $B:=A$, i.e., $B_{s}:=A_{s}=\{0,1,2\}(s \in S)$. Let $\mathbf{B}=\left(B, \Sigma^{\mathbf{B}}\right)$ be the $\left(h, h^{\prime}\right)$-reflection of $\mathbf{A}$ with $h_{s}: B_{s} \rightarrow A_{s}, 0 \mapsto 2,1 \mapsto 1,2 \mapsto 0$ and $h_{s}^{\prime}: A_{s} \rightarrow B_{s}, x \mapsto x$. Then

$$
\begin{aligned}
& f^{\mathbf{B}}\left(b_{1}, \ldots, b_{n}\right)=h^{\prime}\left(f^{\mathbf{A}}\left(h\left(b_{1} 1\right), \ldots, h\left(b_{n}\right)\right)\right)=\psi\left(h\left(b_{1}\right)\right)= \begin{cases}2, & \text { if } b_{1}=0 \\
0, & \text { if } b_{1}=1, \\
0, & \text { if } b_{1}=2,\end{cases} \\
& g^{\mathbf{B}}\left(b_{1}, \ldots, b_{m}\right)=h^{\prime}\left(g^{\mathbf{A}}\left(h\left(b_{1}\right), \ldots, h\left(b_{m}\right)\right)\right)=\varphi\left(h\left(b_{1}\right)\right)= \begin{cases}0, & \text { if } b_{1}=0 \\
1, & \text { if } b_{1}=1, \\
0, & \text { if } b_{1}=2 .\end{cases}
\end{aligned}
$$

Consequently,

$$
f^{\mathbf{B}}\left(g^{\mathbf{B}}\left(b_{1}, \ldots, b_{m}\right), c_{2}, \ldots, c_{n}\right)=\psi\left(h\left(\varphi\left(h\left(b_{1}\right)\right)\right)\right)= \begin{cases}2, & \text { if } b_{1}=0 \\ 0, & \text { if } b_{1}=1 \\ 2, & \text { if } b_{1}=2\end{cases}
$$

Hence $f^{\mathbf{B}}\left(g^{\mathbf{B}}\left(b_{1}, \ldots, b_{m}\right), c_{2}, \ldots, c_{n}\right) \neq f^{\mathbf{B}}\left(g^{\mathbf{B}}\left(b_{1}^{\prime}, \ldots, b_{m}^{\prime}\right), c_{2}^{\prime}, \ldots, c_{n}^{\prime}\right)$ if $b_{1}=0$ and $b_{1}^{\prime}=1$. Therefore $\mathbf{B} \not \models \mu$, i.e., $\mathbf{B} \notin \mathcal{K}$, so $\mathcal{K}$ is not R -closed, that is, condition (1) of Theorem 6.3 does not hold. We conclude that conditions (1)(3) of Theorem 6.3 are not equivalent for $\mathcal{K}$.

Remark 6.6. According to Theorem 6.3, the only HSP-varieties of a non-composable type $\tau$ that are not RP-varieties are the ones satisfying an identity of the form $x_{1}^{s} \approx_{S} x_{2}^{s}$ for some $s \in S$. Using identities of this form, we can express the fact that a sort $s$ is trivial in an algebra $\mathbf{A}=\left(A, \Sigma^{\mathbf{A}}\right)$, in the sense that $A_{s}$ is empty or a singleton.

## References

[1] Adámek, J., Rosický, J., Vitale, E.M.: Birkhoff's variety theorem in many sorts. Algebra Univers. 68, 39-42 (2012)
[2] Barto, L., Opršal, J., Pinsker, M.: The wonderland of reflections. Israel J. Math. 223, 363-398 (2018)
[3] Birkhoff, G., Lipson, J.D.: Heterogeneous algebras. J. Combin. Theory 8, 115133 (1970)
[4] Čupona, G., Markovski, S.: Free objects in primitive varieties of $n$-groupoids. Publ. Inst. Math. (Beograd) (N.S.) 57(71), 147-154 (1995)
[5] Čupona, Ǵ., Markovski, S.: Primitive varieties of algebras. Algebra Univers. 38, 226-234 (1997)
[6] Čupona, G., Markovski, S., Popeska, Ž.: Primitive $n$-identities. In: Contributions to General Algebra, vol. 9, pp. 107-116. Hölder-Pichler-Tempsky, Vienna (1995)
[7] Higgins, P.J.: Algebras with a scheme of operators. Math. Nachr. 27, 115-132 (1963)
[8] Manca, V., Salibra, A.: Soundness and completeness of the Birkhoff equational calculus for many-sorted algebras with possibly empty carrier sets. Theoret. Comput. Sci. 94, 101-124 (1992)
[9] Taylor, W.: Characterizing Mal'cev conditions. Algebra Univers. 3, 351-397 (1973)
[10] Wechler, W.: Universal Algebra for Computer Scientists. EATCS Monogr. Theoret. Comput. Sci., vol. 25. Springer, Berlin (1992)

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