Analysis on the two-element set

Tamás Waldhauser
joint work with Miguel Couceiro and Jean-Luc Marichal

University of Tartu

17 October 2013

Outline

Algebras and clones

Partial derivatives

Local monotonicity

Shadows

Algebras and clones

Partial derivatives

Local monotonicity

Shadows

Algebraic structures

- Analysis = study of functions
- Algebra = study of algebraic structures

Definition

An algebraic structure is a set equipped with some operations:

$$\mathbb{A} = (A; f_1, f_2, \dots), \text{ where}$$
$$f_i \colon A^{n_i} \to A, \ \ (a_1, \dots, a_{n_i}) \mapsto f(a_1, \dots, a_{n_i}).$$

Examples

- groups: $(G; \cdot)$ or rather $\left(G; \cdot, ^{-1}, 1\right)$
- rings: $(R; +, \cdot)$
- lattices: $(L; \wedge, \vee)$

Lattices

Definition

Let $(L; \leq)$ be a partially ordered set in which every two elements have a greatest common lower bound (gclb) and a least common upper bound (lcub). Let us endow L with these two operations:

meet	$\wedge \colon L^2 \to L,\ (x, y) \mapsto x \wedge y = \operatorname{gclb}(x, y) = \inf\{x, y\}\,;$
join	$\vee \colon L^2 \to L,\ (x, y) \mapsto x \vee y = \operatorname{lcub}(x, y) = \sup\{x, y\}\,.$

The resulting algebraic structure $(L; \wedge, \vee)$ is called a lattice.

Examples

real numbers	$x \wedge y = \min(x, y)$	$x \vee y = \max(x, y)$
natural numbers	$x \wedge y = \gcd(x, y)$	$x \vee y = \operatorname{lcm}(x, y)$
2^A (power set of A)	$x \wedge y = x \cap y$	$x \vee y = x \cup y$
subspaces	$x \wedge y = x \cap y$	$x \vee y = x + y$
subgroups	$x \wedge y = x \cap y$	$x \vee y = \langle x \cup y \rangle$
normal subgroups	$x \wedge y = x \cap y$	$x \vee y = \langle x \cup y \rangle = xy$

Algebras and functions

Let $\mathbb{A} = (A; f_1, f_2, \ldots)$ be an arbitrary algebra.
Composing the basic opearations f_i, we can build expressions like

$$g(x_1, x_2, x_3) := f_1(x_1, f_2(a, f_1(x_2, x_1, b)), f_1(f_2(x_2, x_3), x_3, a)).$$

In the case of rings or fields, the resulting functions are called polynomial functions.

Definition

We say that a function $g \colon A^n \to A$ is a polynomial function of the algebra $\mathbb{A}$ if g can be built from variables and constants via finitely many applications of the basic operations of $\mathbb{A}$.

Definition

We say that a function $g \colon A^n \to A$ is a term function of the algebra $\mathbb{A}$ if g can be built from variables via finitely many applications of the basic operations of $\mathbb{A}$.

Malcev conditions

Fact

Many properties of an algebra depend only on its term functions, and not on the particular basic operations.

Example

Groups have a ternary term function p satisfying the identities

$$p(x, x, y) = y = p(y, x, x),$$

namely $p(x, y, z) = xy^{-1}z$. This is "the" reason why normal subgroups of a group satisfy the following modular law:

$$\forall L, M, N \lhd G: \; L \leq N \implies L \vee (M \wedge N) = (L \vee M) \wedge N.$$

This modular law lies behind several results of group theory (e.g., Schreier, Jordan-Hölder, Krull-Schmidt). These results have been extended to arbitrary algebras that have a term function p satisfying the identities above.

Clones

If $\mathbb{A}$ is an algebra and $\mathcal{C}$ is the set of its term functions, then

- $\mathcal{C}$ is closed under composition of functions, and
- $\mathcal{C}$ contains the projections $p_i^{(n)} : (x_1, \ldots, x_n) \mapsto x_i$.

Such a closed class of functions is called a clone on A.

The clone lattice

The set of all clones on a fixed underlying set is a lattice with the lattice operations

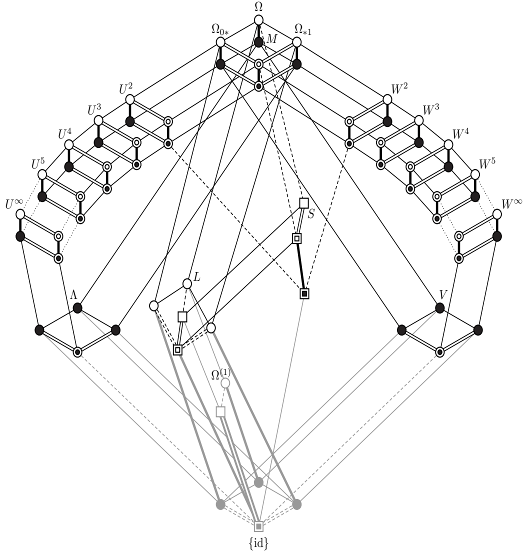
$$\mathcal{C}_1 \wedge \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C}_2$$
$$\mathcal{C}_1 \vee \mathcal{C}_2 = \langle \mathcal{C}_1 \cup \mathcal{C}_2 \rangle \,,$$

where $\langle \cdot \rangle$ denotes closure under composition.

One approach to investigate algebras is to study clones and the clone lattices.

A prominent result in this direction is Post's description of all clones over $A = \{0, 1\}$ around 1920 ...

The Post lattice



Some famous functions on $\{0, 1\}$

x	$\neg x$
0	1
1	0

x	y	$x \wedge y$	$x \vee y$	$x \longrightarrow y$	$x \leftrightarrow y$	$x \oplus y$
0	0	0	0	1	1	0
0	1	0	1	1	0	1
1	0	0	1	0	0	1
1	1	1	1	1	1	0

Some observations:

$\neg x = x \oplus 1$;

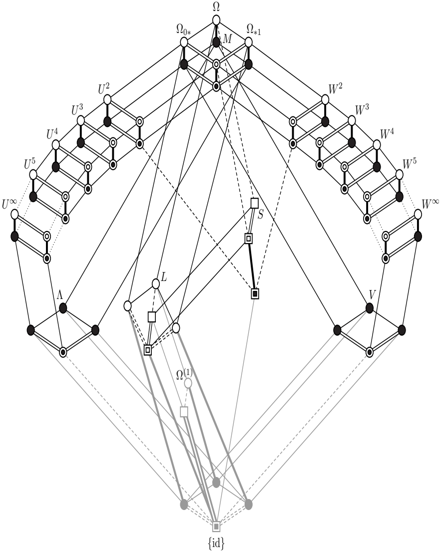
$x \leftrightarrow y = x \oplus y \oplus 1 = \neg (x \oplus y)$;

$\vee$ is the dual of $\wedge$:

$$x \vee y = \neg (\neg x \ \wedge \ \neg y) .$$

x	y	z	$m(x, y, z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

The Post lattice



- Ω = all functions on $\{0,1\}$
- $\Omega_{0*} = \{f : f(\mathbf{0}) = 0\}$
- $\Omega_{*1} = \{f : f(\mathbf{1}) = 1\}$
- $L = \langle \oplus, 0, 1 \rangle = \{x_1 \oplus \cdots \oplus x_n \oplus c\}$
- $M = \langle \wedge, \vee \rangle = \{\text{isotone functions}\} = \{f : \mathbf{x} \leq \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})\}$
- $S = \{\text{selfdual functions}\} = \{f : f(\mathbf{x}) = \neg f(\neg \mathbf{x})\}$
- $S \cap M = \langle m \rangle$
- $W^\infty = \langle \rightarrow \rangle$
- $\Lambda = \langle \wedge, 0, 1 \rangle = \{x_1 \wedge \cdots \wedge x_n, 0, 1\}$
- $V = \langle \vee, 0, 1 \rangle = \{x_1 \vee \cdots \vee x_n, 0, 1\}$
- $\{\mathrm{id}\} = \{\text{projections}\}$

Clones on the three-element set

! LaTeX Error:
File ‘clone_lat_3.eps’ not found.

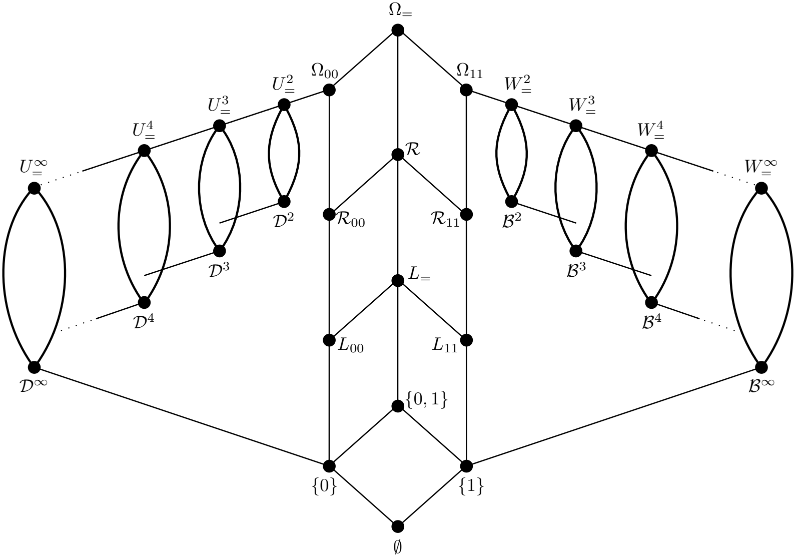
Clones on finite sets

Theorem (Janov, Mučnik, 1959)

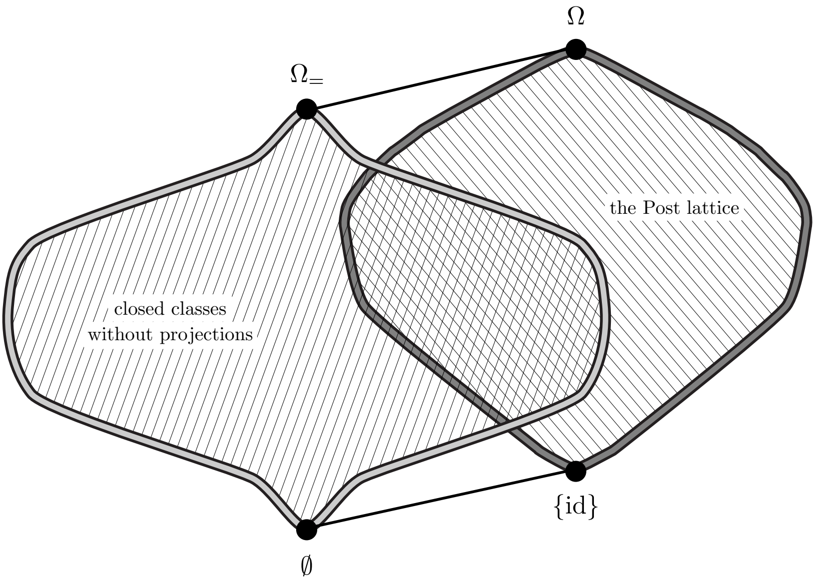
If A is a finite set with at least three elements, then the lattice of clones on A has continuum cardinality.

- Jablonskiĭ (1958): maximal clones on $\{0, 1, 2\}$
- Rosenberg (1970): maximal clones on finite sets
- Csákány (1983): minimal clones on $\{0, 1, 2\}$
- Rosenberg (1983): classification of minimal clones on finite sets (five types, complete description only for types I and IV)
- Szczepara (1995): minimal clones of type II on $\{0, 1, 2, 3\}$
- W. (2000): minimal clones of type III on $\{0, 1, 2, 3\}$
- Schölzel (2013): minimal clones of type V on $\{0, 1, 2, 3\}$

"Clones" without projections on $\{0,1\}$



The lattice of all closed classes on $\{0, 1\}$



Algebras and clones

Partial derivatives

Local monotonicity

Shadows

Boolean and pseudo-Boolean functions

Definition

- Boolean function: $f\colon \{0,1\}^n \to \{0,1\}$ (cf. Post)
- pseudo-Boolean function: $f\colon \{0,1\}^n \to \mathbb{R}$

Applications: 

- computer science
- voting theory
- decision making
- cooperative games
- etc.

Partial derivatives

Definition

The partial derivative of $f\colon \{0,1\}^n \to \mathbb{R}$ w.r.t. x_k is the function $\Delta_k f\colon \{0,1\}^n \to \mathbb{R}$ defined by

$$\begin{aligned}\Delta_k f(\mathbf{x}) &:= f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0) \\ &= f\left(x_1, \ldots, 1, \ldots, x_n\right) - f\left(x_1, \ldots, 0, \ldots, x_n\right).\end{aligned}$$

Observe that $\Delta_k f$ does not depend on x_k.

Example

The partial derivatives of the Boolean sum $f\left(x_1, x_2\right) = x_1 \oplus x_2 = x_1 + x_2 - 2x_1x_2$ are

$$\begin{aligned}\Delta_1 f(x_1, x_2) &= f(1, x_2) - f(0, x_2) = 1 - 2x_2, \\ \Delta_2 f(x_1, x_2) &= f(x_1, 1) - f(x_1, 0) = 1 - 2x_1.\end{aligned}$$

Lattice derivatives

Definition

We define the partial lattice derivatives of $f\colon \{0,1\}^n \to \mathbb{R}$ with respect to x_k by

$$\wedge_k f\colon \{0,1\}^n \to \mathbb{R},\ \wedge_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1) = \min\left(f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\right),$$
$$\vee_k f\colon \{0,1\}^n \to \mathbb{R},\ \vee_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1) = \max\left(f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\right).$$

Example

The lattice derivatives of the Boolean sum $f(x_1, x_2) = x_1 \oplus x_2$ are

$$\wedge_1 f(x_1, x_2) = f(1, x_2) \wedge f(0, x_2) = (1 \oplus x_2) \wedge x_2 = 0,$$
$$\vee_1 f(x_1, x_2) = f(1, x_2) \vee f(0, x_2) = (1 \oplus x_2) \vee x_2 = 1.$$

The second-order lattice derivatives are

$$\vee_2 \wedge_1 f(x_1, x_2) = \vee_2 0 = 0,$$
$$\wedge_1 \vee_2 f(x_1, x_2) = \wedge_1 1 = 1.$$

Lattice derivatives

Proposition

For any functions $f, g\colon \{0,1\}^n \to \mathbb{R}$ *and* $j \neq k \in [n]$, *the following hold:*

- $\wedge_k \wedge_k f = \wedge_k f$ *and* $\vee_k \vee_k f = \vee_k f$;
- *if* $f \leq g$, *then* $\wedge_k f \leq \wedge_k g$ *and* $\vee_k f \leq \vee_k g$;
- $\wedge_j \wedge_k f = \wedge_k \wedge_j f$ *and* $\vee_j \vee_k f = \vee_k \vee_j f$;
- $\vee_k \wedge_j f \leq \wedge_j \vee_k f$.

Proof.

Trivial, except for the last one, which follows from the inequality

$$(a \wedge b) \vee (c \wedge d) \leq (a \vee c) \wedge (b \vee d)\,.$$

□

Permutable lattice derivatives

Theorem

For any Boolean function f, the following conditions are equivalent:

- $\vee_k \wedge_j f = \wedge_j \vee_k f$ *for all* $j \neq k$*;*
- $\left|\Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y})\right| \leq \sum_{i \neq k} |x_i - y_i|$*;*
- $|\Delta_{jk} f| \leq 1$ *for all* $j \neq k$*.*

Definition

We say that $f\colon \{0,1\}^n \to \mathbb{R}$ has p-permutable lattice derivatives, if

$$O_{k_1} \cdots O_{k_p} f = O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f$$

holds for every p-element set $\{k_1, \ldots, k_p\} \subseteq [n]$, for all operators $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ and for every permutation $\pi \in S_p$.

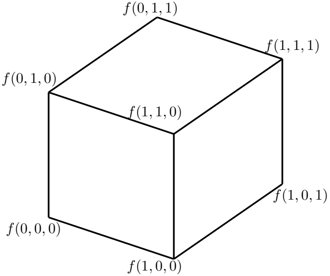
Theorem

If a function has $(p+1)$-permutable lattice derivatives, then it has p-permutable lattice derivatives.

Sections

A section of a function f is any function g that can be obtained from f by substituting constants to some of the variables of f.

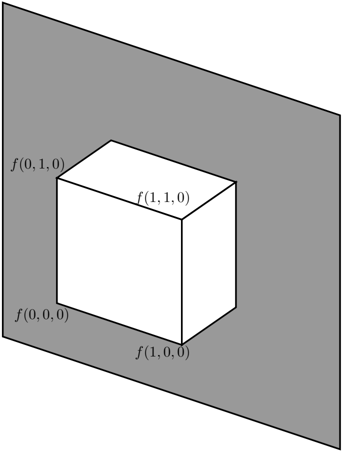
For example, if $f\colon \{0,1\}^3 \to \mathbb{R}$, then $g\colon \{0,1\}^2 \to \mathbb{R},\ g(x_1, x_2) := f(x_1, x_2, 0)$ is a section of f.



Sections

A section of a function f is any function g that can be obtained from f by substituting constants to some of the variables of f.

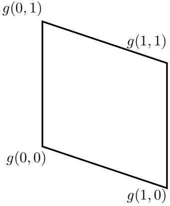
For example, if $f\colon \{0,1\}^3 \to \mathbb{R}$, then
$g\colon \{0,1\}^2 \to \mathbb{R},\ g(x_1, x_2) := f(x_1, x_2, 0)$ is a section of f.



Sections

A section of a function f is any function g that can be obtained from f by substituting constants to some of the variables of f.

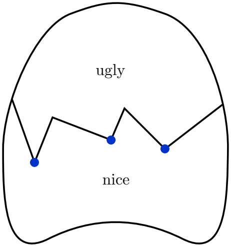
For example, if $f\colon \{0,1\}^3 \to \mathbb{R}$, then $g\colon \{0,1\}^2 \to \mathbb{R},\ g(x_1, x_2) := f(x_1, x_2, 0)$ is a section of f.



Forbidden sections

Theorem

If a function is nice, then all of its sections are also nice, where "nice" can stand for various properties.



Corollary

A function is nice if and only if none of the minimal ugly functions appear among its sections.

Theorem

A Boolean function has 2-permutable lattice derivatives if and only if neither $x_1 \oplus x_2$ nor $x_1 \oplus x_2 \oplus 1$ appears among its sections.

Algebras and clones

Partial derivatives

Local monotonicity

Shadows

Definitions

- $f, g\colon \{0,1\}^n \to \mathbb{R}$ are equivalent, if they can be obtained from each other by negating some of the variables, i.e.,
$$f(x_1, \ldots, x_n) = g(x_1 \oplus \varepsilon_1, \ldots, x_n \oplus \varepsilon_n)$$
for suitable $\varepsilon_1, \ldots, \varepsilon_n \in \{0,1\}$.

- f is isotone (nondecreasing) in x_k if
  - $f(\mathbf{x}_k^1) \geq f(\mathbf{x}_k^0)$ for all $\mathbf{x} \in \{0,1\}^n$, or equivalently
  - $\Delta_k f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \{0,1\}^n$.

- f is antitone (nonincreasing) in x_k if
  - $f(\mathbf{x}_k^1) \leq f(\mathbf{x}_k^0)$ for all $\mathbf{x} \in \{0,1\}^n$, or equivalently
  - $\Delta_k f(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \{0,1\}^n$.

- f is monotone in x_k if
  - f is either isotone or antitone in x_k, or equivalently
  - $\Delta_k f(\mathbf{x})$ does not change sign.

- f is isotone (antitone, monotone) if f is isotone (antitone, monotone) in every variable.

Some facts

Fact

- *A pseudo-Boolean function is monotone if and only if it is equivalent to an isotone function.*
- *All unary functions are monotone.*
- *The only non-monotone binary Boolean functions are*

$$x_1 \oplus x_2 \quad \textit{and} \quad x_1 \oplus x_2 \oplus 1.$$

- *A Boolean function is isotone if and only if* $x \oplus 1$ *does not appear among its sections.*

Local monotonicities

Definition

We say that $f\colon \{0,1\}^n \to \mathbb{R}$ is p-locally monotone, if its partial derivatives do not change sign between two points that are at distance less then p from each other.
Formally: for every $k \in [n]$ and every $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, we have

$$\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| < p \quad \Rightarrow \quad \Delta_k f(\mathbf{x}) \Delta_k f(\mathbf{y}) \geq 0.$$

Fact

- *p-local monotonicity implies $(p-1)$-local monotonicity.*
- *Every function is 1-locally monotone.*
- *An n-ary function is n-locally monotone if and only if it is monotone.*

2-local monotonicity

Theorem

For any Boolean function f, the following conditions are equivalent:

- *f is 2-locally monotone;*
- *f has 2-permutable lattice derivatives;*
- $\left|\Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y})\right| \leq \sum_{i \neq k} |x_i - y_i|$;
- $|\Delta_{jk} f| \leq 1$ *for all* $j \neq k$.

Local monotonicities vs. permutable lattice derivatives

Theorem

If a function is p-locally monotone, then it has p-permutable lattice derivatives.

Example

Let $f\colon \{0,1\}^n \to \{0,1\}$ be the function that takes the value 0 on all tuples of the form

$$(\overbrace{1,\ldots,1}^{m},0,\ldots,0) \text{ with } 0 \leq m \leq n,$$

and takes the value 1 everywhere else. Then f has n-permutable lattice derivatives, but it is only 2-locally monotone.

Theorem

For symmetric functions, p-local monotonicity is equivalent to p-permutability of lattice derivatives.

Algebras and clones

Partial derivatives

Local monotonicity

Shadows

Shadows of pseudo-Boolean functions

Let us fix $\mathbf{x} \in \{0,1\}^n$, and let

$$O_k = \begin{cases} \wedge_k & \text{if } x_k = 0, \\ \vee_k & \text{if } x_k = 1. \end{cases}$$

Applying these operators to $f\colon \{0,1\}^n \to \mathbb{R}$ in the order given by a permutation $\pi \in S_n$, we get a constant function

$$\widehat{f}_\pi(\mathbf{x}) := O_{\pi(1)} \cdots O_{\pi(n)} f.$$

Definition

The lower shadow and the upper shadow of $f\colon \{0,1\}^n \to \mathbb{R}$ are the functions defined by

$$f_{\vee\wedge}\colon \{0,1\}^n \to \mathbb{R},\ \mathbf{x} \mapsto \bigwedge_{\pi \in S_n} \widehat{f}_\pi(\mathbf{x}),$$

$$f_{\wedge\vee}\colon \{0,1\}^n \to \mathbb{R},\ \mathbf{x} \mapsto \bigvee_{\pi \in S_n} \widehat{f}_\pi(\mathbf{x}).$$

Shadows of pseudo-Boolean functions

Proposition

For any $f\colon \{0,1\}^n \to \mathbb{R}$ *and* $\mathbf{x} \in \{0,1\}^n$, *we have*

$$f_{\vee\wedge}(\mathbf{x}) = \vee_{g_1} \cdots \vee_{g_r} \wedge_{b_1} \cdots \wedge_{b_s} f,$$
$$f_{\wedge\vee}(\mathbf{x}) = \wedge_{b_1} \cdots \wedge_{b_s} \vee_{g_1} \cdots \vee_{g_r} f,$$

where

- $G := \{g_1, \ldots, g_r\} = \{k \in [n] : x_k = 1\}$,
- $B := \{b_1, \ldots, b_s\} = \{k \in [n] : x_k = 0\}$.

Definition

If $f_{\vee\wedge} = f_{\wedge\vee}$, then we say that f has a unique shadow; otherwise we say that f is skew.

Fact

A function $f\colon \{0,1\}^n \to \mathbb{R}$ *has a unique shadow if and only if it has n-permutable lattice derivatives.*

Good guys, bad guys

Let $f: 2^{[n]} \to \mathbb{R}$ be a cooperative game, and let $[n] = G \dot{\cup} B$ be a partition of the set of players into good (maximizing) and bad (minimizing) players. We can regard this as a two-player zero-sum game.

The good guys can ensure that the outcome will be at least

$$\max_{G_0 \subseteq G} \min_{B_0 \subseteq B} f\left(G_0 \cup B_0\right) = f_{\vee\wedge}\left(G\right),$$

whereas the bad guys can ensure that the outcome will be at most

$$\min_{B_0 \subseteq B} \max_{G_0 \subseteq G} f\left(G_0 \cup B_0\right) = f_{\wedge\vee}\left(G\right).$$

These two values coincide (i.e., the game is strictly determined) for all partitions $[n] = G \dot{\cup} B$, if and only if f has a unique shadow.

Two extremal cases

Example (skewest functions)

Let $f(x_1, \ldots, x_n) = x_1 \oplus \cdots \oplus x_n$. Then we have

$$f_{\vee\wedge}(\mathbf{x}) = x_1 \wedge \cdots \wedge x_n,$$
$$f_{\wedge\vee}(\mathbf{x}) = x_1 \vee \cdots \vee x_n.$$

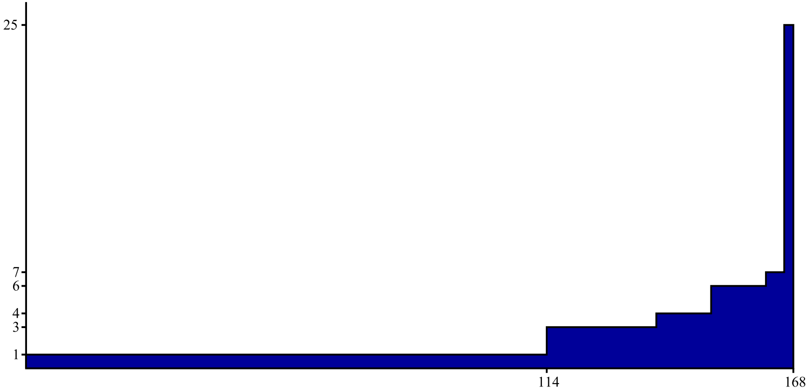
Theorem

The shadows are always isotone. Moreover, if f is monotone, then $f_{\vee\wedge} = f_{\wedge\vee}$, and there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, such that

$$f(x_1, \ldots, x_n) = f_{\vee\wedge}(x_1 \oplus \varepsilon_1, \ldots, x_n \oplus \varepsilon_n).$$

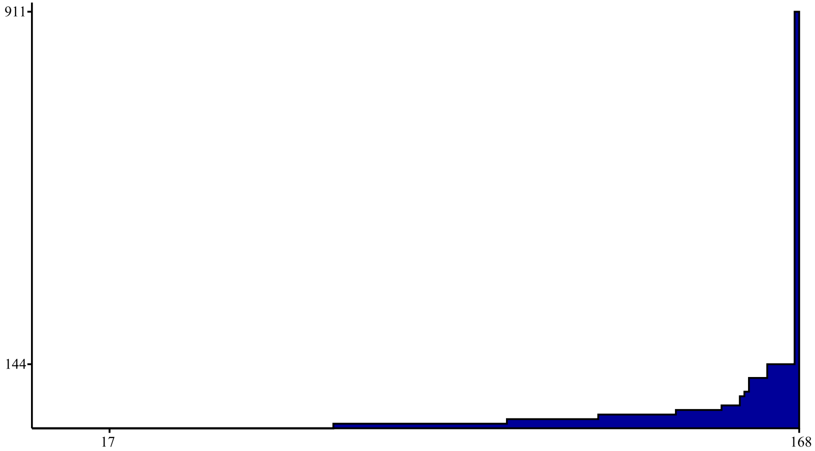
Some statistics for $n = 4$: unique shadows

- 4336 Boolean functions up to equivalence
- 384 of them have a uniqe shadow
- 168 possibilities for the shadow



Some statistics for $n = 4$: lower shadows

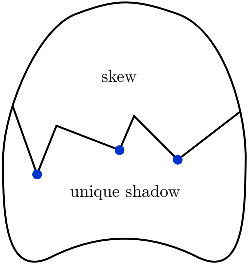
- 4336 Boolean functions up to equivalence
- 168 possibilities for the lower shadow



Forbidden sections

Theorem

If a function has a unique shadow, then all of its sections have a unique shadow as well.



Corollary

A function has a unique shadow if and only if none of the minimal skew functions appear among its sections.

Minimal skew functions

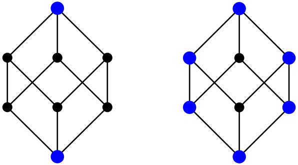
Let $g_n \colon \{0,1\}^n \to \{0,1\}$ be the Boolean function defined by

$$g_n(x_1, \ldots, x_n) = (x_1 \wedge \cdots \wedge x_n) \oplus (x_1 \vee \cdots \vee x_n),$$

and let $h_n \colon \{0,1\}^n \to \{0,1\}$ be the function that takes the value 0 on all tuples of the form

$$(1, \ldots, 1, 0, \ldots, 0) \quad \text{and} \quad (0, \ldots, 0, 1, \ldots, 1),$$

and takes the value 1 everywhere else.



Conjecture

A Boolean function $f \colon \{0,1\}^n \to \{0,1\}$ is a minimal skew function iff f is equivalent to one of the functions $g_n,\ h_n,\ g_n \oplus 1,\ h_n \oplus 1$.

Read these!

- M. Couceiro, J.-L. Marichal, T. Waldhauser,
  Locally monotone Boolean and pseudo-Boolean functions,
  Discrete Appl. Math. **160** (2012), 1651–1660.
  arXiv:1107.1161.
- T Waldhauser,
  On composition-closed classes of Boolean functions
  J. Mult.-Valued Logic Soft Comput. **19** (2012), 493-518.
  arXiv:1102.4355.