A PRÍMSZÁMTÉTEL BIZONYÍTÁSÁNAK FŐBB LÉPÉSEI

$$\boxed{\frac{\psi(x)}{x} \sim 1 \implies \pi(x) \sim \frac{x}{\log x}}$$

$$\psi(x) = \sum_{p^k \le x} \log p = \sum_{p \le x} \log p \left[\frac{\log x}{\log p}\right] \asymp \sum_{p \le x} \log x = \pi(x) \log x$$

$$\psi(x) \le \sum_{p^k \le x} \log p \, \frac{\log x}{\log p} = \pi(x) \log x \qquad \psi(x) \ge \sum_{p \le x} \log p \ge \sum_{x^{1-\varepsilon} < p \le x} \log p \ge (\pi(x) - \underbrace{\pi(x^{1-\varepsilon})}_{\le x^{1-\varepsilon}}) \log x^{1-\varepsilon} \ge (1-\varepsilon)\pi(x)\log x - \overbrace{(1-\varepsilon) x^{1-\varepsilon} \log x}^{o(x)}$$

$$\implies \underbrace{\frac{\psi(x)}{x}}_{\downarrow \, 1} \le \frac{\pi(x)\log x}{x} \le \underbrace{\frac{1}{1-\varepsilon}\,\frac{\psi(x)}{x} + \frac{\log x}{x^{\varepsilon}}}_{\downarrow \, \frac{1}{1-\varepsilon}} \implies 1 \le \liminf \frac{\pi(x)\log x}{x} \le \limsup \frac{\pi(x)\log x}{x} \le \frac{1}{1-\varepsilon} \quad (\forall \varepsilon > 0)$$

$$\boxed{-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = \int_0^{\infty} \psi(x) \cdot s \cdot x^{-s-1}\, dx \qquad (\operatorname{Re} s > 1)}$$

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \implies \log \zeta(s) = -\sum_p \log\left(1 - \frac{1}{p^s}\right) = \sum_{p,k} \frac{1}{k p^{ks}} \implies -\frac{\zeta'(s)}{\zeta(s)} = \sum_{p,k} \frac{\log p}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad \text{(vagy konvolúcióval)}$$

U_{p^k}: ——$\circ$—— $\log p$ (ugrás p^k-ban) $\qquad \psi(x) = \sum_{p,k} U_{p^k}(x)$ (véges összeg)

$$\int_0^{\infty} U_{p^k}(x) \cdot s \cdot x^{-s-1}\, dx = \log p \int_{p^k}^{\infty} s\, x^{-s-1}\, dx = \log p \left[s \frac{x^{-s}}{-s} \right]_{p^k}^{\infty} = \log p \cdot p^{-ks} \;\overset{\sum_{p,k}}{\implies}\; \int_0^{\infty} \psi(x) \cdot s\, x^{-s-1}\, dx = \sum_{p,k} \frac{\log p}{p^{ks}} = -\frac{\zeta'(s)}{\zeta(s)}$$

$$\boxed{\psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} -\frac{\zeta'(s)}{s \cdot \zeta(s)} \cdot x^s\, ds \qquad (\sigma > 1)}$$

$$-\frac{\zeta'(s)}{\zeta(s) s} = \int_0^{\infty} \psi(x)\, x^{-s} \frac{dx}{x} \overset{u = \log x}{=} \int_{-\infty}^{\infty} \psi(e^u) e^{-su}\, du = \int_{-\infty}^{\infty} \overbrace{\psi(e^u) e^{-\sigma u}}^{f(u)} \cdot e^{-itu}\, du = \hat{f}(t) \qquad (s = \sigma + it,\ \sigma \text{ rögzített})$$

Fourier-inv.: $$\psi(e^u) e^{-\sigma u} = f(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{itu}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{\zeta'(s)}{\zeta(s) s} e^{itu}\, dt$$

$$\psi(x) \cdot x^{-\sigma} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{\zeta'(s)}{\zeta(s) \cdot s} x^{it}\, dt \underset{\left(dt = \frac{1}{i} ds\right)}{\implies} \psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} -\frac{\zeta'(s)}{\zeta(s) \cdot s} x^s\, ds$$

$$\boxed{\begin{aligned} \psi(x) &= x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} = \\ &= x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2} \log\left(1 - \frac{1}{x^2}\right) - \log 2\pi \end{aligned}}$$

$$\psi(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} -\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s}\, ds = \sum_{\operatorname{Re} z \le \sigma} \operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s};\ z_0\right)$$

$$\operatorname{Res}\left(\frac{x^s}{s};\ 0\right) = x^0 = 1 \implies \operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s};\ 0\right) = -\frac{\zeta'(0)}{\zeta(0)} = -\log 2\pi$$

$$\operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s};\ 1\right) = -(-1) \cdot \frac{x^1}{1} = x$$

$$\operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s};\ \varrho\right) = -1 \cdot \frac{x^{\varrho}}{\varrho}$$

$$\operatorname{Res}\left(-\frac{\zeta'(s)}{\zeta(s)} \cdot \frac{x^s}{s};\ -2n\right) = -1 \cdot \frac{x^{-2n}}{-2n} = \frac{x^{-2n}}{2n} \qquad \sum \frac{x^{-2n}}{2n} = \frac{1}{2} \sum \frac{(x^{-2})^n}{n} = \frac{1}{2} \cdot \left(-\log(1 - x^{-2})\right)$$

$$\boxed{\zeta(1 + it) \ne 0}$$

$$\log|\zeta(s)| = \operatorname{Re} \log \zeta(s) = \operatorname{Re} \sum_{p,k} \frac{1}{k p^{ks}} = \sum_{p,k} \frac{1}{k p^{k\sigma}} \cos(t \log p^k) \quad (\operatorname{Re} s > 1)$$

$$\log|\zeta(\sigma)^3 \cdot \zeta(\sigma + it)^4 \cdot \zeta(\sigma + 2it)| = \sum_{p,k} \frac{1}{k p^{k\sigma}} \underbrace{\left(3 + 4\cos(t \log p^k) + \cos(2t \log p^k)\right)}_{\ge 0} \ge 0$$

$$\implies |\zeta(\sigma)^3 \cdot \zeta(\sigma + it)^4 \cdot \zeta(\sigma + 2it)| \ge 1 \quad (\sigma > 1)$$

tfh. $\zeta(\sigma + it) = 0$, $\sigma \to 1$, t rögzített

$$\left| \underbrace{(\sigma - 1)}_{\downarrow\, 0} \underbrace{\left[\zeta(\sigma)(\sigma - 1)\right]^3}_{\downarrow\, 1} \cdot \underbrace{\left[\frac{\zeta(\sigma + it)}{\sigma - 1}\right]^4}_{\downarrow\, \zeta'(1 + it)} \cdot \underbrace{\zeta(\sigma + 2it)}_{\zeta(1 + 2it)} \right| \ge 1$$ ↯

Puska

1. Hatványozás

$$
\begin{aligned}
s &= \sigma + it \\
w &= e^s = e^\sigma (\cos t + i \sin t) \implies |w| = e^\sigma \\
n^s &= e^{\log n \cdot s} = n^\sigma (\cos(\log n \cdot t) + i \sin(\log n \cdot t)) \\
s &= \log w = \sigma + it = \log|w| + i \arg w \implies \log|w| = \operatorname{Re}\log w
\end{aligned}
$$

2. Fourier-transzformáció

$$
\begin{aligned}
\widehat{f}(t) &= \int_{-\infty}^{\infty} f(u) \cdot e^{-itu}\, du \\
f(u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(t) \cdot e^{-itu}\, dt
\end{aligned}
$$

3. Logaritmus hatványsora

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

4. Reziduumtétel és argumentum-elv

$$
\begin{aligned}
f(z) &= \sum_{n=N}^{\infty} a_n (z-z_0)^n = (z-z_0)^N \cdot g(z) \\
\operatorname{Res}(f, z_0) &= a_{-1} \\
\operatorname{Res}(f'/f, z_0) &= N = \nu(f, z_0) \\
\frac{1}{2\pi i} \int_{\partial D} f(z)\, dz &= \sum_{z_0 \in D} \operatorname{Res}(f, z_0) \\
\frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)}\, dz &= \sum_{z_0 \in D} \nu(f, z_0)
\end{aligned}
$$

5. Riemann-féle ζ függvény

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\operatorname{Re} s > 1)$$

szingularitás: $s = 1$ egyszeres pólus, $\operatorname{Res}(\zeta, 1) = 1$

triviális zérushelyek: $-2, -4, -6, \ldots$ (mindegyik egyszeres)

nemtriviális zérushelyek: $0 \le \operatorname{Re}\rho \le 1$ (mindegyik egyszeres)

1737: Euler $\sum \frac{1}{p} = \infty$

1792: Gauss $li(x)$

~~Euler $\sum \frac{1}{p} = \infty$~~

1850: Csebisev $\psi \leftrightarrow \pi$
alsó, felső becslés
ha $\exists$ lim, akkor az 1

1859: Riemann

1893: Hadamard-szorzat

1896: Hadamard prímtétel

1896: de la Vallée-Poussin, prímtétel
$\exists c > 0$, $\sigma < 1 - \frac{c}{\log t}$ $(|t| > K)$

$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$ $(\operatorname{Re} s > 0)$

$\Gamma(n) = (n-1)!$

$0, -1, -2, -3, \ldots$ egyszeres pólus

nincs zérushely

$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$

$\xi(s) = \xi(1-s)$

nincs pólus

zérushelyei a ζ [illegible]