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Posets of minors of functions in multiple-valued logic

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***Abstract*—We study the structure of the partially ordered set of minors of an arbitrary function of several variables. We give an abstract characterization of such "minor posets" in terms of colorings of partition lattices, and we also present infinite families of examples as well as constructions that can be used to build new minor posets.**

I. Introduction

Traditionally, multiple-valued logic deals with truth functions of the form $f\colon A^n \to A$, where A is a finite set of truth values. We slightly generalize this setup by allowing the domain and the codomain of the function to be different sets, and we do not assume that they are finite sets. We investigate the partially ordered set of functions that can be obtained from an arbitrary n-variable function $f\colon A^n \to B$ via identifications of variables. Such functions are called minors of f, and they are naturally partially ordered, since some minors of f can be also minors of each other; we shall use the symbol $\downarrow f$ to denote this poset of minors of the function f. In fact, the minor relation is a partial order on the set $\mathcal{F}_{AB}$ of all functions of several variables from A to B, if we regard functions differing only in inessential variables and/or in the order of their variables as equivalent. Our goal is to characterize the principal ideals $\downarrow f$ of this poset up to isomorphism (see Figure 2). We give the precise definitions in Section II; here we present only an illustrative example.

Example 1. *Let us consider the function $f(x_1, x_2, x_3, x_4) = x_1x_3 + x_2 + x_4$ over the 2-element field. Identifying the first two variables, we obtain the minor $g(x, y, z) = f(x, x, y, z) = xy + x + z$. If we identify the first and the fourth variable, then we get $f(x, y, z, x) = xz + y + x$, which is the same as $g(x, z, y)$, hence we consider this minor to be the same as (or equivalent to) g. On the other hand, identifying the first and third variables of f, we obtain a new minor $f(x, y, x, z) = x + y + z$, and one can verify that there are no other 3-variable minors of f. Identification of the second and fourth variables yields the minor $h(x, y, z) = f(x, y, z, y) = xz$, which has formally 3 variables, but depends only on 2 of them. Note that $g(x, y, x) = xy$ is equivalent to h, hence h is a minor of g. Examining all possible variable identifications, we see that f has altogether 6 minors up to equivalence, which form the poset shown in Figure 1.*

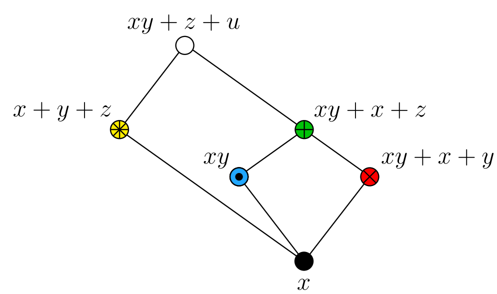

Fig. 1.

Looking only at the Hasse diagram of Figure 1 (ignoring the labels), it is not at all clear, whether there is a function whose minors give this poset, and this is exactly the problem that we consider in this paper. After recalling the necessary definitions and introducing some formalism for minors in Section II, we present a characterization of such "minor posets" by means of admissible colorings of partition lattices in Section III. Then, in Section IV we use this characterization to give some infinite families of examples of minor posets, and we also present some operations that allow us to construct new minor posets from known ones. However, it still remains an open problem to find a finite bounded poset that is *not* the poset of minors of any function, if there is such a poset at all.

II. Preliminaries

A. Posets

For a bounded poset P, let $\perp_P$ and $\top_P$ denote its least and greatest elements; we drop the subscript when there is no danger of ambiguity. The *dual* of a poset P is the poset P^d obtained by reversing the ordering of P (drawing the Hasse diagram of P upside down). The *interval* $[a; b]$ in P is the set $\{x \in P : a \leq x \leq b\}$. The *principal ideal* generated by $a \in P$ is the interval $\downarrow a := [\perp_P; a]$, and the *principal filter* generated by a is the interval $[a; \top_P]$.

We denote the n-element chain by $\mathbf{n}$, and M_n denotes the bounded poset (in fact, lattice) of size $n + 2$ with no comparabilities among its elements except for the top and bottom elements. The *ordinal sum* (linear sum) of posets P

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and Q is the poset $P \oplus Q$ obtained by putting Q "on top of" P. With this notation we have $\mathbf{n} = \underbrace{\mathbf{1} \oplus \cdots \oplus \mathbf{1}}_{n}$ and $M_n = \mathbf{1} \oplus \big(\underbrace{\mathbf{1} \cup \cdots \cup \mathbf{1}}_{n}\big) \oplus \mathbf{1}$.

By a *coloring* of a poset we mean an onto map $c\colon P \to C$, where C is an arbitrary nonempty set, whose elements are referred to as colors. Given such a coloring, we can introduce a relation ρ on C by $u \rho v \iff \exists a, b \in P\colon a \leq b$ and $c(a) = u, c(b) = v$. If ρ is a partial order (which is not always the case), we obtain the "poset of colors" $(C; \rho)$, and in this case we will use the symbol $\leq$ instead of ρ. Note that $(C; \leq)$ can be naturally identified with the poset of equivalence classes with respect to the kernel of the map c, hence we shall denote this *quotient poset* by $P/\ker c$.

B. Partitions

For any nonempty set V, let Π_V denote the set of all partitions of V; if $V = [n] := \{1, \ldots, n\}$ then we simply write Π_n. Each partition $\alpha \in \Pi_V$ corresponds naturally to an equivalence relation $\rho_\alpha \subseteq V \times V$. For notational convenience, we will sometimes use the same symbol for a partition and the corresponding equivalence relation, when there is no risk of ambiguity. For example, we denote the block of $\alpha \in \Pi_V$ containing $v \in V$ by v/α instead of the more usual notation v/ρ_α. Similarly, we use the symbol $\ker h$ not only for the kernel of a map $h\colon V \to A$, but also for the corresponding partition in Π_V.

For $\alpha, \beta \in V$, we say that α is a *refinement* of β and β is a *coarsening* of α (denoted by $\alpha \leq \beta$) if every block of α is a subset of some block of β (equivalently, $\rho_\alpha \subseteq \rho_\beta$). The poset $(\Pi_V; \leq)$ is a lattice, where $\alpha \wedge \beta$ is the partition corresponding to $\rho_\alpha \cap \rho_\beta$ and $\alpha \vee \beta$ is the partition corresponding to the transitive closure of $\rho_\alpha \cup \rho_\beta$. The top element of Π_V is $\top = \{V\}$ and the bottom element is $\bot = \{\{v\} : v \in V\}$. If $\alpha < \beta$ and there is no partition ξ with $\alpha < \xi < \beta$ then β is an *upper cover* of α (α is a *lower cover* of β), and we shall denote this by $\alpha \prec \beta$. Note that in this case β is obtained from α by merging two blocks; in particular, $\vartheta \prec \top$ holds if and only if ϑ has exactly two blocks.

Ore proved in [4] that every automorphism of Π_V is induced by a permutation of V. It follows immediately that every isomorphism between partition lattices is induced by a bijection between the underlying sets. More precisely, let V and W be nonempty sets, and let $\pi\colon V \to W$ be a bijection. For any partition $\alpha = \{V_1, \ldots, V_k\} \in \Pi_V$, let $\widetilde{\pi}(\alpha) = \{\pi(V_1), \ldots, \pi(V_k)\} \in \Pi_W$. Obviously, $\widetilde{\pi}\colon \Pi_V \to \Pi_W$ is an isomorphism. With this notation we can recast Ore's theorem in the following form.

Theorem 2 ([4]). *For arbitrary sets V and W, every isomorphism between Π_V and Π_W is of the form $\widetilde{\pi}$ for some bijection $\pi\colon V \to W$.*

Although Π_V is not a modular lattice if $|V| > 3$, the following special case of the isomorphism theorem for perspective intervals in modular lattices does hold.

Fact 3. *Let $\alpha, \gamma, \vartheta \in \Pi_V$ with $\alpha \leq \vartheta \prec \top$ and $\alpha \prec \gamma \not\leq \vartheta$. If one of the blocks of α is also a block of ϑ, then the following two maps are mutually inverse isomorphisms between the intervals $[\alpha; \vartheta]$ and $[\gamma; \top]$:*

$$[\alpha; \vartheta] \to [\gamma; \top], \ \xi \mapsto \xi \vee \gamma;$$
$$[\gamma; \top] \to [\alpha; \vartheta], \ \xi \mapsto \xi \wedge \vartheta.$$

Remark 4. *The intervals $[\alpha; \vartheta]$ and $[\gamma; \top]$ in Fact 3 are both isomorphic to the partition lattice on $|\alpha| - 1 = |\gamma|$ elements, hence from Theorem 2 we see that up to permutations of blocks of α, the only isomorphism from $[\alpha; \vartheta]$ to $[\gamma; \top]$ is $\xi \mapsto \xi \vee \gamma$.*

C. Functions and their minors

A *function of several variables* is a map of the form $f\colon A^n \to B$, where A and B are arbitrary nonempty sets, and n is a natural number, called the *arity* of f. To avoid degenerate cases, the sets A and B will be assumed to have at least two elements. The set of all such functions (of arbitrary arities) is denoted by $\mathcal{F}_{AB}$. We say that the i-th variable of f is *essential* (or that f *depends on* its i-th variable) if there exist tuples $\mathbf{a}, \mathbf{a}' \in A^n$ differing only in their i-th coordinate such that $f(\mathbf{a}) \neq f(\mathbf{a}')$.

For $f, g \in \mathcal{F}_{AB}$, we say that g is a *minor* of f (notation: $g \leq f$), if there is a map $\sigma\colon [n] \to [m]$ such that $g(x_1, \ldots, x_m) = f\big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big)$, where n and m denote the arities of f and g, respectively. It is easy to see that $g \leq f$ holds if and only if g can be obtained from f by identification of variables, permutation of variables and/or addition or deletion of inessential variables. The minor relation is a quasiorder on $\mathcal{F}_{AB}$, and the corresponding *equivalence* of functions is defined and denoted by $f \equiv g \iff f \leq g$ and $g \leq f$. Two functions are equivalent if and only if they can be obtained from each other by permutation of variables and/or addition or deletion of inessential variables, whereas to form a *proper minor* $g < f$ (meaning $g \leq f$ but $g \not\equiv f$), one must identify at least two essential variables. Considering functions only up to equivalence, as we shall do in this paper, one obtains the poset $(\mathcal{F}_{AB}/{\equiv}; \leq)$, which is our main object of study. The structure of this poset is quite complicated; for instance, it has been shown in [1] that it contains a copy of the poset of finite subsets of a countable set (hence a copy of every finite poset) even in the simplest case $A = B = \{0, 1\}$ (i.e., in the case of Boolean functions). In fact, $(\mathcal{F}_{AB}/{\equiv}; \leq)$ is universal for the class of countable posets with finite principal ideals, whenever $|B| \geq \min(3, |A|)$ [3].

Here we deal with principal ideals of $(\mathcal{F}_{AB}/{\equiv}; \leq)$. The principal ideal $\downarrow f$ generated by a function f consists of the minors of f (up to equivalence), hence we call it the *poset of minors* of f, and we also say that P is a *minor poset* if there exists a function $f\colon A^n \to B$ for some sets A, B and for some natural number n, such that $P \cong \downarrow f$. Clearly $\downarrow f$ is a finite poset with largest element $f/\equiv$. Although $\mathcal{F}_{AB}/\equiv$ has no least element (but it has several minimal elements), every function f has a least minor, namely the unary function $f(x, \ldots, x)$; see Figure 2. Therefore, every minor poset is a finite bounded poset. We shall denote the

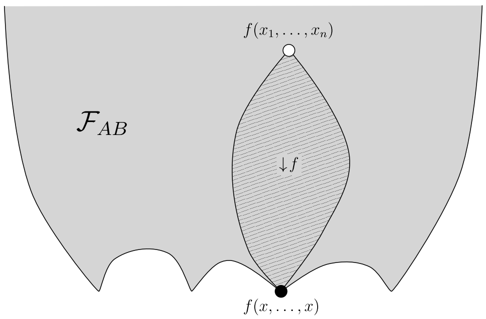

Fig. 2.

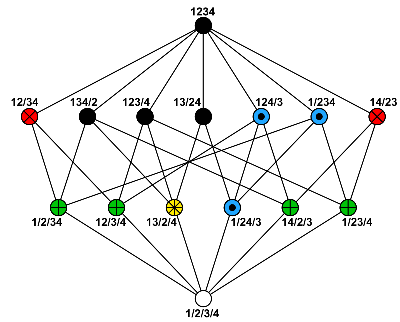

Fig. 3.

class of all minor posets by $\mathcal{M}$, and our main goal is to characterize members of $\mathcal{M}$ by means of a necessary and sufficient condition that does not refer to the existence of a suitable function f. In Corollary 17 we establish such a "function-free" characterization; however, this involves quite an intricate property that is not easy to verify for a concrete poset. Therefore, in spite of this characterization, it is still not clear whether all finite bounded posets are minor posets or not. In Section IV we present some infinite families of minor posets, and we prove that $\mathcal{M}$ is closed under certain poset constructions.

In order to present the promised characterization, we need to introduce some more abstract formalism for tuples, functions and minors (see [5]). An n-ary function from A to B can be viewed as a map $f\colon A^V \to B$, where V is an arbitrary n-element set (whose elements are considered to be the variables of f), and the elements of A^V are maps of the form $\mathbf{a}\colon V \to A$ (evaluations of variables). Note that in the special case $V = [n]$, the elements of A^V can be naturally identified with n-tuples, and in this case we get back the usual notion of a function of several variables. We will formulate our results in this usual setting, but in the proofs we will also need the more abstract view of functions allowing arbitrary finite sets as the set of variables.

For $\mathbf{a} \in A^W$ and $\sigma\colon V \to W$, we can define the composition $\mathbf{a} \circ \sigma \in A^V$ by $(\mathbf{a} \circ \sigma)(v) = \mathbf{a}(\sigma(v))$. Minors of f are functions $g\colon A^W \to B$ that can be given in the form $g(\mathbf{a}) = f(\mathbf{a} \circ \sigma)$ for some map $\sigma\colon V \to W$. If $\alpha \in \Pi_V$ is a partition, then let nat_α denote the natural surjection $\mathrm{nat}_\alpha\colon V \to \alpha, v \mapsto v/\alpha$. The map nat_α induces a minor $f_\alpha\colon A^\alpha \to B$, which is given by $f_\alpha(\mathbf{a}) = f(\mathbf{a} \circ \mathrm{nat}_\alpha)$ for all $\mathbf{a} \in A^\alpha$. Observe that f_α is obtained from f by identifying variables belonging to the same block of α. Conversely, for every map $\sigma\colon V \to W$, the minor $g(\mathbf{a}) = f(\mathbf{a} \circ \sigma)$ is equivalent to f_α with $\alpha = \ker \sigma$. This shows that it suffices to work with minors of the form f_α, and we shall record this fact here for reference.

Fact 5. *If $f\colon A^V \to B$ and $g\colon A^W \to B$ are arbitrary functions, then*

$$g \le f \iff \exists \alpha \in \Pi_V : g \equiv f_\alpha.$$

III. Admissible colorings

According to Fact 5, every minor of an n-variable function f is equivalent to a function f_α for some $\alpha \in \Pi_n$. This means that we can encode all information about minors of f into a "coloring" c of the partition lattice Π_n, where the color of a partition α is $c(\alpha) = f_\alpha/\equiv$. Actually, the only relevant property of this coloring is that two minors receive the same color if and only if they are equivalent. Clearly, we have $\beta \ge \alpha \implies f_\beta \le f_\alpha$. The following easy observation formulates a kind of converse of this statement, showing that we can recover the poset $\downarrow f$ as the quotient of Π_n by the kernel of the aforementioned coloring c.

Proposition 6. *For every function $f\colon A^n \to B$ and for all $\alpha, \beta \in \Pi_n$, the function f_β is a minor of f_α if and only if there exists a partition $\gamma \ge \alpha$ such that $f_\gamma \equiv f_\beta$.*

Proof: The "if" part of the statement is obvious. For the "only if" part, assume that $f_\beta \le f_\alpha$. By Fact 5, this means that there exists a partition $\delta \in \Pi_\alpha$ such that $f_\beta \equiv (f_\alpha)_\delta$. Let $\gamma \in \Pi_n$ be the partition obtained by merging the blocks of α that belong to the same block of δ. (More precisely, two elements $u, v \in [n]$ are ρ_γ-related if and only if the α-blocks u/α and v/α are ρ_δ-related.) Clearly, $\gamma \ge \alpha$ and $(f_\alpha)_\delta \equiv f_\gamma$, hence $f_\beta \equiv f_\gamma$. ■

Corollary 7. *For every function $f\colon A^n \to B$, the poset of minors of f is dually isomorphic to $\Pi_n/\ker c$ for the natural coloring $c\colon \Pi_n \to \downarrow f$, $\alpha \mapsto f_\alpha/\equiv$.*

Example 8. *Let us consider the function $f(x_1, x_2, x_3, x_4) = x_1 x_3 + x_2 + x_4$ of Example 1 once more. We have computed there that $f_{12/3/4} \equiv f_{14/2/3} \equiv g > h \equiv f_{1/24/3}$ and $f_{13/2/4}$ is incomparable to g and h. (Here we use a simplified, but hopefully clear notation for partitions.) Calculating f_α for all the 15 partitions of $[4]$, we get a coloring of Π_4 with 6 colors, as shown in Figure 3. The partial order induced on the 6 colors is the dual of the poset of Figure 1.*

Corollary 7 shows that we can obtain each minor poset as a "poset of colors", where the order on the colors is induced by a suitable coloring of a partition lattice. Therefore, our main goal is to characterize those colorings that can arise from a function.

We define an abstract property of colorings of partition lattices, called admissibility (see Definition 10), and in Corollary 17 we prove that admissibility is indeed a necessary and sufficient condition for the existence of a function f such that the given coloring is induced by f (as in Corollary 7).

Proposition 9. *Let $c\colon \Pi_n \to C$ be a coloring, and let $\alpha, \beta, \vartheta \in \Pi_n$ such that $\alpha \leq \vartheta \prec \top$ and $\alpha \prec \beta \nleq \vartheta$. Then the following two conditions are equivalent.*

(i) *For every $\gamma \in \Pi_n$ with $\alpha \prec \gamma \nleq \vartheta$ (in particular, for $\gamma = \beta$), the map*

$$\varphi_\gamma\colon [\alpha; \vartheta] \to [\gamma; \top], \ \xi \mapsto \xi \vee \gamma$$

is a color-preserving isomorphism (cf. Fact 3).

(ii) *One of the blocks of α is also a block of ϑ and*

$$\forall \xi \in [\alpha; \top] : c(\xi) = c(\xi \wedge \vartheta).$$

Definition 10. *Let $c\colon \Pi_n \to C$ be an arbitrary coloring, and let $\alpha, \beta \in \Pi_n$.*

(i) *We write $\alpha \sim \beta$ if the intervals $[\alpha; \top]$ and $[\beta; \top]$ are isomorphic as colored posets, i.e., there is an isomorphism φ from $[\alpha; \top]$ to $[\beta; \top]$ such that $c(\xi) = c(\varphi(\xi))$ for all $\xi \in [\alpha; \top]$.*

(ii) *We write $\alpha \rightsquigarrow_1 \beta$ if $\alpha \prec \beta$ and there is a partition $\vartheta \in \Pi_n$ with $\alpha \leq \vartheta \prec \top$ and $\beta \nleq \vartheta$ such that the equivalent conditions of Proposition 9 are satisfied.*

Let $\rightsquigarrow$ be the reflexive-transitive closure of $\rightsquigarrow_1$, i.e., $\alpha \rightsquigarrow \beta$ if and only if there exist partitions $\alpha_0, \ldots, \alpha_k \in \Pi_n$ for some $k \geq 0$ such that $\alpha = \alpha_0 \rightsquigarrow_1 \alpha_1 \rightsquigarrow_1 \cdots \rightsquigarrow_1 \alpha_k = \beta$ (this includes the case $\alpha = \beta$ when $k = 0$).
We say that the coloring c is admissible*, if for all $\alpha, \beta \in \Pi_n$, we have*

$$c(\alpha) = c(\beta) \implies \exists \alpha', \beta' \in \Pi_n\colon \ \alpha \rightsquigarrow \alpha' \sim \beta' \leftsquigarrow \beta. \quad (1)$$

Remark 11. *Note that if $\alpha \sim \beta$ or $\alpha \rightsquigarrow \beta$, then $c(\alpha) = c(\beta)$. Thus the reverse implication of (1) always holds.*

Proposition 12. *Let $f\colon A^n \to B$ be an arbitrary function, and let $c(\alpha) = f_\alpha / \equiv$ for all $\alpha \in \Pi_n$. Then c is an admissible coloring of Π_n.*

Proof: Let $\alpha = \{V_1, \ldots, V_k\} \in \Pi_n$ be an arbitrary partition of size $k \geq 2$, and assume that V_1 is an inessential variable of f_α. Let $\vartheta = \{V_1, V_2 \cup \cdots \cup V_k\}$ and suppose that $\alpha \prec \gamma \nleq \vartheta$. Clearly, γ is obtained from α by merging V_1 with another block V_j. This means that we get f_γ from f_α by identifying the inessential variable V_1 with another variable, hence we have $f_\alpha \equiv f_\gamma$, that is $c(\alpha) = c(\gamma)$. Similarly, for any $\xi \in [\alpha; \vartheta]$, denoting by $\varphi(\xi) = \xi \vee \gamma$ the partition obtained from ξ by merging V_1 (which must be a block of ξ) with the block containing V_j, we have $c(\xi) = c(\varphi(\xi))$, therefore condition (i) of Proposition 9 is satisfied. Thus we can conclude that $\alpha \rightsquigarrow_1 \gamma$ for every $\gamma \in \Pi_n$ such that $\alpha \prec \gamma \nleq \vartheta$.

We have proved that if f_α has an inessential variable, then there exists an upper cover γ of α such that $\alpha \rightsquigarrow_1 \gamma$. Proceeding this way (always identifying an inessential variable with another variable as long as there is an inessential variable), we finally arrive at a partition α' such that $\alpha \rightsquigarrow \alpha'$ and all variables of $f_{\alpha'}$ are essential.

Now we are ready to prove that c is an admissible coloring. Assume that $c(\alpha) = c(\beta)$, i.e., $f_\alpha \equiv f_\beta$, and use the above procedure to find partitions α' and β' such that $\alpha \rightsquigarrow \alpha'$ and $\beta \rightsquigarrow \beta'$ with $f_{\alpha'}$ and $f_{\beta'}$ depending on all their variables. Since $f_{\alpha'} \equiv f_\alpha \equiv f_\beta \equiv f_{\beta'}$, the functions $f_{\alpha'}$ and $f_{\beta'}$ are equivalent, and this implies that they can be obtained from each other by a permutation of variables, since both functions have only essential variables. This permutation of variables induces naturally a color-preserving isomorphism between the intervals $[\alpha'; \top]$ and $[\beta'; \top]$, showing that $\alpha' \sim \beta'$. Thus we have $\alpha \rightsquigarrow \alpha' \sim \beta' \leftsquigarrow \beta$, and this proves that (1) is satisfied. ■

Remark 13. *If f depends on all of its variables, then $c(\bot) = f/\equiv$ appears only at $\bot$ in the coloring of Proposition 12. Therefore, one may always assume without loss of generality that $\bot$ is the unique element of Π_n with color $c(\bot)$. On the other hand, one cannot assume the same about the color of $\top$: a function can have several minors that are equivalent to $f_\top = f(x, \ldots, x)$ (see also Remark 20).*

Next we would like to prove the following converse of Proposition 12: for any admissible coloring $c\colon \Pi_n \to C$, there is a function $f\colon A^n \to B$ such that two partitions of $[n]$ have the same color if and only if the corresponding minors of f are equivalent. To construct this function, let A be any set with at least n elements, let $B = C$, and define $f\colon A^n \to B$ by $f(\mathbf{a}) := c(\ker \mathbf{a})$ for all $\mathbf{a} \in A^n$. Here $\ker \mathbf{a}$ denotes the (partition corresponding to the) kernel of the map $\mathbf{a}\colon [n] \to A, \ i \mapsto a_i$. All partitions of $[n]$ with at most $|A|$ blocks arise in the form $\ker \mathbf{a}$, therefore our assumption $|A| \geq n$ guarantees that in fact every element of Π_n will occur. We will show in Theorem 16 that the above function has the desired property, thus we can conclude that every poset that appears as the poset of minors of a function can be represented by a function f having the special property that $f(\mathbf{a})$ is determined by the kernel of $\mathbf{a}$.

Let f be the function defined above, and let us consider an arbitrary minor f_α. From the definition of a minor we have that $f_\alpha(\mathbf{a}) = f(\mathbf{a} \circ \mathrm{nat}_\alpha) = c(\ker(\mathbf{a} \circ \mathrm{nat}_\alpha))$ for all $\mathbf{a} \in A^\alpha$. Observe that the partition $\ker(\mathbf{a} \circ \mathrm{nat}_\alpha)$ is a coarsening of α (merging two blocks of α if and only if $\mathbf{a}$ assigns the same value to them). Moreover, the assumption $|A| \geq n$ ensures that we obtain every coarsening of α (every element of the interval $[\alpha; \top]$) this way. This observation will be of key importance in the next two lemmas, which prepare the proof of Theorem 16, our main result in this section.

Lemma 14. *Let $c\colon \Pi_n \to C$ be an arbitrary coloring, and let the function $f\colon A^n \to C$ be defined by $f(\mathbf{a}) = c(\ker \mathbf{a})$ for all $\mathbf{a} \in A^n$, where A is a finite set with at least n elements. For arbitrary partitions $\alpha, \beta \in \Pi_n$, the minors f_α and f_β can be obtained from each other by a permutation of variables if and only if $\alpha \sim \beta$.*

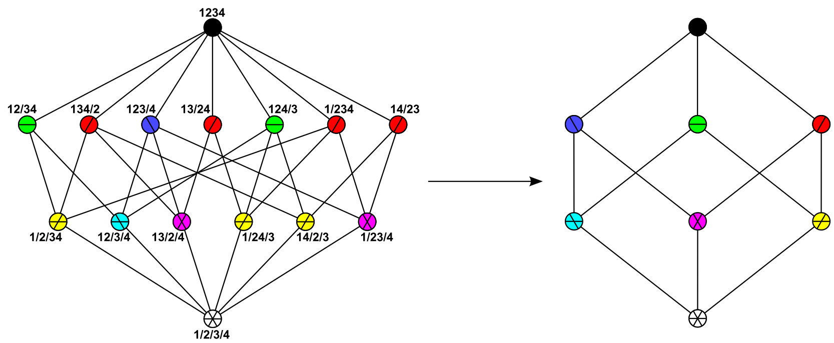

Fig. 4.

Lemma 15. *Let $c\colon \Pi_n \to C$ be an arbitrary coloring, and let the function $f\colon A^n \to C$ be defined by $f(\mathbf{a}) = c(\ker \mathbf{a})$ for all $\mathbf{a} \in A^n$, where A is a finite set with at least n elements. For arbitrary partitions $\alpha, \beta \in \Pi_n$, the minor f_β can be obtained from f_α by identifying an inessential variable with another variable if and only if $\alpha \rightsquigarrow_1 \beta$.*

Theorem 16. *Let $c\colon \Pi_n \to C$ be an admissible coloring, and let the function $f\colon A^n \to C$ be defined by $f(\mathbf{a}) = c(\ker \mathbf{a})$ for all $\mathbf{a} \in A^n$, where A is a finite set with at least n elements. Then for every $\alpha, \beta \in \Pi_n$, we have $f_\alpha \equiv f_\beta$ if and only if $c(\alpha) = c(\beta)$.*

Proof: Let us suppose first that $f_\alpha \equiv f_\beta$. Let $\alpha = \{V_1, \ldots, V_k\} \in \Pi_n$ and assume that $V_1, \ldots, V_\ell$ are inessential variables and $V_{\ell+1}, \ldots, V_k$ are essential variables in f_α. If $\alpha' = \{V_1 \cup \cdots \cup V_\ell \cup V_{\ell+1}, V_{\ell+2}, \ldots, V_k\}$, then $f_{\alpha'}$ depends on all of its variables, and $f_{\alpha'}$ can be obtained from f_α by repeatedly identifying an inessential variable with an essential one. Similarly, let $f_{\beta'}$ be the "essential minor" of f_β. Clearly, $f_\alpha \equiv f_\beta$ implies that $f_{\alpha'}$ and $f_{\beta'}$ can be obtained from each other by a permutation of variables. Now Lemma 14 and Lemma 15 yield $\alpha \rightsquigarrow \alpha' \sim \beta' \leftsquigarrow \beta$, and then $c(\alpha) = c(\beta)$ follows (see Remark 11).

Conversely, if $c(\alpha) = c(\beta)$, then, by the admissibility of the coloring c, there exist $\alpha', \beta' \in \Pi_n$ such that $\alpha \rightsquigarrow \alpha' \sim \beta' \leftsquigarrow \beta$. Lemma 15 shows that $f_\alpha \equiv f_{\alpha'}$ and $f_\beta \equiv f_{\beta'}$, and Lemma 14 shows that $f_{\alpha'} \equiv f_{\beta'}$. Therefore, we can conclude that f_α and f_β are equivalent. ∎

Corollary 17. *A poset P belongs to $\mathcal{M}$ (i.e., isomorphic to the poset of all minors of some function f) if and only if there is an admissible coloring $c\colon \Pi_n \to C$ for some $n \in \mathbb{N}$ and for some nonempty set C such that $P^d \cong \Pi_n / \ker c$.*

IV. Constructions and examples

In this section we give some (families) of examples of minor posets, and we also present some constructions which allow us to build new minor posets from known ones.

Theorem 18. *The following are minor posets for every natural number n:*

(i) *the dual of the partition lattice Π_n;*
(ii) *the n-element chain $\mathbf{n}$;*
(iii) *the n-dimensional cube (Boolean lattice) $\mathbf{2}^n$;*
(iv) *the lattice M_n.*

Proof: In each case we give an admissible coloring of a partition lattice such that the corresponding quotient is dually isomorphic to the desired poset. We leave it to the reader to verify that these colorings are indeed admissible.

(i) If $c\colon \Pi_n \to \Pi_n$ is the identity map, then clearly $\Pi_n / \ker c$ is dually isomorphic to Π_n^d.
(ii) For the coloring $c\colon \Pi_n \to [n]$, $\alpha \mapsto |\alpha|$, the quotient $\Pi_n / \ker c$ is (the dual of) an n-element chain.
(iii) Let c be the coloring that assigns to every partition $\alpha = \{V_1, \ldots, V_k\} \in \Pi_{n+1}$ the set of minimal elements of the blocks of α (under the natural ordering $1 < \cdots < n+1$), that is, $c(\alpha) := \{\min V_1, \ldots, \min V_k\}$. The image of c consists of those subsets of $[n+1]$ that contain the element 1, and we have $\alpha > \beta \implies c(\alpha) \subset c(\beta)$. Moreover, if $1 \in M \subset N \subseteq [n+1]$, then one can find partitions $\alpha, \beta \in \Pi_{n+1}$ with $\alpha > \beta$ and $c(\alpha) = M, c(\beta) = N$. This implies that $\Pi_{n+1} / \ker c$ is isomorphic to the lattice of subsets of $[n+1]$ containing 1, which is (dually) isomorphic to $\mathbf{2}^n$. Figure 4 illustrates the coloring and the corresponding quotient for $n = 3$.
(iv) Let us choose a natural number m such that $\binom{m}{2} \geq n$, and let us color Π_m as follows. The bottom element of Π_m is white, the atoms (i.e., the upper covers of $\perp$) receive n different colors (different from white and black) in an arbitrary way, and all the other elements of Π_m are black. Then $\Pi_m / \ker c$ is (dually) isomorphic to M_n.

∎

Proposition 19. *The class $\mathcal{M}$ is closed under taking principal ideals: if $P \in \mathcal{M}$ and $a \in P$, then the principal ideal $[\perp_P; a]$ is also a member of $\mathcal{M}$.*

Proof: If f is a function such that $\downarrow f$ is isomorphic to P, and $a \in P$, then f has a minor f_α corresponding to a, and $\downarrow f_\alpha$ is isomorphic to the principal ideal $[\perp_P; a]$. ∎

Remark 20. *A natural idea to prove an analogous statement for principal filters would be the following. Take an admissible coloring c of Π_n such that $\Pi_n / \ker c \cong P^d$, and let α correspond to a under this isomorphism. Change all colors outside of $\downarrow\alpha$ to the color of α; then the resulting quotient poset of Π_n will be dually isomorphic to the principal filter $[a; \top_P]$ of P. However, unfortunately, this modified coloring will not be admissible in general. Nevertheless, it might be still true that $\mathcal{M}$ is closed under taking principal filters, but a different argument would be needed to prove this.*

In the following theorem we prove that one can always add a new top element to a minor poset. Recalling that the ordinal sum of posets is denoted by $\oplus$, the poset obtained by adding a new top element to P can be written as $P \oplus \mathbf{1}$.

Theorem 21. *If $P \in \mathcal{M}$, then $P \oplus \mathbf{1} \in \mathcal{M}$.*

Proof: By Corollary 17, there is an admissible coloring $c\colon \Pi_{n-1} \to C$ for some natural number n and for some nonempty set C such that $\Pi_{n-1} / \ker c \cong P^d$. For any $\xi \in \Pi_n$, let us simply write $\xi - n$ for the partition that is obtained from ξ by deleting the element n. More precisely, if $\xi = \{V_1, \ldots, V_k\}$, and, say, $n \in V_k$, then let $\xi - n = \{V_1, \ldots, V_k \setminus \{n\}\} \in \Pi_{n-1}$, discarding the block $V_k \setminus \{n\}$ if it is empty. Define $c^*\colon \Pi_n \to C$ by $c^*(\xi) = c(\xi - n)$. One can verify that c^* is admissible and that the quotient poset $\Pi_n / \ker c^*$ is isomorphic to P^d (we omit the details).

Now let us introduce a new color $* \notin C$ and modify the coloring c^* by changing the color of $\bot$ to $*$ (the colors of the other elements remain the same). Clearly, this new coloring is also admissible, and the corresponding quotient of Π_n is isomorphic to $\mathbf{1} \oplus P^d$ (note that the "old" color $c^*(\bot)$ does still appear, for instance as $c^*(\{\{1\}, \{2\}, \ldots, \{n-1, n\}\})$. Therefore, $P \oplus \mathbf{1}$ (the dual of $\mathbf{1} \oplus P^d$) belongs to $\mathcal{M}$ by Corollary 17. ∎

Remark 22. *It is a natural question whether $P \in \mathcal{M}$ implies $\mathbf{1} \oplus P \in \mathcal{M}$. A simple proof could be obtained by changing the color of $\top$ to a new color $*$ at the end of the previous proof. Unfortunately, this new coloring is not necessarily admissible, and it remains an open problem whether adding a new bottom element to a minor poset yields a minor poset or not.*

Next we describe a construction of "gluing" two posets together, and we show that $\mathcal{M}$ is closed under this construction. For finite bounded posets P_1 and P_2, let $P_1 * P_2$ denote the poset obtained from the disjoint union (parallel sum) of P_1 and P_2 by identifying the top elements as well as the bottom elements (see Figure 5).

Theorem 23. *If $P_1, P_2 \in \mathcal{M}$, then $P_1 * P_2 \in \mathcal{M}$.*

Proof: Suppose that $P_1, P_2 \in \mathcal{M}$, and let $c_i\colon \Pi_{W_i} \to C_i$ be admissible colorings such that $\Pi_{W_i} / \ker c_i \cong P_i^d$ for $i = 1, 2$. We assume that the sets W_1 and W_2 are disjoint, and also that $c_1(\bot_1) = c_2(\bot_2) = \spadesuit$, $c_1(\top_1) = c_2(\top_2) = \heartsuit$ but apart from these two colors, there is no common color used in c_1 and c_2. (Here $\bot_i$ and $\top_i$ denote the bottom and top elements of Π_{W_i}.) By Remark 13, we may also suppose that the color $\spadesuit$ appears only at the bottom in c_1 as well as in c_2. We shall construct an admissible coloring $c\colon \Pi_W \to C$ with $W = W_1 \cup W_2$ and $C = C_1 \cup C_2$ such that $\Pi_W / \ker c \cong (P_1 * P_2)^d$.

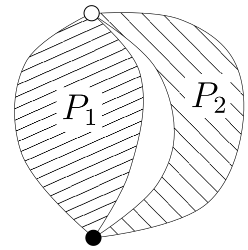

Fig. 5.

For $i = 1, 2$, let $\omega_i \in \Pi_W$ be the partition of W whose only non-singleton block is W_i, and let $\iota_i\colon \Pi_{W_i} \to \Pi_W$ be the natural embedding that maps Π_{W_i} isomorphically onto $[\bot_i; \omega_i]$. We define the desired coloring c by

$$c(\xi) = \begin{cases} c_i\left(\iota_i^{-1}(\xi)\right), & \text{if } \xi \in [\bot_i; \omega_i] \text{ for some } i \in \{1, 2\}; \\ \heartsuit, & \text{if } \xi \notin [\bot_i; \omega_1] \cup [\bot_i; \omega_2]. \end{cases}$$

Note that c is well defined, as the intervals $[\bot_1; \omega_1]$ and $[\bot_2; \omega_2]$ intersect only at the bottom, and $c_1(\bot_1) = c_2(\bot_2)$. One can check that c is admissible (again, we omit the technical details), and it is clear that $\Pi_W / \ker c \cong (P_1 * P_2)^d$. ∎

Starting with the examples of Theorem 18, one can build many minor posets using the constructions of Theorems 21 and 23. For example, the poset of Figure 1 can be constructed as $\mathbf{3} * (M_2 \oplus \mathbf{1})$. We have verified (using these techniques as well as ad hoc colorings) that all bounded posets up to 6 elements are minor posets. It might even be the case that every finite bounded poset is a minor poset, but the (dis)proof of this statement still eludes us.

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