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On the shape of solution sets of systems of (functional) equations

Endre Tóth and Tamás Waldhauser

Abstract. Solution sets of systems of linear equations over fields are characterized as being affine subspaces. But what can we say about the "shape" of the set of all solutions of other systems of equations? We study solution sets over arbitrary algebraic structures, and we give a necessary condition for a set of n-tuples to be the set of solutions of a system of equations in n unknowns over a given algebra. In the case of Boolean equations we obtain a complete characterization, and we also characterize solution sets of systems of Boolean functional equations.

Mathematics Subject Classification. Primary 06E30; Secondary 08A40, 39B52, 39B72.

Keywords. Systems of equations, Functional equations, Solution sets, Clones, Boolean functions.

1. Introduction

A basic fact from undergraduate linear algebra: solution sets of systems of homogeneous linear equations in n variables over a field K are precisely the subspaces of the vector space K^n, i.e., sets of n-tuples that are closed under linear combinations. Similarly, solution sets of systems of arbitrary linear equations are characterized by being closed under affine combinations. In this paper we propose an abstract framework that encompasses the aforementioned two well-known situations and allows us to study sets of solutions of systems of equations in great generality. Our aim is to determine the "shape" of solution sets by giving necessary and sufficient conditions for a set of tuples to arise as the set of all solutions of a system of equations. We establish a universal necessary condition, and prove that it is also sufficient for Boolean equations, i.e., for equations over the two-element set $\{0, 1\}$. We also present examples showing that this is not the case for domains with at least three elements. For functional equations such a general framework was established in [2]; here we

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prove that the necessary condition found there actually characterizes sets of solutions of Boolean functional equations.

To make this more precise, let us fix a nonempty set A and a set F of operations on A that we are allowed to use in our equations (for example, the unary operations ax ($a \in K$) and the binary operation $x + y$ as well as constants $c \in K$ in the case of linear equations over a field K). Since we can use these operations several times, we can build composite operations (for example $a_1x_1 + \cdots + a_nx_n + c$). This means that every equation in n variables can be written as $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$, where f and g are obtained as compositions of operations from F. The set of all such operations is denoted by $[F]$, and it is called the *clone* generated by F (see Sect. 2 for the precise definitions). Elements of the clone $[F]$ are also called *term functions* of the algebraic structure $\mathbb{A} = (A; F)$, and our equations are the same as equations over $\mathbb{A}$ in the sense of universal algebra. However, in universal algebra the focus is on (the complexity of) finding one solution or deciding if there is a solution at all, whereas here we study the structure of the set of all solutions.

If two sets of operations generate the same clone, then they produce the same equations, thus it is natural to speak about equations over a clone C. This leads to the main problem of this paper: given a clone C, characterize sets $T \subseteq A^n$ that can appear as the set of all solutions of a system of equations over C. After introducing the required notions and notations in Sect. 2, we give a general necessary condition in Sect. 3 (see Theorem 3.1). More precisely, we prove that for every clone C, one can assign a clone C^* (called the *centralizer* of C) such that if $T \subseteq A^n$ is the set of all solutions of a system of equations over C, then T is closed under C^*. In certain special cases, such as in the case of (homogeneous) linear equations (see Example 3.2), being closed under C^* is sufficient for being the solution set of a system of C-equations. Unfortunately, as we show in Example 3.3, there are other "non-linear" clones for which this is not true. However, we will prove in Sect. 4 that for Boolean functions (i.e., for $A = \{0, 1\}$) the condition given in Theorem 3.1 is sufficient. Thus we obtain a complete characterization of solution sets of systems of Boolean equations in terms of closure conditions, which is similar in spirit to the "linear" examples mentioned in the first paragraph (Theorem 4.1). We will use this result in Sect. 5 to characterize solution sets of systems of Boolean equations, solving the main problem of [2] in the Boolean case (Theorem 5.1).

2. Preliminaries

2.1. Operations and clones

Let A be an arbitrary set with at least two elements. By an *operation* on A we mean a map $f\colon A^n \to A$; the nonnegative integer n is called the *arity*

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of the operation f. (We allow nullary operations: since A^0 is a singleton, an operation of arity zero can be naturally identified with the unique element in its image set.) The set of all operations on A is denoted by $\mathcal{O}_A$. Operations on $A = \{0,1\}$ are called *Boolean functions*, and we will also use the notation $\Omega = \mathcal{O}_{\{0,1\}}$ for the set of all Boolean functions (see the appendix for some background on Boolean functions). For a set $F \subseteq \mathcal{O}_A$ of operations, by $F^{(n)}$ we mean the set of n-ary members of F. In particular, $\mathcal{O}_A^{(n)}$ stands for the set of all n-ary operations on A.

We will denote tuples by boldface letters, and we will use the corresponding plain letters with subscripts for the components of the tuples. For example, if $\mathbf{a} \in A^n$, then a_i denotes the i-th component of $\mathbf{a}$, i.e., $\mathbf{a} = (a_1, \ldots, a_n)$. In particular, if $f \in \mathcal{O}_A^{(n)}$, then $f(\mathbf{a})$ is a short form for $f(a_1, \ldots, a_n)$. In accordance with the above, we denote the n-tuple $(1, 1, \ldots, 1)$ by $\mathbf{1}$, and similarly the n-tuple $(0, 0, \ldots, 0)$ by $\mathbf{0}$ (the length of the tuple shall be clear from the context). If $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)} \in A^n$ and $f \in \mathcal{O}_A^{(m)}$, then $f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})$ denotes the n-tuple obtained by applying f to the tuples $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}$ componentwise:

$$f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) = \big(f(t_1^{(1)}, \ldots, t_1^{(m)}), \ldots, f(t_n^{(1)}, \ldots, t_n^{(m)})\big).$$

We say that $T \subseteq A^n$ is *closed under* C, if for all $m \in \mathbb{N}, \mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)} \in T$ and for all $f \in C^{(m)}$ we have $f(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$.

Let $f \in \mathcal{O}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{O}_A^{(k)}$. By the *composition* of f by $g_1, \ldots, g_n$ we mean the operation $h \in \mathcal{O}_A^{(k)}$ defined by

$$h(\mathbf{x}) = f\big(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})\big) \text{ for all } \mathbf{x} \in A^k.$$

If a class $C \subseteq \mathcal{O}_A$ of operations is closed under composition and contains the *projections* $(x_1, \ldots, x_n) \mapsto x_i$ for all $1 \leq i \leq n \in \mathbb{N}$, then C is said to be a *clone* (notation: $C \leq \mathcal{O}_A$). Notable examples include all continuous operations on a topological space, all monotone operations on an ordered set, all polynomial operations of a ring (or any algebraic structure), etc. (see also Example 2.1). For an arbitrary set F of operations on A, there is a least clone $[F]$ containing F, called the clone *generated* by F. The elements of this clone are those operations that can be obtained from members of F and from projections by finitely many compositions.

The set of all clones on A is a lattice under inclusion; the greatest element of this lattice is $\mathcal{O}_A$, and the least element is the *trivial clone* consisting of projections only. There are countably infinitely many clones on the two-element set; these have been described by Post [4], hence the lattice of clones on $\{0,1\}$ is called the *Post lattice*. In the appendix we present the Post lattice and we define Boolean clones that we need in the proof of our main results. If A is a finite set with at least three elements, then there is a continuum of clones on A, and it is a very difficult open problem to describe all clones on A even for $|A| = 3$.

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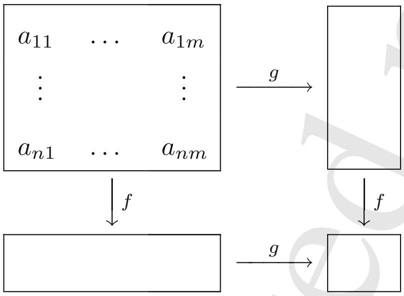
2.2. Centralizer clones

We say that the operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ *commute* (notation: $f \perp g$) if

$$f\big(g(a_{11}, a_{12}, \ldots, a_{1m}), \ldots, g(a_{n1}, a_{n2}, \ldots, a_{nm})\big)$$
$$= g\big(f(a_{11}, a_{21}, \ldots, a_{n1}), \ldots, f(a_{1m}, a_{2m}, \ldots, a_{nm})\big)$$

holds for all $a_{ij} \in A$ $(1 \leq i \leq n, 1 \leq j \leq m)$. This can be visualized as follows: for every $n \times m$ matrix $Q = (a_{ij})$, first applying g to the rows of Q and then applying f to the resulting column vector yields the same result as first applying f to the columns of Q and then applying g to the resulting row vector:



Denoting by $\mathbf{c}_j \in A^n$ $(j = 1, \ldots, m)$ the j-th column vector of Q, we can express the commutation property more compactly:

$$f(g(\mathbf{c}_1, \ldots, \mathbf{c}_m)) = g(f(\mathbf{c}_1), \ldots, f(\mathbf{c}_m)). \tag{2.1}$$

It is easy to verify that if $f, g_1, \ldots, g_n$ all commute with an operation h, then the composition $f(g_1, \ldots, g_n)$ also commutes with h. This implies that for any $F \subseteq \mathcal{O}_A$, the set $F^* := \{g \in \mathcal{O}_A \mid f \perp g \text{ for all } f \in F\}$ is a clone, called the *centralizer* of F. Clones arising in this form are called *primitive positive clones*; such clones seem to be quite rare: there are only finitely many primitive positive clones over any finite set [1]. It is useful to note that if $C = [F]$, then $C^* = F^*$. This implies that in order to compute the centralizer of a clone C, it is sufficient to determine the operations commuting with a (preferably small) generating set of C.

Example 2.1. Let K be a field, and let L be the clone of all operations over K that are represented by a linear polynomial:

$$L := \{a_1x_1 + \cdots + a_kx_k + c \mid k \geq 0, a_1, \ldots, a_k, c \in K\}.$$

Since L is generated by the operations $x + y$, ax $(a \in K)$ and the constants $c \in K$, the centralizer L^* consists of those operations f over K that commute

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with $x + y$ and ax (i.e., f is additive and homogeneous), and also commute with the constants (i.e., $f(c, \dots, c) = c$ for all $c \in K$):

$$L^* := \{a_1x_1 + \cdots + a_kx_k \mid k \geq 1, a_1, \dots, a_k \in K \text{ and } a_1 + \cdots + a_k = 1\}.$$

Similarly, one can verify that $L_0^* = L_0$ for the clone

$$L_0 := \{a_1x_1 + \cdots + a_kx_k \mid k \geq 0, a_1, \dots, a_k \in K\}.$$

2.3. Equations and solution sets

Let us fix a clone $C \leq \mathcal{O}_A$ and a natural number n. By an n-ary *equation over* C (*C-equation* for short) we mean an equation of the form $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$, where $f, g \in C^{(n)}$. We will often simply write this equation as a pair (f, g). A *system of C-equations* is a finite set of C-equations of the same arity:

$$\mathcal{E} := \big\{(f_1, g_1), \dots, (f_t, g_t)\big\}, \text{ where } f_i, g_i \in C^{(n)} \ (i = 1, \dots, t).$$

We define the *set of solutions of* $\mathcal{E}$ as the set

$$\operatorname{Sol}(\mathcal{E}) := \big\{\mathbf{a} \in A^n \mid f_i(\mathbf{a}) = g_i(\mathbf{a}) \text{ for } i = 1, \dots, t\big\}.$$

For $\mathbf{a} \in A^n$ we denote by $\operatorname{Eq}_C(\mathbf{a})$ the set of C-equations satisfied by $\mathbf{a}$:

$$\operatorname{Eq}_C(\mathbf{a}) := \big\{(f, g) \mid f, g \in C^{(n)} \text{ and } f(\mathbf{a}) = g(\mathbf{a})\big\}.$$

Let $T \subseteq A^n$ be an arbitrary set of tuples. We denote by $\operatorname{Eq}_C(T)$ the set of C-equations satisfied by T:

$$\operatorname{Eq}_C(T) := \bigcap_{\mathbf{a} \in T} \operatorname{Eq}_C(\mathbf{a}).$$

Example 2.2. Considering the "linear" clones of Example 2.1, L-equations are linear equations and L_0-equations are homogeneous linear equations.

3. A general necessary condition

Looking for a characterization of solution sets by means of closure conditions, we would like to determine operations under which solution sets of C-equations are closed. The following theorem shows that the solution set is always closed under operations in the centralizer C^*.

Theorem 3.1. *For any clone $C \leq \mathcal{O}_A$, the set of all solutions of a system of C-equations is closed under C^*.*

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Proof. Let $C \leq \mathcal{O}_A$ be a clone and let $\mathcal{E}$ be a system of n-ary C-equations with solution set $T = \mathrm{Sol}(\mathcal{E}) \subseteq A^n$. Let $\Phi \in C^*$ be an arbitrary m-ary operation, and let $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)} \in T$; we need to prove that $\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$. Consider an arbitrary equation $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ from $\mathcal{E}$. Since $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}$ are solutions of $\mathcal{E}$, we have $f(\mathbf{t}^{(j)}) = g(\mathbf{t}^{(j)})$ for $j = 1, \ldots, m$. This implies that

$$\Phi(f(\mathbf{t}^{(1)}), \ldots, f(\mathbf{t}^{(m)})) = \Phi(g(\mathbf{t}^{(1)}), \ldots, g(\mathbf{t}^{(m)})). \tag{3.1}$$

Let us consider the $n \times m$ matrix $Q = (t_i^{(j)})$ obtained by writing the tuples $\mathbf{t}^{(j)}$ next to each other as column vectors. Then the left hand side of (3.1) is obtained by applying f to the columns of Q and then applying Φ to the resulting row vector. Since Φ and f commute, we get the same by applying first Φ row-wise and then applying f column-wise, and the result in this case is $f(\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}))$ (cf. also (2.1)). Rewriting similarly the right hand side of (3.1), we can conclude that

$$f(\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})) = g(\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})).$$

This means that the tuple $\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})$ also satisfies the equation (f, g). This holds for every equation of $\mathcal{E}$, thus we have $\Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$. □

Example 3.2. Let us consider once more the case of linear equations (we use the notation of Examples 2.1 and 2.2). A set of tuples (vectors) $T \subseteq K^n$ is closed under the clone L^* if and only if T is an affine subspace of K^n, and T is closed under $L_0^* = L_0$ if and only if T is a subspace of K^n. Thus in this case T is the solution set of a system of L-equations (L_0-equations) if and only if T is closed under L^* (L_0^*).

Theorem 3.1 gives a necessary condition for a set $T \subseteq A^n$ to be the set of all solutions of a system of C-equations. In the case of (homogeneous) linear equations this condition is sufficient as well (see the example above). In the next section we prove that if A is a two-element set then for every clone $C \leq \mathcal{O}_A$, every set of tuples that is closed under C^* is the solution set of some system of C-equations. However, for a three-element underlying set this is not always the case.

Example 3.3. Let us consider the (nonassociative) binary operation $f(x, y) = x \otimes y$ on $A = \{0, 1, 2\}$ defined by the following operation table:

$\otimes$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	0

Observe that $x \otimes x = 0$ and $x \otimes 0 = 0 \otimes x = 0$ hold identically, hence the only unary operations in the clone $C = [f]$ are $g_0(x) = 0$ and $g_1(x) = x$.

Therefore, the only nontrivial C-equation of arity $n = 1$ is (g_0, g_1), whose solution set is $\{0\}$. Thus there are only two subsets $T \subseteq A$ that are solution sets of (systems of) unary C-equations, namely $T = \{0\}$ and $T = \{0, 1, 2\}$. However, the set $\{0, 1\}$ is also closed under C^*. Indeed, if $\Phi \in C^*$ is an m-ary operation and $a_1, \ldots, a_m \in \{0, 1\}$, then, observing that $a_i = a_i \otimes 2$, we can compute $\Phi(\mathbf{a}) = \Phi(a_1, \ldots, a_m)$ as follows:

$$\Phi(\mathbf{a}) = \Phi(a_1 \otimes 2, \ldots, a_m \otimes 2) = \Phi(\mathbf{a}) \otimes \Phi(\mathbf{2}) = f(\Phi(\mathbf{a}), \Phi(\mathbf{2})). \qquad (3.2)$$

Since the range of f contains only the elements 0 and 1, we see that the right hand side of (3.2) belongs to $\{0, 1\}$. We can conclude that the set $\{0, 1\}$ is closed under C^*, yet it is not the solution set of any system of C-equations.

4. Boolean equations

In this section we consider exclusively Boolean equations, that is, from now on our underlying set is $A = \{0, 1\}$. We will use the notation of the appendix; in particular, $\Omega = \mathcal{O}_{\{0,1\}}$ stands for the set of all Boolean functions. By proving a converse of Theorem 3.1, we will establish the following characterization of solution sets of Boolean equations.

Theorem 4.1. *For any Boolean clone $C \leq \Omega$ and $T \subseteq \{0, 1\}^n$, the following two conditions are equivalent:*

(i) *there is a system $\mathcal{E}$ of C-equations such that $T = \mathrm{Sol}(\mathcal{E})$;*
(ii) *T is closed under C^*.*

The implication (i) $\implies$ (ii) follows from Theorem 3.1, so we only need to prove that (ii) implies (i). Since all Boolean clones are known (see the appendix), we could do this one by one for every single Boolean clone. However, many clones have the same centralizer, therefore, as the following remark shows, it suffices to prove Theorem 4.1 for a few clones (note that this remark is valid for any set A, not just for the two-element set).

Remark 4.2. Let $C_1 \leq C_2 \leq \mathcal{O}_A$ and $C_1^* = C_2^* = C$. Assume that Theorem 4.1 is true for C_1, and let $T \subseteq A^n$ be closed under C. Then there is a system of C_1-equations such that $T = \mathrm{Sol}(\mathcal{E})$. From $C_1 \subseteq C_2$ it follows that $\mathcal{E}$ is also a system of C_2-equations. Thus Theorem 4.1 holds for C_2 as well.

We can further reduce the number of cases by considering Boolean functions up to duality. The *dual* of $f \in \Omega^{(n)}$ is the Boolean function f^d defined by $f^d(x_1, \ldots, x_n) = \neg f(\neg x_1, \ldots, \neg x_n)$, and the dual of a Boolean clone C is $C^d = \{f^d \mid f \in C\}$. Note that dualizing means just interchanging 0 and 1, hence if Theorem 4.1 holds for C, then it is obviously valid for C^d, too.

Considering the observations above as well as the list of centralizers of Boolean clones given in the appendix, it suffices to prove the implication (ii) $\implies$ (i) of Theorem 4.1 for the following 18 cases:

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1. $L^* = L_{01}$, $L_0^* = L_0$, $L_{01}^* = L$, $SL^* = SL$;
2. $M^* = [x]$, $(U^\infty M)^* = [0]$, $(U_{01}^\infty M)^* = [0,1]$, $S^* = [\neg]$, $SM^* = \Omega^{(1)}$;
3. $\Lambda^* = \Lambda_{01}$, ${\Lambda_0}^* = \Lambda_0$, ${\Lambda_1}^* = \Lambda_1$, ${\Lambda_{01}}^* = \Lambda$;
4. $(\Omega^{(1)})^* = S_{01}$, $[\neg]^* = S$, $[0,1]^* = \Omega_{01}$, $[0]^* = \Omega_0$, $[x]^* = \Omega$.

We will present the proof through a sequence of 18 lemmas. These are grouped into four subsections by the methods used in their proofs, according to the numbering above.

4.1. Linear clones

Lemma 4.3. *If $T \subseteq \{0,1\}^n$ is closed under the clone ${L_0}^* = L_0$, then there exists a system $\mathcal{E}$ of L_0-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. This is a special case of Example 3.2 for the two-element field. □

Lemma 4.4. *If $T \subseteq \{0,1\}^n$ is closed under the clone ${L_{01}}^* = L$, then there exists a system $\mathcal{E}$ of L_{01}-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${L_{01}}^* = L$. Since T is closed under $L = [x+y,1]$, it is a subspace in $\{0,1\}^n$, and we also have $\mathbf{1} \in T$. Therefore there exists a system of homogeneous linear equations $\mathcal{E}$ such that the set of solutions of $\mathcal{E}$ is exactly T. It only remains to verify that $\mathcal{E}$ is equivalent to a system of L_{01}-equations. Recall that $L_{01} = \{x_1 + \cdots + x_n \mid n \text{ is odd}\}$.

An equation in $\mathcal{E}$ is of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_m} = 0$. Since $\mathbf{1} \in T$, the tuple $\mathbf{1}$ satisfies this equation, hence it follows that $2 \mid m$. Adding x_{i_1} to both sides, we obtain the equivalent equation $x_{i_2} + \cdots + x_{i_m} = x_{i_1}$. Since there is an odd number of variables on both sides, this is an L_{01}-equation. □

Lemma 4.5. *If $T \subseteq \{0,1\}^n$ is closed under the clone $L^* = L_{01}$, then there exists a system $\mathcal{E}$ of L-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. This is a special case of Example 3.2 for the two-element field. □

Lemma 4.6. *If $T \subseteq \{0,1\}^n$ is closed under the clone $SL^* = SL$, then there exists a system $\mathcal{E}$ of SL-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $SL^* = SL$. Note that

$$SL = [x+y+z, x+1] = \{x_1 + \cdots + x_n + c \mid n \text{ is odd, and } c \in \{0,1\}\}.$$

Since $SL \supseteq L_{01}$ we see that T is an affine subspace in $\{0,1\}^n$, hence there exists a system $\mathcal{E}$ of linear equations such that $T = \mathrm{Sol}(\mathcal{E})$. Moreover, since $x+1 \in SL$, we have $\mathbf{x} \in T \Rightarrow \neg\mathbf{x} \in T$. It only remains to verify that $\mathcal{E}$ is equivalent to a system of SL-equations.

An equation in $\mathcal{E}$ is of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_m} = c$. Since $\mathbf{x} \in T$ implies that $\neg\mathbf{x} \in T$, it follows that $2 \mid m$. Our equation is equivalent to

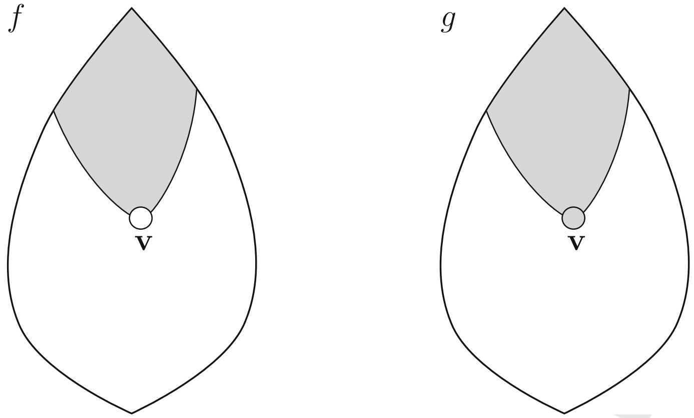

FIGURE 1. The functions f and g in the proof of Lemma 4.7

$x_{i_2} + \cdots + x_{i_m} = x_{i_1} + c$, and since at both sides of the equation there is an odd number of variables, it follows that this is an SL-equation. $\square$

4.2. Clones with unary centralizers

Lemma 4.7. *If $T \subseteq \{0,1\}^n$ is closed under the clone $M^* = [x]$, then there exists a system $\mathcal{E}$ of M-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Note that every subset of $\{0,1\}^n$ is closed under $[x]$. For every $T \subsetneq \{0,1\}^n$, we have

$$T = \bigcap_{\mathbf{v} \notin T} T_{\mathbf{v}}, \tag{4.1}$$

where $T_{\mathbf{v}} = \{0,1\}^n \setminus \{\mathbf{v}\}$. Therefore it suffices to show that for every $\mathbf{v} \in \{0,1\}^n$, there exists an M-equation (f,g) such that $T_{\mathbf{v}} = \mathrm{Sol}(\{(f,g)\})$.

Let $\mathbf{v} \in \{0,1\}^n$ be an arbitrary n-tuple. Let f and g be the following functions:

$$f(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} > \mathbf{v}; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad g(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \geq \mathbf{v}; \\ 0, & \text{otherwise.} \end{cases}$$

Figure 1 shows a schematic view of the Hasse diagram of $\{0,1\}^n$. Grey color indicates points where the value of the corresponding function is 1; on the remaining tuples the values are 0. It is easy to see that $f, g \in M$ and that for all $\mathbf{v} \in \{0,1\}^n$, we have $f(\mathbf{x}) = g(\mathbf{x})$ if and only if $\mathbf{x} \neq \mathbf{v}$, therefore the set of solutions of $f(\mathbf{x}) = g(\mathbf{x})$ is indeed $T_{\mathbf{v}}$. $\square$

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Lemma 4.8. *If* $T \subseteq \{0,1\}^n$ *is closed under the clone* $(U^\infty M)^* = [0]$*, then there exists a system* $\mathcal{E}$ *of* $U^\infty M$*-equations such that* $T = \mathrm{Sol}(\mathcal{E})$.

Proof. A set $T \subseteq \{0,1\}^n$ is closed under $[0]$ if and only if $\mathbf{0} \in T$. Thus, similarly to the proof of Lemma 4.7, it suffices to show that for every $\mathbf{v} \in \{0,1\}^n \backslash \{\mathbf{0}\}$ there exists a $U^\infty M$-equation (f,g) such that $T_\mathbf{v} = \mathrm{Sol}(\{(f,g)\})$. (We can exclude $\mathbf{v} = \mathbf{0}$ from the intersection (4.1) because $\mathbf{0} \in T$.)

Let $\mathbf{v} \in \{0,1\}^n \backslash \{\mathbf{0}\}$ be an arbitrary n-tuple, and let f and g be the same functions, as defined in the proof of Lemma 4.7. We have seen that f and g are monotone and $\mathrm{Sol}(\{(f,g)\}) = T_\mathbf{v}$. Hence it only remains to verify that $f, g \in U^\infty$, that is, there exists a $k \in \mathbb{N}$ such that for all $\mathbf{x} \in \{0,1\}^n$, if $f(\mathbf{x}) = 1$ ($g(\mathbf{x}) = 1$), then $x_k = 1$. We may assume (after a permutation of coordinates) that $\mathbf{v}$ is of the form $(0,0,\dots,0,1,1,\dots,1)$. Since $\mathbf{v} \neq \mathbf{0}$, at least one 1 appears in $\mathbf{v}$, i.e., $v_n = 1$. If $f(\mathbf{x}) = 1$, then $\mathbf{x} > \mathbf{v}$, hence $x_n = 1$, thus $f \in U^\infty$. Similarly, $x_n = 1$ whenever $g(\mathbf{x}) = 1$, so $g \in U^\infty$. □

Lemma 4.9. *If* $T \subseteq \{0,1\}^n$ *is closed under the clone* $(U_{01}^\infty M)^* = [0,1]$*, then there exists a system* $\mathcal{E}$ *of* $U_{01}^\infty M$*-equations such that* $T = \mathrm{Sol}(\mathcal{E})$.

Proof. The proof is almost identical to those of the previous two lemmas. Here we have $\mathbf{0}, \mathbf{1} \in T$, hence we can assume that $\mathbf{v} \notin \{\mathbf{0}, \mathbf{1}\}$, and we only need to show that in this case the functions f and g defined in the proof of Lemma 4.7 are 0-preserving as well as 1-preserving. By the definition of the functions f and g, it is obvious that $f(\mathbf{0}) = 0$ and $g(\mathbf{1}) = 1$. Moreover, $\mathbf{v} \neq \mathbf{0}$ implies that $g(\mathbf{0}) = 0$ and $\mathbf{v} \neq \mathbf{1}$ implies that $f(\mathbf{1}) = 1$. Thus $f, g \in U_{01}^\infty M$, as claimed. □

Lemma 4.10. *If* $T \subseteq \{0,1\}^n$ *is closed under the clone* $S^* = [\neg]$*, then there exists a system* $\mathcal{E}$ *of* S*-equations such that* $T = \mathrm{Sol}(\mathcal{E})$.

Proof. For every $T \subsetneq \{0,1\}^n$ that is closed under the clone $[\neg]$, we have

$$T = \bigcap_{\mathbf{v} \notin T} T_\mathbf{v},$$

where $T_\mathbf{v} = \{0,1\}^n \backslash \{\mathbf{v}, \neg\mathbf{v}\}$. (Note that we are changing the notation of the previous three lemmas.) Therefore it suffices to show that for every $\mathbf{v} \in \{0,1\}^n$ there exists an S-equation (f,g) such that $T_\mathbf{v} = \mathrm{Sol}(\{(f,g)\})$.

Let $\mathbf{v} \in \{0,1\}^n$ be an arbitrary n-tuple, and let $f \in S$ be an arbitrary n-ary self-dual function. Define the function g by

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \notin \{\mathbf{v}, \neg\mathbf{v}\}; \\ \neg f(\mathbf{x}), & \text{if } \mathbf{x} \in \{\mathbf{v}, \neg\mathbf{v}\}. \end{cases}$$

Clearly, the set of solutions of $f(\mathbf{x}) = g(\mathbf{x})$ is indeed $T_\mathbf{v}$, and it is straightforward to verify that g is self-dual. □

Lemma 4.11. *If* $T \subseteq \{0,1\}^n$ *is closed under the clone* $SM^* = \Omega^{(1)}$*, then there exists a system* $\mathcal{E}$ *of* SM*-equations such that* $T = \mathrm{Sol}(\mathcal{E})$.

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Proof. Using the notation of Lemma 4.10, we need to show that for every $\mathbf{v} \in \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$ there exists an SM-equation (f, g) such that $T_{\mathbf{v}} = \mathrm{Sol}(\{(f, g)\})$. (We exclude $\mathbf{0}$ and $\mathbf{1}$ since T is closed under $\Omega^{(1)} = [0, 1, \neg x]$.)

Let $\mathbf{v} \in \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$, and let $h \in SM$ be an arbitrary n-ary self-dual monotone function. Define the function f by

$$f(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \leq \mathbf{v} \text{ or } \mathbf{x} < \neg\mathbf{v}; \\ 1, & \text{if } \mathbf{x} > \mathbf{v} \text{ or } \mathbf{x} \geq \neg\mathbf{v}; \\ h(\mathbf{x}), & \text{otherwise.} \end{cases}$$

Since $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$, the tuples $\mathbf{v}$ and $\neg\mathbf{v}$ are incomparable, hence the three cases in the definition of f are mutually exclusive and thus f is well defined. Define the function g by

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } \mathbf{x} \notin \{\mathbf{v}, \neg\mathbf{v}\}; \\ \neg f(\mathbf{x}), & \text{if } \mathbf{x} \in \{\mathbf{v}, \neg\mathbf{v}\}. \end{cases}$$

Let H be the set of tuples $\mathbf{x} \in \{0,1\}^n$ that are incomparable to both $\mathbf{v}$ and $\neg\mathbf{v}$. (Note that H is closed under negation.) The colors on Figure 2 indicate the value of the corresponding function as in the proof of Lemma 4.7. The striped area represents the set H. From the definition of the function g it is clear that the set of solutions of $f(\mathbf{x}) = g(\mathbf{x})$ is indeed $T_{\mathbf{v}}$.

It only remains to verify that $f, g \in SM$, that is, f and g are both monotone and self-dual. We present the details for f only; the proof for g is similar.

Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary n-tuples with $\mathbf{x} \leq \mathbf{y}$. To verify that $f \in M$, we consider four cases:

1. If $\mathbf{x}, \mathbf{y} \in H$, then $f(\mathbf{x}) = h(\mathbf{x}) \leq h(\mathbf{y}) = f(\mathbf{y})$, as $h \in SM$.
2. If $\mathbf{x}, \mathbf{y} \notin H$, then from the definition of the function f we have $f(\mathbf{x}) \leq f(\mathbf{y})$.
3. If $\mathbf{x} \in H$ and $\mathbf{y} \notin H$, then $\mathbf{y}$ is comparable to $\mathbf{v}$ or $\neg\mathbf{v}$. If $f(\mathbf{y}) = 1$, then obviously $f(\mathbf{x}) \leq f(\mathbf{y})$. If $f(\mathbf{y}) = 0$, then $\mathbf{y} \leq \mathbf{v}$ or $\mathbf{y} < \neg\mathbf{v}$. However, in this case $\mathbf{x} \leq \mathbf{y}$ implies that $\mathbf{x}$ is comparable to $\mathbf{v}$ or to $\neg\mathbf{v}$, contradicting the assumption $\mathbf{x} \in H$.
4. The case $\mathbf{x} \notin H$, $\mathbf{y} \in H$ can be verified similarly to the previous case.

For self-duality, let $\mathbf{x} \in \{0,1\}^n$ be an arbitrary n-tuple; we need to show that $f(\mathbf{x}) = \neg f(\neg\mathbf{x})$. We distinguish two cases:

1. If $\mathbf{x} \notin H$, then $\neg\mathbf{x} \notin H$. If $f(\mathbf{x}) = 0$, then either $\mathbf{x} \leq \mathbf{v}$ or $\mathbf{x} < \neg\mathbf{v}$. In the first case, we have $\neg\mathbf{x} \geq \neg\mathbf{v}$, and in the second case, we have $\neg\mathbf{x} > \mathbf{v}$. In both cases, $f(\neg\mathbf{x}) = 1$. Similarly, $f(\mathbf{x}) = 1$ implies that $f(\neg\mathbf{x}) = 0$.
2. If $\mathbf{x} \in H$, then $\neg\mathbf{x} \in H$, therefore $f(\mathbf{x}) = h(\mathbf{x}) = \neg h(\neg\mathbf{x}) = \neg f(\neg\mathbf{x})$, as $h \in SM$. ☐

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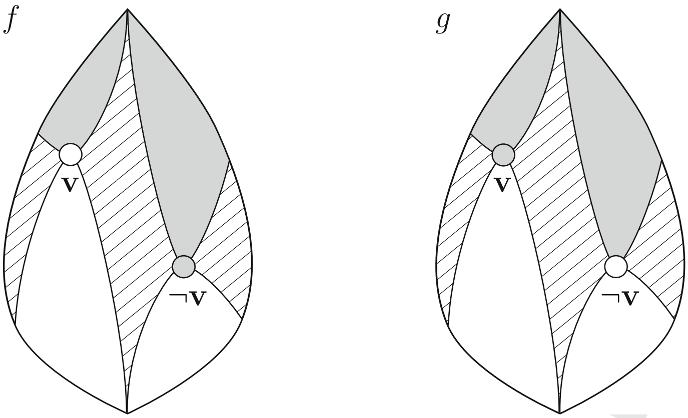

Figure 2. The functions f and g in the proof of Lemma 4.11

4.3. Clones generated by conjunctions and constants

Lemma 4.12. *If $T \subseteq \{0,1\}^n$ is closed under the clone $\Lambda^* = \Lambda_{01}$, then there exists a system $\mathcal{E}$ of Λ-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Note that $\Lambda = [x \wedge y, 0, 1]$, and that $\Lambda_{01} = [x \wedge y]$. Let $T \subseteq \{0,1\}^n$ be closed under the clone $\Lambda^* = \Lambda_{01}$, and let $\mathcal{E} = \mathrm{Eq}_\Lambda(T)$. We will show that $T = \mathrm{Sol}(\mathcal{E})$. Since $T \subseteq \mathrm{Sol}(\mathcal{E})$ is trivial, it suffices to prove that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ implies $\mathbf{v} \in T$ for all $\mathbf{v} \in \{0,1\}^n$.

Let $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$, and suppose first that $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$. We may assume without loss of generality that $\mathbf{v}$ is of the form $(1,1,\ldots,1,0,0,\ldots,0)$, where $v_1 = \cdots = v_k = 1$ and $v_{k+1} = \cdots = v_n = 0$ $(k \in \{1,\ldots,n-1\})$. Let us consider the following Λ-equation:

$$x_1 \wedge \cdots \wedge x_k = x_1 \wedge \cdots \wedge x_k \wedge x_{k+1}. \tag{4.2}$$

It is clear that $\mathbf{v}$ does not satisfy (4.2), thus the Eq. (4.2) does not appear in $\mathcal{E}$. Hence, there exists an n-tuple $\mathbf{t}^{(1)} \in T$ such that $\mathbf{t}^{(1)}$ does not satisfy (4.2), i.e., $t_1^{(1)} = \cdots = t_k^{(1)} = 1$ and $t_{k+1}^{(1)} = 0$. Similarly, for all $m \in \{1,\ldots,n-k\}$ we may consider the Λ-equation

$$x_1 \wedge \cdots \wedge x_k = x_1 \wedge \cdots \wedge x_k \wedge x_{k+m}. \tag{4.3}$$

Just like (4.2), the equation (4.3) does not appear in $\mathcal{E}$, thus there exists $\mathbf{t}^{(m)} \in T$ such that $t_1^{(m)} = \cdots = t_k^{(m)} = 1$ and $t_{k+m}^{(m)} = 0$. We know that T is closed under the clone Λ_{01}, in particular, T is closed under conjunctions. Therefore $\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(n-k)} \in T$ implies that

$$\mathbf{t}^{(1)} \wedge \cdots \wedge \mathbf{t}^{(n-k)} = (1,1,\ldots,1,0,0,\ldots,0) = \mathbf{v} \in T.$$

It only remains to consider the cases $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} = \mathbf{1}$. If $\mathbf{v} = \mathbf{0}$ satisfies $\mathcal{E}$, then let us consider the following Λ-equations for all $i \in \{1, \dots, n\}$:

$$x_i = 1. \tag{4.4}$$

Since $\mathbf{v} = \mathbf{0}$ does not satisfy (4.4), this equation does not belong to $\mathcal{E}$. Thus T contains a counterexample $\mathbf{t}^{(i)}$ to (4.4) such that $t_i^{(i)} = 0$. Therefore we have

$$\mathbf{t}^{(1)} \wedge \cdots \wedge \mathbf{t}^{(n)} = (0, 0, \dots, 0) = \mathbf{v} \in T.$$

If $\mathbf{v} = \mathbf{1}$ satisfies $\mathcal{E}$, then we consider the following Λ-equation:

$$x_1 \wedge \cdots \wedge x_n = 0. \tag{4.5}$$

Similarly as above, T contains a counterexample to (4.5), and the only such counterexample is $\mathbf{1}$. □

Lemma 4.13. *If $T \subseteq \{0,1\}^n$ is closed under the clone ${\Lambda_0}^* = \Lambda_0$, then there exists a system $\mathcal{E}$ of Λ_0-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${\Lambda_0}^* = \Lambda_0$, and define $\mathcal{E}$ as $\mathcal{E} = \mathrm{Eq}_{\Lambda_0}(T)$. If $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ and $\mathbf{v} \neq \mathbf{0}$, then the same argument as in Lemma 4.12 proves that $\mathbf{v} \in T$. It only remains to consider the case $\mathbf{v} = \mathbf{0}$. Since T is closed under the clone Λ_0 and $\mathbf{0} \in \Lambda_0$, it follows that $\mathbf{0} \in T$. □

Lemma 4.14. *If $T \subseteq \{0,1\}^n$ is closed under the clone ${\Lambda_1}^* = \Lambda_1$, then there exists a system $\mathcal{E}$ of Λ_1-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${\Lambda_1}^* = \Lambda_1$, and define $\mathcal{E}$ as $\mathcal{E} = \mathrm{Eq}_{\Lambda_1}(T)$. If $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ and $\mathbf{v} \neq \mathbf{1}$, then the same argument as in Lemma 4.12 proves that $\mathbf{v} \in T$. Since T is closed under the clone Λ_1 and $\mathbf{1} \in \Lambda_1$, it follows that $\mathbf{1} \in T$. □

Lemma 4.15. *If $T \subseteq \{0,1\}^n$ is closed under the clone ${\Lambda_{01}}^* = \Lambda$, then there exists a system $\mathcal{E}$ of Λ_{01}-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone ${\Lambda_{01}}^* = \Lambda$, and define $\mathcal{E}$ as $\mathcal{E} = \mathrm{Eq}_{\Lambda_{01}}(T)$. If $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ and $\mathbf{v} \neq \mathbf{0}, \mathbf{1}$, then the same argument as in Lemma 4.12 proves that $\mathbf{v} \in T$. Since T is closed under the clone Λ and $\mathbf{0}, \mathbf{1} \in \Lambda$, it follows that $\mathbf{0}, \mathbf{1} \in T$. □

4.4. Unary clones

Lemma 4.16. *If $T \subseteq \{0,1\}^n$ is closed under the clone $[x]^* = \Omega$, then there exists a system $\mathcal{E}$ of $[x]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $[x]^* = \Omega$, and let $\mathcal{E} = \mathrm{Eq}_{[x]}(T)$. We will show that $T = \mathrm{Sol}(\mathcal{E})$. Since $T \subseteq \mathrm{Sol}(\mathcal{E})$ is trivial, it suffices to prove that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$ implies $\mathbf{v} \in T$ for all $\mathbf{v} \in \{0,1\}^n$.

Let $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$, and let $T = \{\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}\}$, where $m = |T|$. Let us consider the matrix $Q = (t_i^{(j)}) \in \{0,1\}^{n \times m}$ whose j-th column vector is $\mathbf{t}^{(j)}$. Let $\mathbf{r}_i = (t_i^{(1)}, \ldots, t_i^{(m)})$ be the i-th row of Q, and let $R = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ be the set of row vectors of Q. Define the m-ary function Φ by

$$\Phi(\mathbf{x}) = \begin{cases} v_i, & \text{if } \mathbf{x} = \mathbf{r}_i; \\ 0, & \text{if } \mathbf{x} \notin R. \end{cases}$$

Note that Φ is defined in such a way that $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)})$. However, we need to verify that Φ is a well-defined function. Assume that $\mathbf{r}_i = \mathbf{r}_j$ and $v_i \neq v_j$ for some $i, j \in \{1, \ldots, n\}$. From $\mathbf{r}_i = \mathbf{r}_j$ it follows that T satisfies the $[x]$-equation $x_i = x_j$, hence this equation belongs to $\mathcal{E}$. On the other hand, $\mathbf{v}$ satisfies $\mathcal{E}$, thus $v_i = v_j$, which is a contradiction. Therefore the function Φ is well defined, and obviously $\Phi \in \Omega$. The set T is closed under the clone Ω, hence $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$. □

Lemma 4.17. *If $T \subseteq \{0,1\}^n$ is closed under the clone $[0]^* = \Omega_0$, then there exists a system $\mathcal{E}$ of $[0]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $[0]^* = \Omega_0$, let $\mathcal{E} = \mathrm{Eq}_{[0]}(T)$, and assume that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$. Define Q, $\mathbf{r}_i$, R and Φ as in the proof of Lemma 4.16. The proof of Lemma 4.16 shows that Φ is well defined; we only need to verify that $\Phi \in \Omega_0$. If $\mathbf{0} \notin R$, then $\Phi(\mathbf{0}) = 0$ follows from the definition of Φ. If $\mathbf{r}_i = \mathbf{0}$ for some i, then the $[0]$-equation $x_i = 0$ holds in T, thus $(x_i, 0) \in \mathcal{E}$. Therefore $\mathbf{v}$ satisfies this equation as well, hence $\Phi(\mathbf{0}) = \Phi(\mathbf{r}_i) = v_i = 0$. This shows that $\Phi \in \Omega_0$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under Ω_0. □

Lemma 4.18. *If $T \subseteq \{0,1\}^n$ is closed under the clone $[0,1]^* = \Omega_{01}$, then there exists a system $\mathcal{E}$ of $[0,1]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. The proof is almost identical to that of Lemma 4.17; we just need to modify the definition of Φ such that $\Phi(\mathbf{1}) = 1$ if $\mathbf{1} \notin R$. Taking equations of the form $x_i = 0$ and $x_i = 1$ into account, we can prove that $\Phi \in \Omega_{01}$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \ldots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under Ω_{01}. □

Lemma 4.19. *If $T \subseteq \{0,1\}^n$ is closed under the clone $[\neg]^* = S$, then there exists a system $\mathcal{E}$ of $[\neg]$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0,1\}^n$ be closed under the clone $[\neg]^* = S$, let $\mathcal{E} = \mathrm{Eq}_{[\neg]}(T)$, and assume that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$. Define Q, $\mathbf{r}_i$ and R as in the proof of Lemma 4.16

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and let $R' = \{\neg \mathbf{r}_1, \dots, \neg \mathbf{r}_n\}$. Let $h \in S$ be an arbitrary m-ary self-dual function and define the function $\Phi \in \Omega^{(m)}$ by

$$\Phi(\mathbf{x}) = \begin{cases} v_i, & \text{if } \mathbf{x} = \mathbf{r}_i; \\ \neg v_i, & \text{if } \mathbf{x} = \neg \mathbf{r}_i; \\ h(\mathbf{x}), & \text{if } \mathbf{x} \notin R \cup R'. \end{cases}$$

We show that the function Φ is well defined. We distinguish two cases:

1. If $\mathbf{r}_i = \mathbf{r}_j$ and $v_i \neq v_j$ for some $i, j \in \{1, \dots, n\}$, then T satisfies the $[\neg]$-equation $x_i = x_j$, hence this equation belongs to $\mathcal{E}$. On the other hand, $\mathbf{v}$ satisfies $\mathcal{E}$, thus $v_i = v_j$, which is a contradiction.
2. If $\mathbf{r}_i = \neg \mathbf{r}_j$ and $v_i \neq \neg v_j$ for some $i, j \in \{1, \dots, n\}$, then T satisfies the $[\neg]$-equation $x_i = \neg x_j$, hence this equation appears in $\mathcal{E}$. On the other hand, $\mathbf{v}$ satisfies $\mathcal{E}$, thus $v_i = \neg v_j$, which is a contradiction.

It only remains to verify that $\Phi \in S$. Let $\mathbf{a}$ be an arbitrary n-tuple. If $\mathbf{a} \notin R \cup R'$, then $\Phi(\mathbf{a}) = h(\mathbf{a}) = \neg h(\neg \mathbf{a}) = \neg \Phi(\neg \mathbf{a})$, since the function h is self-dual. If $\mathbf{a} = \mathbf{r}_i$ for some $i \in \{1, \dots, n\}$, then $\neg \mathbf{a} = \neg \mathbf{r}_i$, thus $\Phi(\neg \mathbf{a}) = \neg v_i = \neg \Phi(\mathbf{a})$. This shows that $\Phi \in S$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under S. □

Lemma 4.20. *If $T \subseteq \{0, 1\}^n$ is closed under the clone $(\Omega^{(1)})^* = S_{01}$, then there exists a system $\mathcal{E}$ of $\Omega^{(1)}$-equations such that $T = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $T \subseteq \{0, 1\}^n$ be closed under the clone $(\Omega^{(1)})^* = S_{01}$, let $\mathcal{E} = \mathrm{Eq}_{\Omega^{(1)}}(T)$, and assume that $\mathbf{v} \in \mathrm{Sol}(\mathcal{E})$. Define Q, $\mathbf{r}_i$, R and R' as in the proof of Lemma 4.19, and let us also define Φ in the same way as there, but this time choosing the function h from S_{01}. We can follow the same argument as before, but we also need to verify that $\Phi \in \Omega_{01}$. If $\mathbf{0} \notin R \cup R'$, then $\Phi(\mathbf{0}) = 0$, since $h \in S_{01}$. If $\mathbf{0} \in R$, and $\mathbf{0} = \mathbf{r}_i$, then the $\Omega^{(1)}$-equation $x_i = 0$ holds in $\mathcal{E}$, thus $v_i = 0$. Therefore, from the definition of the function Φ, we have $\Phi(\mathbf{0}) = 0$. If $\mathbf{0} \in R'$, and $\mathbf{0} = \neg \mathbf{r}_i$, then the $\Omega^{(1)}$-equation $\neg x_i = 0$ holds in $\mathcal{E}$, thus $\neg v_i = 0$, hence $\Phi(\mathbf{0}) = 0$. This proves that $\Phi \in \Omega_0$, and a similar argument shows that $\Phi \in \Omega_1$. Therefore $\Phi \in S_{01}$, and then $\mathbf{v} = \Phi(\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(m)}) \in T$ follows, as T is closed under S_{01}. □

5. Boolean functional equations

A framework for functional equations was presented in [2], which includes many classical functional equations as special cases (see the examples in [2]). The problem of characterizing solution sets of functional equations was posed there, and a general necessary condition was also established, which is similar to our Theorem 3.1. Here we prove that for Boolean functions that condition is also sufficient, thus we obtain a complete characterization of solution sets of Boolean functional equations.

First let us recall the abstract definition of a functional equation proposed in [2]. Let $\mathcal{A}$ and $\mathcal{B}$ be clones on sets A and B, respectively. A $(\mathcal{B}, \mathcal{A})$ *-equation* is a functional equation of the form

$$\begin{aligned} & u(\mathbf{f}(g_{11}, \ldots, g_{1n}), \ldots, \mathbf{f}(g_{r1}, \ldots, g_{rn})) \\ & \quad = v(\mathbf{f}(h_{11}, \ldots, h_{1n}), \ldots, \mathbf{f}(h_{s1}, \ldots, h_{sn})), \end{aligned} \tag{5.1}$$

where $r, s, n \geq 0$, $u \in \mathcal{B}^{(r)}, v \in \mathcal{B}^{(s)}$, each g_{ij} and h_{ij} is a function in $\mathcal{A}^{(m)}$, $m \geq 0$, and $\mathbf{f}$ is an n-ary function symbol. Observe that if we interpret the function symbol $\mathbf{f}$ by a function $f\colon A^n \to B$, then each side of (5.1) becomes an m-ary function from A to B. If these two functions coincide, then f is a solution of the equation. We can define systems of functional equations and solution sets in a natural way (similarly to Sect. 2.3).

The following theorem gives the promised characterization of solution sets of functional equations in the case of Boolean functions (i.e., for $A = B = \{0, 1\}$).

Theorem 5.1. *A class $\mathcal{K}$ of n-ary Boolean functions is the solution set of a system of $(\mathcal{B}, \mathcal{A})$-equations if and only if the following two conditions hold:*

(A) *for every $f \in \mathcal{K}$ and $\varphi \in (\mathcal{A}^*)^{(1)}$ we have $f(\varphi(x_1), \ldots, \varphi(x_n)) \in \mathcal{K}$, and*
(B) *for every $\ell \geq 0, f_1, \ldots, f_\ell \in \mathcal{K}$ and $\Phi \in (\mathcal{B}^*)^{(\ell)}$ we have $\Phi(f_1, \ldots, f_\ell) \in \mathcal{K}$.*

The "only if" part was proved in Proposition 5 of [2] for arbitrary functions (not only for Boolean functions). For the "if" part, we need to show that if $\mathcal{K} \subseteq \Omega^{(n)}$ satisfies the two conditions of the theorem, then it is the set of all solutions of some system of $(\mathcal{B}, \mathcal{A})$-equations, or, using the terminology of [2], $\mathcal{K}$ is *definable* by $(\mathcal{B}, \mathcal{A})$-equations. We present the proof through several lemmas. First we show how to use our Theorem 4.1 and condition (B) to find a system of functional equations (but not $(\mathcal{B}, \mathcal{A})$-equations yet) whose solution set is $\mathcal{K}$.

Lemma 5.2. *If $\mathcal{K} \subseteq \Omega^{(n)}$ satisfies condition (B), then there is a system of $(\mathcal{B}, [0, 1])$-equations such that $\mathcal{K} = \mathrm{Sol}(\mathcal{E})$.*

Proof. Let $N = 2^n$, and let $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\} = \{0, 1\}^n$. To every function $f \in \Omega^{(n)}$ we can assign a tuple $\vec{f} \in \{0, 1\}^N$ by listing all the values of the function: $\vec{f} := (f(\mathbf{a}_1), \ldots, f(\mathbf{a}_N))$. Condition (B) implies that the set $\overrightarrow{\mathcal{K}} := \left\{\vec{f} \mid f \in \mathcal{K}\right\} \subseteq \{0, 1\}^N$ is closed under the clone $\mathcal{B}$ (cf. Example 6 of [2]). Therefore, by Theorem 4.1, $\overrightarrow{\mathcal{K}}$ is definable by a system of $\mathcal{B}$-equations. Let (u, v) be one of the defining equations of $\overrightarrow{\mathcal{K}}$ (where $u, v \in \mathcal{B}^{(N)}$), and let us rewrite it as a functional equation:

$$u(\mathbf{f}(\mathbf{a}_1), \ldots, \mathbf{f}(\mathbf{a}_N)) = v(\mathbf{f}(\mathbf{a}_1), \ldots, \mathbf{f}(\mathbf{a}_N)). \tag{5.2}$$

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For example, if $n=2$, then (5.2) takes this form:

$$u(\mathbf{f}(0,0),\,\mathbf{f}(0,1),\,\mathbf{f}(1,0),\,\mathbf{f}(1,1)) = v(\mathbf{f}(0,0),\,\mathbf{f}(0,1),\,\mathbf{f}(1,0),\,\mathbf{f}(1,1)).$$

Rewriting all the defining equations of $\overrightarrow{\mathcal{K}}$ this way, we get a system $\mathcal{E}$ of functional equations such that $\mathrm{Sol}(\mathcal{E}) = \mathcal{K}$. Regarding the entries of the tuples $\mathbf{a}_i$ in (5.2) as constant functions (which play the role of the functions g_{ij} and h_{ij} in (5.1)), we see that (5.2) is a $(\mathcal{B},[0,1])$-equation and thus $\mathcal{E}$ is a system of $(\mathcal{B},[0,1])$-equations. □

The next step in the proof is to translate the system $\mathcal{E}$ of $(\mathcal{B},[0,1])$-equations found in Lemma 5.2 into a system of $(\mathcal{B},\mathcal{A})$-equations. Condition (A) will play a key role during this translation. Using the list of centralizer clones given in the appendix, it is easy to compute $(\mathcal{A}^*)^{(1)}$ for each Boolean clone $\mathcal{A}$ (one may also use the Post lattice to compute the unary part of $\mathcal{A}^*$ as the intersection $\mathcal{A}^* \cap \Omega^{(1)}$). Up to duality, we have the following possibilities (in the second and the third item $k = 2, 3, \ldots, \infty$):

1. $(\mathcal{A}^*)^{(1)} = \{x\}$ for $\mathcal{A} = \Omega,\ M,\ L,\ \Lambda,\ \Omega^{(1)},\ [0,1]$;
2. $(\mathcal{A}^*)^{(1)} = \{x, 0\}$ for $\mathcal{A} = \Omega_0,\ M_0,\ L_0,\ U^k,\ U^k M,\ \Lambda_0,\ [0]$;
3. $(\mathcal{A}^*)^{(1)} = \{x, 0, 1\}$ for $\mathcal{A} = \Omega_{01},\ M_{01},\ U^k_{01},\ U^k_{01}M,\ \Lambda_{01}$;
4. $(\mathcal{A}^*)^{(1)} = \{x, \neg\}$ for $\mathcal{A} = S,\ SL,\ [\neg]$;
5. $(\mathcal{A}^*)^{(1)} = \{x, 0, 1, \neg\}$ for $\mathcal{A} = S_{01},\ SM,\ L_{01},\ [x]$.

Similarly to Remark 4.2, it is useful to observe that if $\mathcal{A}_1 \leq \mathcal{A}_2$ and $(\mathcal{A}_1^*)^{(1)} = (\mathcal{A}_2^*)^{(1)}$, then condition (A) is the same for $\mathcal{A}_1$ and $\mathcal{A}_2$, and if a class $\mathcal{K}$ is definable by $(\mathcal{B},\mathcal{A}_1)$-equations, then $\mathcal{K}$ is also definable by $(\mathcal{B},\mathcal{A}_2)$-equations. This means that in each of the five lists of clones above, it suffices to prove Theorem 5.1 for the last clone $\mathcal{A}$ in the list, since it is contained in the previous ones (one can verify this with the help of the Post lattice). In the first list this l(e)ast(!) clone is $[0,1]$, hence we have nothing to do: the $(\mathcal{B},[0,1])$-equations of Lemma 5.2 are already $(\mathcal{B},\mathcal{A})$-equations. Thus we only have four cases, and we deal with them one by one in the following four lemmas.

Lemma 5.3. *Let $\mathcal{K} \subseteq \Omega^{(n)}$, $\mathcal{A} = [0]$, and $\mathcal{B} \leq \Omega$. If $\mathcal{K}$ satisfies conditions (A) and (B), then $\mathcal{K}$ is definable by $(\mathcal{B},\mathcal{A})$-equations.*

Proof. First let us note that condition (A) with $\varphi(x) = 0$ means that $f \in \mathcal{K}$ implies that the constant function $f(\mathbf{0})$, regarded as an n-ary function, also belongs to $\mathcal{K}$. According to Lemma 5.2, there is a system $\mathcal{E}$ of $(\mathcal{B},[0,1])$-equations such that $\mathcal{K} = \mathrm{Sol}(\mathcal{E})$, and every equation in $\mathcal{E}$ is of the form (5.2) with $u, v \in \mathcal{B}^{(N)}$. If E is one such equation, then let $\widetilde{E}$ denote the equation obtained from E by replacing each occurrence of 1 in the tuples $\mathbf{a}_i$ by x. For example, if $n=2$, then $\widetilde{E}$ is of the form

$$u(\mathbf{f}(0,0),\,\mathbf{f}(0,x),\,\mathbf{f}(x,0),\,\mathbf{f}(x,x)) = v(\mathbf{f}(0,0),\,\mathbf{f}(0,x),\,\mathbf{f}(x,0),\,\mathbf{f}(x,x)).$$

Since $0, x \in \mathcal{A}$, the functional equation $\widetilde{E}$ is a $(\mathcal{B}, \mathcal{A})$-equation. We claim that $\mathcal{K}$ is the set of all solutions of the system $\widetilde{\mathcal{E}} := \{\widetilde{E} \mid E \in \mathcal{E}\}$.

For each $E \in \mathcal{E}$, the equation $\widetilde{E}$ is formally stronger than E: if a function f satisfies $\widetilde{E}$, then, setting $x = 1$ in $\widetilde{E}$, we see that f also satisfies E. This shows that $\mathrm{Sol}(\widetilde{\mathcal{E}}) \subseteq \mathrm{Sol}(\mathcal{E}) = \mathcal{K}$. Conversely, assume that $f \in \mathcal{K}$ and let $\widetilde{E} \in \widetilde{\mathcal{E}}$; we may assume without loss of generality that E is of the form (5.2). Clearly, f satisfies $\widetilde{E}$ in the case $x = 1$; we need to verify that f satisfies $\widetilde{E}$ for $x = 0$ as well, i.e.,

$$u(f(\mathbf{0}), \dots, f(\mathbf{0})) = v(f(\mathbf{0}), \dots, f(\mathbf{0})). \tag{5.3}$$

Let $g \in \Omega^{(n)}$ be the constant function defined by $g(x_1, \dots, x_n) = f(\mathbf{0})$. As observed at the beginning of the proof, $f \in \mathcal{K}$ implies that $g \in \mathcal{K}$. Since $\mathcal{K} = \mathrm{Sol}(\mathcal{E})$, the function g satisfies every equation in $\mathcal{E}$. In particular, g satisfies E, and this means exactly that (5.3) holds. This proves that f satisfies each equation $\widetilde{E} \in \widetilde{\mathcal{E}}$, hence $f \in \mathrm{Sol}(\widetilde{\mathcal{E}})$. Thus, we have shown that $\mathcal{K} \subseteq \mathrm{Sol}(\widetilde{\mathcal{E}})$, and this completes the proof. □

Lemma 5.4. *Let $\mathcal{K} \subseteq \Omega^{(n)}$, $\mathcal{A} = \Lambda_{01}$, and $\mathcal{B} \leq \Omega$. If $\mathcal{K}$ satisfies conditions (A) and (B), then $\mathcal{K}$ is definable by $(\mathcal{B}, \mathcal{A})$-equations.*

Proof. We start with the system $\mathcal{E}$ of $(\mathcal{B}, [0,1])$-equations defining $\mathcal{K}$, which was constructed in the proof of Lemma 5.2. For each equation $E \in \mathcal{E}$, let $\widetilde{E}$ be the equation obtained from E by replacing each occurrence of 0 by $x \wedge y$ and each occurrence of 1 by x in the tuples $\mathbf{a}_i$. For example, if $n = 2$, then $\widetilde{E}$ is of the form

$$\begin{aligned} &u(\mathbf{f}(x \wedge y, x \wedge y), \mathbf{f}(x \wedge y, x), \mathbf{f}(x, x \wedge y), \mathbf{f}(x, x)) \\ &\quad = v(\mathbf{f}(x \wedge y, x \wedge y), \mathbf{f}(x \wedge y, x), \mathbf{f}(x, x \wedge y), \mathbf{f}(x, x)). \end{aligned} \tag{5.4}$$

Since $x, x \wedge y \in \mathcal{A}$, the set $\widetilde{\mathcal{E}} := \{\widetilde{E} \mid E \in \mathcal{E}\}$ is a system of $(\mathcal{B}, \mathcal{A})$-equations.

Just like in the proof of the previous lemma, it is clear that $\mathrm{Sol}(\widetilde{\mathcal{E}}) \subseteq \mathcal{K}$. To prove the reversed inclusion, let $f \in \mathcal{K}$ and $\widetilde{E} \in \widetilde{\mathcal{E}}$ [again, E is assumed to be in the form (5.2)]. We need to verify that f satisfies $\widetilde{E}$. If $x = 0$, then $\widetilde{E}$ reduces to (5.3), which is true since $\mathcal{K}$ satisfies (A) with $\varphi(x) = 0 \in (\mathcal{A}^*)^{(1)}$. Similarly, (A) with $\varphi(x) = 1 \in (\mathcal{A}^*)^{(1)}$ shows that $\widetilde{E}$ is valid for $x = y = 1$. Finally, if $x = 1$ and $y = 0$, then $\widetilde{E}$ holds because f satisfies E. Thus $f \in \mathrm{Sol}(\widetilde{\mathcal{E}})$, and this proves that $\mathcal{K} \subseteq \mathrm{Sol}(\widetilde{\mathcal{E}})$. □

Lemma 5.5. *Let $\mathcal{K} \subseteq \Omega^{(n)}$, $\mathcal{A} = [\neg]$, and $\mathcal{B} \leq \Omega$. If $\mathcal{K}$ satisfies conditions (A) and (B), then $\mathcal{K}$ is definable by $(\mathcal{B}, \mathcal{A})$-equations.*

Proof. Similarly to the proofs of the previous two lemmas, we translate the system $\mathcal{E}$ of $(\mathcal{B}, [0,1])$-equations from Lemma 5.2 into a system of $(\mathcal{B}, \mathcal{A})$-equations. This time, we replace 0 with x and 1 with $\neg x$ in every tuple $\mathbf{a}_i$ in every equation in $\mathcal{E}$. Let us illustrate this again in the case $n = 2$:

$$u(\mathbf{f}(x,x),\ \mathbf{f}(x,\neg x),\ \mathbf{f}(\neg x,x),\ \mathbf{f}(\neg x,\neg x))$$
$$= v(\mathbf{f}(x,x),\ \mathbf{f}(x,\neg x),\ \mathbf{f}(\neg x,x),\ \mathbf{f}(\neg x,\neg x)).$$

Since $x, \neg x \in \mathcal{A}$, we obtain a system $\widetilde{\mathcal{E}}$ of $(\mathcal{B},\mathcal{A})$-equations this way, and we need to show that $\mathcal{K} \subseteq \mathrm{Sol}(\widetilde{\mathcal{E}})$, as the other containment is obvious.

Assume that $f \in \mathcal{K}$ and let $\widetilde{E} \in \widetilde{\mathcal{E}}$. If $x = 0$ then $\widetilde{E}$ is equivalent to E, which is satisfied by f, as $f \in \mathcal{K} = \mathrm{Sol}(\mathcal{E})$. If $x = 1$, then $\widetilde{E}$ takes the form

$$u(\mathbf{f}(\neg \mathbf{a}_1), \ldots, \mathbf{f}(\neg \mathbf{a}_N)) = v(\mathbf{f}(\neg \mathbf{a}_1), \ldots, \mathbf{f}(\neg \mathbf{a}_N)).$$

This equation for $\mathbf{f} = f$ is the same as E for the function $\mathbf{f} = g$, where $g(x_1, \ldots, x_n) = f(\neg x_1, \ldots, \neg x_n)$. Condition (A) with $\varphi(x) = \neg x$ shows that $g \in \mathcal{K} = \mathrm{Sol}(\mathcal{E})$, hence g satisfies E, and this implies that f satisfies $\widetilde{E}$ for $x = 1$. □

Lemma 5.6. *Let $\mathcal{K} \subseteq \Omega^{(n)}$, $\mathcal{A} = [x]$, and $\mathcal{B} \leq \Omega$. If $\mathcal{K}$ satisfies conditions (A) and (B), then $\mathcal{K}$ is definable by $(\mathcal{B},\mathcal{A})$-equations.*

Proof. The proof is very similar to the previous ones, so we omit the details. We translate $\mathcal{E}$ to a system $\widetilde{\mathcal{E}}$ of $(\mathcal{B},\mathcal{A})$-equations by replacing every 0 by x and every 1 by y. Let $\widetilde{E} \in \widetilde{\mathcal{E}}$ and $f \in \mathcal{K}$. To prove that f satisfies $\widetilde{E}$, we consider four cases: for $x = 0, y = 1$ we get back E; for $x = 0, y = 0$ we use (A) with $\varphi(x) = 0$; for $x = 1, y = 1$ we use (A) with $\varphi(x) = 1$; for $x = 1, y = 0$ we use (A) with $\varphi(x) = \neg x$. □

Appendix

The Post lattice

E.L. Post proved that there are countably infinitely many Boolean clones (i.e., clones over the set $\{0,1\}$), and described them explicitly in [4]. We define only those clones that we use in this paper; see [5] for the explanation of the notation used in the *Post lattice* below.

- Ω is the clone of all Boolean functions: $\Omega = \mathcal{O}_{01}$.
- Ω_0 and Ω_1 denote the clones of *0-preserving* and *1-preserving* functions, respectively: $\Omega_0 = \{f \in \Omega \mid f(\mathbf{0}) = 0\}$, $\quad \Omega_1 = \{f \in \Omega \mid f(\mathbf{1}) = 1\}$.
- Ω_{01} is the clone of *idempotent* functions: $\Omega_{01} = \Omega_0 \cap \Omega_1$.

In general, if C is a clone, then let $C_0 = C \cap \Omega_0$, $C_1 = C \cap \Omega_1$, and $C_{01} = C_0 \cap C_1$.

- $\Omega^{(1)}$ is the clone of all essentially unary functions: $\Omega^{(1)} = [x, \neg x, 0, 1]$.
- M is the clone of *monotone* functions: $M = \{f \in \Omega \mid \mathbf{x} \leq \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})\}$.

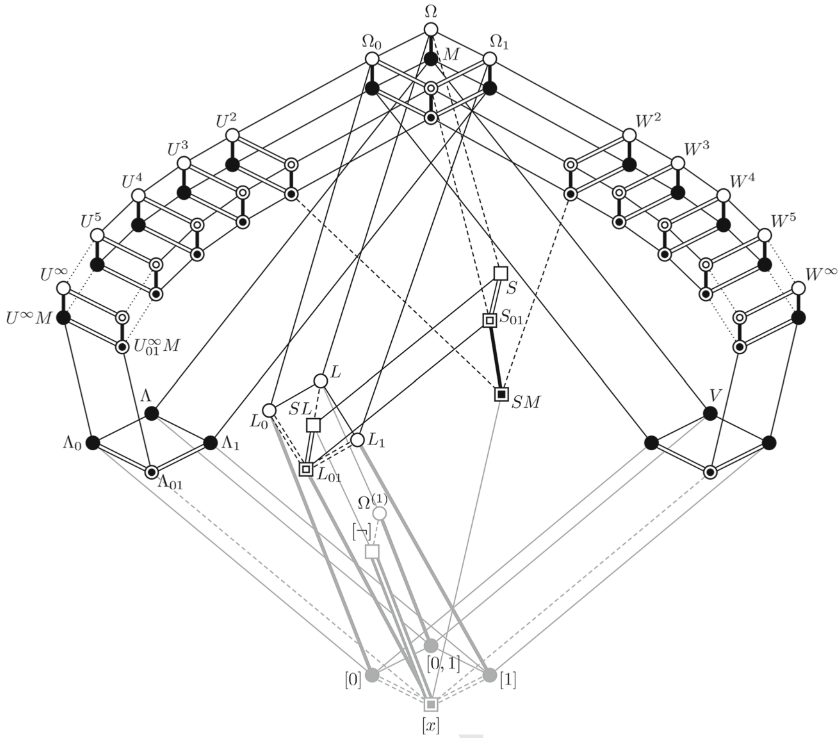

Figure 3. The Post lattice

- $U^\infty = \{f \in \Omega^{(n)} \mid n \in \mathbb{N}_0, \exists k \in \{1, \dots, n\} \colon f(\mathbf{x}) = 1 \implies x_k = 1\}$, and $U^\infty M = U^\infty \cap M$, $U^\infty_{01} M = U^\infty \cap \Omega_{01} \cap M$.
- S is the clone of *self-dual* functions: $S = \{f \in \Omega \mid \neg f(\neg \mathbf{x}) = f(\mathbf{x})\}$.
- $\Lambda = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} \cup [0, 1] = [\wedge, 0, 1]$
- $\Lambda_0 = \Lambda \cap \Omega_0 = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} \cup [0] = [\wedge, 0]$
- $\Lambda_1 = \Lambda \cap \Omega_1 = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} \cup [1] = [\wedge, 1]$
- $\Lambda_{01} = \Lambda \cap \Omega_{01} = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}\} = [\wedge]$
- $L = \{x_1 + \cdots + x_n + c \mid c \in \{0, 1\}, n \in \mathbb{N}_0\} = [x + y, 1]$
- $L_0 = L \cap \Omega_0 = \{x_1 + \cdots + x_n \mid n \in \mathbb{N}_0\} = [x + y]$
- $L_{01} = L \cap \Omega_{01} = \{x_1 + \cdots + x_n \mid n \text{ is odd}\} = [x + y + z]$
- $SL = S \cap L = \big\{x_1 + \cdots + x_n + c \mid n \text{ is odd, and } c \in \{0, 1\}\big\} = [x + y + z, x + 1]$

Centralizer clones of Boolean clones

If a clone D is the centralizer of some clone C, then D is said to be a *primitive positive clone*. All primitive positive Boolean clones are given in [3], but the centralizers of the other (not primitive positive) clones are not given there.

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However, using the Post lattice, one can determine the centralizers of these clones by straightforward calculations. We omit the details and give only the list of all Boolean clones together with their centralizers.

- $[x] = \Omega^* = M^*$
- $[0] = {\Omega_0}^* = {M_0}^* = (U^k)^* = (U^k M)^*$ (for any $k \in \{2, 3, \ldots, \infty\}$)
- $[1] = {\Omega_1}^* = {M_1}^* = (W^k)^* = (W^k M)^*$ (for any $k \in \{2, 3, \ldots, \infty\}$)
- $[0, 1] = {\Omega_{01}}^* = {M_{01}}^* = (U_{01}^k)^* = (U_{01}^k M)^* = (W_{01}^k)^* = (W_{01}^k M)^*$ (for any $k \in \{2, 3, \ldots, \infty\}$)
- $[\neg] = S^*$, $\Omega^{(1)} = {S_{01}}^* = SM^*$
- $L_{01} = L^*$, $L_0 = {L_0}^*$, $L_1 = {L_1}^*$, $L = {L_{01}}^*$, $SL = SL^*$
- $\Lambda_{01} = \Lambda^*$, $\Lambda_0 = {\Lambda_0}^*$, $\Lambda_1 = {\Lambda_1}^*$, $\Lambda = {\Lambda_{01}}^*$
- $V_{01} = V^*$, $V_0 = {V_0}^*$, $V_1 = {V_1}^*$, $V = {V_{01}}^*$
- $S_{01} = (\Omega^{(1)})^*$, $S = [\neg]^*$
- $\Omega_{01} = [0, 1]^*$, $\Omega_0 = [0]^*$, $\Omega_1 = [1]^*$, $\Omega = [x]^*$

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