Analysis on the two-element set

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Outline

Algebras and clones

Partial derivatives

Shadows

Algebraic structures

- Analysis = study of functions
- Algebra = study of algebraic structures

Definition
An algebraic structure is a set equipped with some operations:

$$\mathbb{A} = (A; f_1, f_2, \ldots)\,, \text{ where}$$
$$f_i\colon A^{n_i} \to A,\ \ (a_1, \ldots, a_{n_i}) \mapsto f\,(a_1, \ldots, a_{n_i})\,.$$

Examples

- groups: $(G; \cdot)$ or rather $\left(G; \cdot, ^{-1}, 1\right)$
- rings: $(R; +, \cdot)$
- lattices: $(L; \wedge, \vee)$

Lattices

Definition

Let $(L; \leq)$ be a partially ordered set in which every two elements have a greatest common lower bound (gclb) and a least common upper bound (lcub). Let us endow L with these two operations:

meet	$\wedge\colon L^2 \to L,\ (x, y) \mapsto x \wedge y = \operatorname{gclb}(x, y) = \inf\{x, y\};$
join	$\vee\colon L^2 \to L,\ (x, y) \mapsto x \vee y = \operatorname{lcub}(x, y) = \sup\{x, y\}.$

The resulting algebraic structure $(L; \wedge, \vee)$ is called a lattice.

Examples

real numbers	$x \wedge y = \min(x, y)$	$x \vee y = \max(x, y)$
natural numbers	$x \wedge y = \gcd(x, y)$	$x \vee y = \operatorname{lcm}(x, y)$
2^A (power set of A)	$x \wedge y = x \cap y$	$x \vee y = x \cup y$
subspaces	$x \wedge y = x \cap y$	$x \vee y = x + y$
subgroups	$x \wedge y = x \cap y$	$x \vee y = \langle x \cup y \rangle$
normal subgroups	$x \wedge y = x \cap y$	$x \vee y = \langle x \cup y \rangle = xy$

Algebras and functions

Let $\mathbb{A} = (A; f_1, f_2, \ldots)$ be an arbitrary algebra.
Composing the basic opearations f_i, we can build expressions like

$$g(x_1, x_2, x_3) := f_1(x_1, f_2(a, f_1(x_2, x_1, b)), f_1(f_2(x_2, x_3), x_3, a)).$$

In the case of rings or fields, the resulting functions are called polynomial functions.

Definition

We say that a function $g\colon A^n \to A$ is a polynomial function of the algebra $\mathbb{A}$ if g can be built from variables and constants via finitely many applications of the basic operations of $\mathbb{A}$.

Definition

We say that a function $g\colon A^n \to A$ is a term function of the algebra $\mathbb{A}$ if g can be built from variables via finitely many applications of the basic operations of $\mathbb{A}$.

Malcev conditions

Fact

Many properties of an algebra depend only on its term functions, and not on the particular basic operations.

Example

Groups have a ternary term function p satisfying the identities

$$p(x, x, y) = y = p(y, x, x),$$

namely $p(x, y, z) = xy^{-1}z$. This is “the” reason why normal subgroups of a group satisfy the following modular law:

$$\forall L, M, N \lhd G : \; L \leq N \implies L \vee (M \wedge N) = (L \vee M) \wedge N.$$

This modular law lies behind several results of group theory (e.g., Schreier, Jordan-Hölder, Krull-Schmidt). These results have been extended to arbitrary algebras that have a term function p satisfying the identities above.

Clones

Let $\mathbb{A}$ be an algebra and let $\mathcal{C}$ is the set of its term functions.
Then $\mathcal{C}$ is a class of functions of several variables on A that is closed under composition of functions.
Such a closed class of functions is called a clone on A.

Holy smokes, I've been cloned!

The clone lattice

The set of all clones on a fixed underlying set is a lattice with the lattice operations

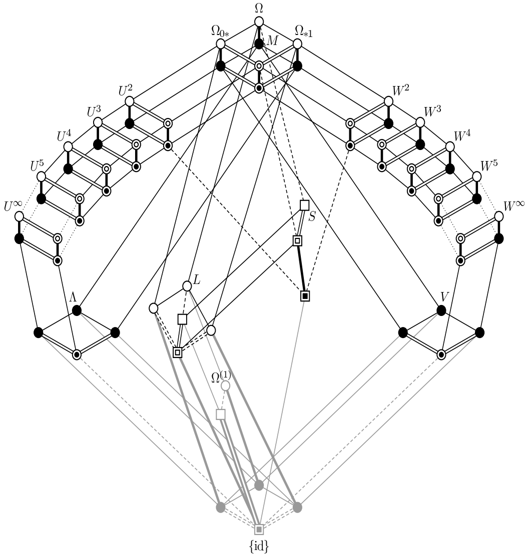
$$\mathcal{C}_1 \wedge \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C}_2$$
$$\mathcal{C}_1 \vee \mathcal{C}_2 = \langle \mathcal{C}_1 \cup \mathcal{C}_2 \rangle \,,$$

where $\langle \cdot \rangle$ denotes closure under composition.

One approach to investigate algebras is to study clones and the clone lattices.

A prominent result in this direction is Post's description of all clones over $A = \{0, 1\}$ around 1920 . . .

The Post lattice



Some famous functions on $\{0,1\}$

x	$\neg x$
0	1
1	0

x	y	$x \wedge y$	$x \vee y$	$x \to y$	$x \leftrightarrow y$	$x \oplus y$
0	0	0	0	1	1	0
0	1	0	1	1	0	1
1	0	0	1	0	0	1
1	1	1	1	1	1	0

Some observations:

$\neg x = x \oplus 1$;

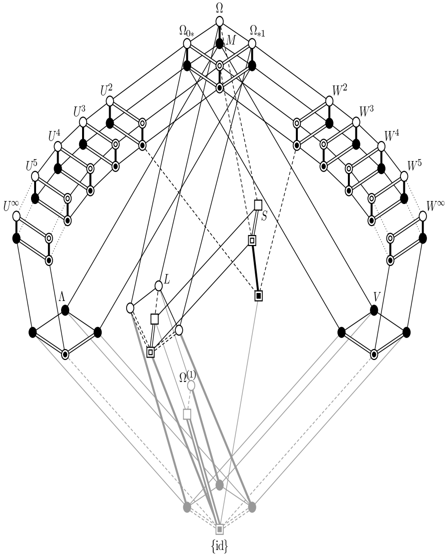
$x \leftrightarrow y = x \oplus y \oplus 1 = \neg\,(x \oplus y)$;

$\vee$ is the dual of $\wedge$:

$$x \vee y = \neg\,(\neg x \;\wedge\; \neg y)\,.$$

x	y	z	$m\,(x,y,z)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

The Post lattice



- Ω = all functions on $\{0,1\}$
- $\Omega_{0*} = \{f : f(\mathbf{0}) = 0\}$
- $\Omega_{*1} = \{f : f(\mathbf{1}) = 1\}$
- $L = \langle \oplus, 0, 1 \rangle = \{x_1 \oplus \cdots \oplus x_n \oplus c\}$
- $M = \{\text{monotone functions}\} = \{f : \mathbf{x} \leq \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})\}$
- $S = \{\text{selfdual functions}\} = \{f : f(\mathbf{x}) = \neg f(\neg \mathbf{x})\}$
- $S \cap M = \langle m \rangle$
- $W^\infty = \langle \to \rangle$
- $\Lambda = \langle \wedge, 0, 1 \rangle = \{x_1 \wedge \cdots \wedge x_n, 0, 1\}$
- $V = \langle \vee, 0, 1 \rangle = \{x_1 \vee \cdots \vee x_n, 0, 1\}$
- $\{\text{id}\} = \{\text{projections}\}$

Clones on the three-element set

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Clones on finite sets

Theorem (Janov, Mučnik, 1959)

If A is a finite set with at least three elements, then the lattice of clones on A has continuum cardinality.

- Jablonskiĭ (1958): maximal clones on $\{0, 1, 2\}$
- Rosenberg (1970): maximal clones on finite sets
- Csákány (1983): minimal clones on $\{0, 1, 2\}$
- Rosenberg (1983): classification of minimal clones on finite sets (five types, complete description only for types I and IV)
- Szczepara (1995): minimal clones of type II on $\{0, 1, 2, 3\}$
- W. (2000): minimal clones of type III on $\{0, 1, 2, 3\}$
- Schölzel (2013): minimal clones of type V on $\{0, 1, 2, 3\}$

Boolean and pseudo-Boolean functions

Definition

- Boolean function: $f\colon \{0,1\}^n \to \{0,1\}$ (cf. Post)
- pseudo-Boolean function: $f\colon \{0,1\}^n \to \mathbb{R}$

Note that $\{0,1\}^n \equiv$ power set of $[n] = \{1,2,\ldots,n\}$, hence a pseudo-Boolean function assigns numbers to (sub)sets.

Applications: 

- computer science
- voting theory
- decision making
- cooperative games
- etc.

Partial derivatives

Definition

The partial derivative of $f\colon \{0,1\}^n \to \mathbb{R}$ w.r.t. x_k is the function $\Delta_k f\colon \{0,1\}^n \to \mathbb{R}$ defined by

$$\begin{aligned}\Delta_k f(\mathbf{x}) &:= f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0) \\ &= f\left(x_1, \ldots, 1, \ldots, x_n\right) - f\left(x_1, \ldots, 0, \ldots, x_n\right).\end{aligned}$$

Observe that $\Delta_k f$ does not depend on x_k.

Example

The partial derivatives of the Boolean sum $f\left(x_1, x_2\right) = x_1 \oplus x_2 = x_1 + x_2 - 2x_1x_2$ are

$$\begin{aligned}\Delta_1 f(x_1, x_2) &= f(1, x_2) - f(0, x_2) = 1 - 2x_2, \\ \Delta_2 f(x_1, x_2) &= f(x_1, 1) - f(x_1, 0) = 1 - 2x_1.\end{aligned}$$

Lattice derivatives

Definition

We define the partial lattice derivatives of $f\colon \{0,1\}^n \to \mathbb{R}$ with respect to x_k by

$$\wedge_k f\colon \{0,1\}^n \to \mathbb{R},\ \wedge_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1) = \min\left(f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\right),$$
$$\vee_k f\colon \{0,1\}^n \to \mathbb{R},\ \vee_k f(\mathbf{x}) = f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1) = \max\left(f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\right).$$

Example

The lattice derivatives of the Boolean sum $f(x_1, x_2) = x_1 \oplus x_2$ are

$$\wedge_1 f(x_1, x_2) = f(1, x_2) \wedge f(0, x_2) = (1 \oplus x_2) \wedge x_2 = 0,$$
$$\vee_1 f(x_1, x_2) = f(1, x_2) \vee f(0, x_2) = (1 \oplus x_2) \vee x_2 = 1.$$

The second-order lattice derivatives are

$$\vee_2 \wedge_1 f(x_1, x_2) = \vee_2 0 = 0,$$
$$\wedge_1 \vee_2 f(x_1, x_2) = \wedge_1 1 = 1.$$

Permutable lattice derivatives

Theorem

For any Boolean function f, the following conditions are equivalent:

- $\vee_k \wedge_j f = \wedge_j \vee_k f$ *for all* $j \neq k$*;*
- $\left|\Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y})\right| \leq \sum_{i \neq k} |x_i - y_i|$*;*
- $|\Delta_{jk} f| \leq 1$ *for all* $j \neq k$*.*

Definition

We say that $f\colon \{0,1\}^n \to \mathbb{R}$ has p-permutable lattice derivatives, if

$$O_{k_1} \cdots O_{k_p} f = O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f$$

holds for every p-element set $\{k_1, \ldots, k_p\} \subseteq [n]$, for all operators $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ and for every permutation $\pi \in S_p$.

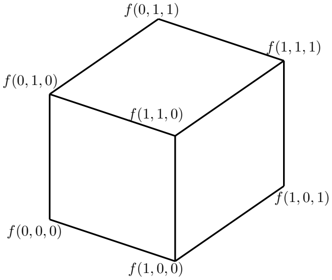
Theorem

If a function has $(p+1)$-permutable lattice derivatives, then it has p-permutable lattice derivatives.

Sections

A section of a function f is any function g that can be obtained from f by substituting constants to some of the variables of f.

For example, if $f\colon \{0,1\}^3 \to \mathbb{R}$, then $g\colon \{0,1\}^2 \to \mathbb{R},\ g(x_1, x_2) := f(x_1, x_2, 0)$ is a section of f.



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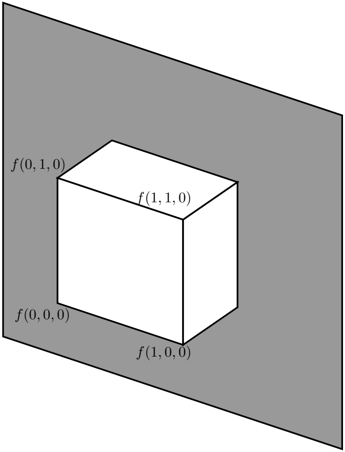

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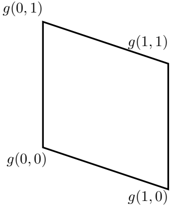
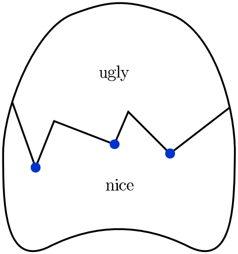


Forbidden sections

Theorem

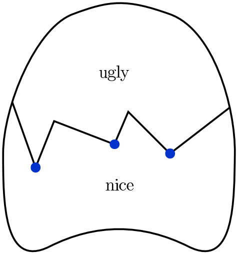
If a function is nice, then all of its sections are also nice, where "nice" can stand for various properties.



Corollary

A function is nice if and only if none of the minimal ugly functions appear among its sections.

Nice = monotone



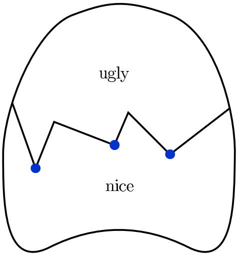
Theorem

The only minimal ugly Boolean function is $x \oplus 1$.

Corollary

A Boolean function is monotone if and only if $x \oplus 1$ *does not appear among its sections.*

Nice = 2-permutable lattice derivatives



Theorem

There are two minimal ugly Boolean functions, namely $x_1 \oplus x_2$ *and* $x_1 \oplus x_2 \oplus 1$.

Corollary

A Boolean function has 2-permutable lattice derivatives if and only if neither $x_1 \oplus x_2$ *nor* $x_1 \oplus x_2 \oplus 1$ *appears among its sections.*

Nice = permutable lattice derivatives

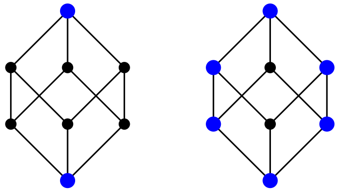
Let $g_n\colon \{0,1\}^n \to \{0,1\}$ be the Boolean function defined by

$$g_n(x_1, \ldots, x_n) = (x_1 \wedge \cdots \wedge x_n) \oplus (x_1 \vee \cdots \vee x_n),$$

and let $h_n\colon \{0,1\}^n \to \{0,1\}$ be the function that takes the value 0 on all tuples of the form

$$(1, \ldots, 1, 0, \ldots, 0) \quad \text{and} \quad (0, \ldots, 0, 1, \ldots, 1),$$

and takes the value 1 everywhere else.



Conjecture

Up to a certain kind of equivalence, the only minimal ugly Boolean functions are g_n and h_n $(n = 1, 2, \ldots)$.

Tovább is van, mondjam még?

MONDJAD!

NE MONDJAD!

There is more, shall I go on?

YES

NO

Shadows of pseudo-Boolean functions

Let $f\colon \{0,1\}^n \to \mathbb{R}$ be a function with permutable lattice derivatives, and let us choose an operator $O_k \in \{\wedge_k, \vee_k\}$ for each $k \in [n]$. This choice of operators can be encoded by a tuple $\mathbf{x} \in \{0,1\}^n$ as follows:

$$x_k := \begin{cases} 0 & \text{if } O_k = \wedge_k, \\ 1 & \text{if } O_k = \vee_k. \end{cases}$$

Applying these operators to a function $f\colon \{0,1\}^n \to \mathbb{R}$, we get a constant function, which we denote by

$$\widehat{f}\,(\mathbf{x}) := O_1 \cdots O_n f.$$

The resulting function $\widehat{f}\colon \{0,1\}^n \to \mathbb{R}$ is called the shadow of f. If the function f does not have permutable lattice derivatives, then the order of the operators does matter, hence $\widehat{f}\,(\mathbf{x})$ cannot be defined.

Example

Let us compute the shadow of $f(x_1, x_2) = x_1 \to x_2$.

$$\begin{aligned}
\widehat{f}(0,0) &= \wedge_1 \wedge_2 f = 0 \\
\widehat{f}(1,0) &= \vee_1 \wedge_2 f = 1 \\
\widehat{f}(0,1) &= \wedge_1 \vee_2 f = 1 \\
\widehat{f}(1,1) &= \vee_1 \vee_2 f = 1
\end{aligned}$$

Thus, $\widehat{f}(x_1, x_2) = x_1 \vee x_2$.

Example

Let us compute the shadow of $g(x_1, x_2) = x_1 \vee x_2$.

$$\begin{aligned}
\widehat{g}(0,0) &= \wedge_1 \wedge_2 g = 0 \\
\widehat{g}(1,0) &= \vee_1 \wedge_2 g = 1 \\
\widehat{g}(0,1) &= \wedge_1 \vee_2 g = 1 \\
\widehat{g}(1,1) &= \vee_1 \vee_2 g = 1
\end{aligned}$$

Thus, $\widehat{g}(x_1, x_2) = x_1 \vee x_2$.

Which functions are shadows?

Theorem

- *The shadows are always monotone.*
- *Every monotone function appears as a shadow: if f is monotone, then $\widehat{f} = f$.*
- *If a function f can be made monotone by negating some of its variables, then $\widehat{f}$ is this monotone function. Formally, if there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ such that*

  $$g(x_1, \ldots, x_n) := f(x_1 \oplus \varepsilon_1, \ldots, x_n \oplus \varepsilon_n)$$

  is a monotone function, then $\widehat{f} = g$.

Example

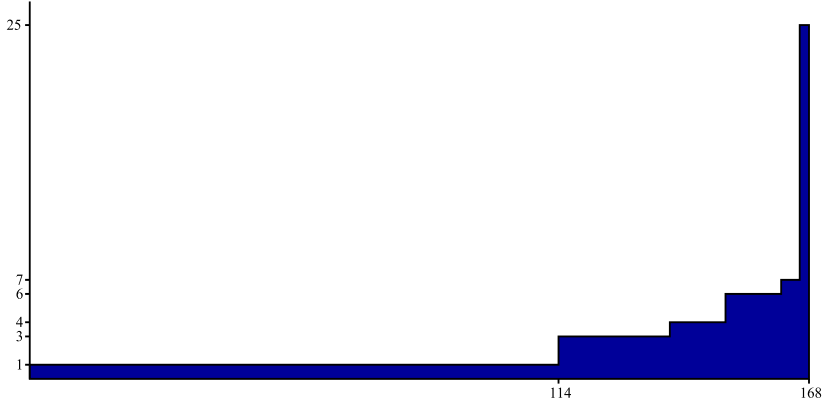
The function $f(x_1, x_2) = x_1 \to x_2$ can be made monotone by negating its first variable:

$$g(x_1, x_2) := f(x_1 \oplus 1, x_2 \oplus 0) = f(\neg x_1, x_2) = \neg x_1 \to x_2 = x_1 \vee x_2.$$

Hence, $\widehat{f}(x_1, x_2) = x_1 \vee x_2$.

Some statistics for $n = 4$

- 4336 Boolean functions up to equivalence
- 384 of them have permutable lattice derivatives
- 168 possibilities for the shadow



Good guys, bad guys

Let $f : 2^{[n]} \to \mathbb{R}$ be a cooperative game, and let $[n] = G \dot{\cup} B$ be a partition of the set of players into good (maximizing) and bad (minimizing) players. We can regard this as a two-player zero-sum game.

The good guys can ensure that the outcome will be at least

$$\max_{G_0 \subseteq G} \min_{B_0 \subseteq B} f\left(G_0 \cup B_0\right)$$

whereas the bad guys can ensure that the outcome will be at most

$$\min_{B_0 \subseteq B} \max_{G_0 \subseteq G} f\left(G_0 \cup B_0\right).$$

These two values coincide for all partitions $[n] = G \dot{\cup} B$, if and only if f has permutable lattice derivatives. In this case the shadow gives the above maximin (=minimax) value:

$$\widehat{f}\left(G\right) = \max_{G_0 \subseteq G} \min_{B_0 \subseteq B} f\left(G_0 \cup B_0\right) = \min_{B_0 \subseteq B} \max_{G_0 \subseteq G} f\left(G_0 \cup B_0\right).$$

Enjoy your night in Szeged!

Photo: Tóth Orsi, www.szeretlekmagyarorszag.hu

Read this!

M. Couceiro, J.-L. Marichal, T. Waldhauser,
Locally monotone Boolean and pseudo-Boolean functions,
Discrete Appl. Math. **160** (2012), 1651–1660.
arXiv:1107.1161.