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MULTIPLICATION OF MATRICES OVER LATTICES

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Dedicated to the memory of Ivo Rosenberg.

Abstract. We study the multiplication operation of square matrices over lattices. If the underlying lattice is distributive, then matrices form a semigroup; we investigate idempotent and nilpotent elements and the maximal subgroups of this matrix semigroup. We prove that matrix multiplication over nondistributive lattices is antiassociative, and we determine the invertible matrices in the case when the least or the greatest element of the lattice is irreducible.

1. Introduction

Multiplication of matrices over a lattice L can be defined in the same way as for matrices over rings, letting the join operation play the role of addition and the meet operation play the role of multiplication. For notational convenience, we will actually write the lattice operations as addition and multiplication. Thus, throughout the paper, $L = (L; +, \cdot)$ denotes a lattice, and $\mathbf{M}_n(L)$ stands for the set of all $n \times n$ matrices over L. To exclude trivial cases, we will always assume without further mention that L has at least two elements and $n \geq 2$. If L has a least and a greatest element (these will be denoted by 0 and 1), then we can define the identity matrix $I \in \mathbf{M}_n(L)$ with ones on the diagonal and zeros everywhere off the diagonal, and it is easy to see that I is indeed the identity element of $\mathbf{M}_n(L)$.

In Section 2 we focus on the semigroup $\mathbf{M}_n(\mathbf{2})$ of $n \times n$ matrices over the two-element lattice $\mathbf{2} = \{0, 1\}$. We can regard a matrix $A \in \mathbf{M}_n(\mathbf{2})$ as the characteristic function of a set $\alpha \subseteq X^2$ where $X := \{1, \ldots, n\}$, thus matrices over $\mathbf{2}$ correspond to binary relations, and $\mathbf{M}_n(\mathbf{2})$ is isomorphic to the semigroup of binary relations on the set X. We recall various results about this semigroup in Section 2, namely, the description of idempotent elements, Green's relations and maximal subgroups. We also present a visual proof of B. Schein's characterization of idempotents of $\mathbf{M}_n(\mathbf{2})$ [20] by interpreting the graph corresponding to a matrix as a transportation network.

Sections 3 and 4 deal with matrices over bounded distributive lattices; these can be viewed as multiple-valued analogues of binary relations. Boundedness is not a serious restriction, since most of the time we shall work in a finitely generated sublattice (for instance, in the sublattice generated by the n^2 entries of an $n \times n$ matrix), and finitely generated distributive lattices are finite. A bounded distributive lattice is a semiring, and matrices over any semiring form a semiring [7]. In particular, $\mathbf{M}_n(L)$ is a semigroup under multiplication for every distributive lattice L; see [1] for an overview of various properties of these semigroups.

Generalizing results of Section 2 to this multiple-valued setting, we describe idempotents and maximal subgroups in some special cases; the full description of maximal subgroups constitutes a topic for further research. We also determine nilpotent matrices over certain distributive lattices, including chains, which are the most important cases from the viewpoint of applications, and then we discuss connections to a problem related to fuzzy relations [9].

Matrix multiplication over arbitrary lattices is not always associative, and if it is not, then we may ask how far it is from being associative. There are several ways to measure associativity; one of them is the associative spectrum introduced in [5]. The number of possibilities of inserting parentheses (or brackets) into a product $x_1 \cdot \ldots \cdot x_n$ is given by the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$. If multiplication is associative, then all these different *bracketings* give the same result, but if the multiplication is not associative, then some of the bracketings may induce different n-variable term functions. The *associative spectrum* of a binary operation is the sequence $\{s_n\}_{n=1}^{\infty}$ that counts the number of different term functions induced by bracketings of the product $x_1 \cdot \ldots \cdot x_n$. Clearly $s_1 = s_2 = 1$, and $1 \leq s_n \leq C_{n-1}$ holds for all natural numbers n, and we can say that the faster the spectrum grows, the less associative the multiplication is. In particular, if the associative spectrum is the sequence of Catalan numbers, then the multiplication is said to be *antiassociative*. Of course, there are plenty of operations that fall between the two extreme cases of being associative or antiassociative; examples of associative spectra of various growth rates can be found in [5, 13].

We shall see in Section 5 that there is a dichotomy for matrix multiplication over lattices: if L is distributive, then $\mathbf{M}_n(L)$ is a semigroup, while if L is not distributive, then the multiplication of $\mathbf{M}_n(L)$ is antiassociative. Nonassociativity has some unfortunate consequences; for example, powers of matrices and inverse matrices are not always well defined. On the other hand, we prove that if L is bounded and 0 is meet-irreducible or 1 is join-irreducible, then inverses are unique (even if L is not distributive), and we describe explicitly the invertible matrices in this case, showing that they form a group isomorphic to the symmetric group S_n. (Recall that an element $a \in L \setminus \{1\}$ is said to be *meet-irreducible* if $a = b \cdot c$ implies that $a = b$ or $a = c$; *join-irreducibility* can be defined dually.)

Some personal remarks from the second author about Ivo Rosenberg: As a graduate student working in clone theory under the supervision of Béla Csákány, I certainly learned the name of Ivo Rosenberg early in my studies. His theorems on maximal and minimal clones are cornerstones of the theory of clones, and I always imagined the discoverer of these theorems as an unapproachable "giant". It is no wonder that I was thrilled to meet him at the AAA58 conference in Vienna in 1999. Unfortunately, it was our first and last personal encounter. We spoke only a few words, and he apologized very kindly for not being able to attend my talk. I was a bit disappointed, but much more astonished for receiving such friendly apologies from this giant of clone theory as a first-year doctoral student. My talk was about measuring associativity, and our joint paper with Béla Csákány about associative spectra appeared in this journal 20 years ago, in the special issue dedicated to the 65th birthday of Ivo Rosenberg. Now this is a special issue for a much more sad occasion, and I can only hope that this modest contribution is worthy to commemorate Ivo Rosenberg.

2. Preliminaries

To each $n \times n$ matrix A over the two-element lattice $\mathbf{2} = \{0, 1\}$, we can associate a binary relation α defined on the set $X := \{1, \ldots, n\}$, by letting $(i, j) \in \alpha \iff a_{ij} = 1$. Matrix multiplication translates to relational product in this interpretation: if the relations corresponding to $A, B \in \mathbf{M}_n(\mathbf{2})$ are α and β, then AB describes the relation

$$\alpha \circ \beta = \{(x, y) \in X \times X : \exists z \in X,\ (x, z) \in \alpha \text{ and } (z, y) \in \beta\}.$$

Therefore, $\mathbf{M}_n(\mathbf{2})$ is isomorphic to the semigroup of binary relations on the n-element set. If $\alpha \subseteq \beta$ holds, then we have $a_{ij} \leq b_{ij}$ $(i, j = 1, \ldots, n)$ for the entries of the corresponding matrices $A, B \in \mathbf{M}_n(\mathbf{2})$; in this case we write $A \leq B$.

Remark 2.1. We can regard the relation $\alpha \subseteq X^2$ corresponding to $A \in \mathbf{M}_n(\mathbf{2})$ as the edge set of a directed graph with vertex set X, having A as its adjacency matrix. We can think of this graph as a transportation network: the vertices are sites (cities, store-houses, etc.), and the edges are (possibly one-way) roads, on which trucks can transport goods between the sites. If $a_{ii} = 0$ (i.e., $(i, i) \notin \alpha$), then trucks are not allowed to stop at site i, while if $a_{ii} = 1$ (i.e., there is a loop $(i, i) \in \alpha$), then there is a parking lot at site i, where trucks can wait as long as they wish. Powers of A

account for routes[1] in our graph: if $A^\ell = (w_{ij})_{i,j=1}^n$, then $w_{ij} = 1$ if and only if there is a directed route of length ℓ from i to j.

The semigroup of binary relations plays a prominent role in semigroup theory; we recall a few of the plethora of results about this semigroup in this section.

2.1. **Idempotent matrices.** The characterization of idempotent elements of $\mathbf{M}_n(\mathbf{2})$ was given by B. Schein [20] in terms of so-called pseudo-orders. A reflexive transitive relation is called a *quasi-order*. The symmetric part $\alpha \cap \alpha^{-1}$ of a quasi-order α is an equivalence relation, and α induces a natural partial order on the blocks of this equivalence relation. We usually use the symbol $\lesssim$ for a quasi-order on the set X; we denote the corresponding equivalence relation by $\sim$, and the partially ordered set (poset, for short) corresponding to the quasi-order $\lesssim$ is $(X/\!\sim; \leq)$. We say that an element $y \in X$ *covers* $x \in X$ (notation: $x \prec y$), if $x/\!\sim$ is strictly less than $y/\!\sim$, and there is no third $\sim$-block between them:

$$x \prec y \iff x \lesssim y, x \nsim y \text{ and } \forall z \in X\colon x \lesssim z \lesssim y \implies x \sim z \text{ or } z \sim y.$$

A pseudo-order relation is obtained from a quasi-order by removing some of the loops (i.e., edges of the form (x, x)) in such a way, that loops can be removed only from singleton $\sim$-blocks, and it is not allowed to remove loops from both members of a covering pair.

Definition 2.2. Let $\alpha \subseteq X^2$ be a binary relation, and let Q_α denote the set of vertices with a loop: $Q_\alpha = \{x \in X : (x, x) \in \alpha\}$. We say that α is a *pseudo-order* if the reflexive closure $\alpha \cup \{(x, x) : x \in X \setminus Q_\alpha\}$ is a quasi-order (as above, we denote this quasi-order by $\lesssim$ and we use the symbols $\sim$ and $\prec$ for the corresponding equivalence relation and cover relation), and Q_α satisfies the following two conditions:

- (a) $\forall x \in X \setminus Q_\alpha\colon\ x/\!\sim = \{x\}$,
- (b) $\forall x, y \in X\colon\ x \prec y \implies x \in Q_\alpha$ or $y \in Q_\alpha$.

Remark 2.3. Let us note that if α is a pseudo-order, then $\alpha \cap \alpha^{-1}$ is the restriction of $\sim$ to Q_α, i.e., $\alpha \cap \alpha^{-1} = \sim \cap Q_\alpha^2$. Indeed, since $\alpha \subseteq \lesssim$, we have $\alpha \cap \alpha^{-1} \subseteq \lesssim \cap \gtrsim = \sim$. Furthermore, if $(x, y) \in \alpha \cap \alpha^{-1}$ and $x \neq y$, then x and y belong to the same non-singleton $\sim$-block, hence condition (a) implies $x, y \in Q_\alpha$, while if $x = y$, then it is obvious from the definition of Q_α that $x \in Q_\alpha$. Conversely, if $x \sim y$ and $x, y \in Q_\alpha$, then $(x, y), (y, x) \in \alpha$, since $\lesssim$ and α differ only on $X \setminus Q_\alpha$.

Remark 2.4. We can interpret pseudo-orders in terms of the transportation network outlined in Remark 2.1 as follows. A relation $\alpha \subseteq X^2$ is a pseudo-order if and only if whenever you drive from site x to site y,

- (a') you can choose a direct route (formally: if there is a route from x to y, then (x, y) is an edge), and
- (b') it is also possible to plan your route so that you will have a chance to take a rest in a parking lot on the way (formally: if there is a route from x to y, then there is a route that includes a vertex with a loop).

Indeed, condition (a) in Definition 2.2 ensures that the removal of loops from the underlying quasi-order $\lesssim$ does not ruin transitivity, thus (a') holds for every pseudo-order. (Observe that (a') is actually equivalent to transitivity.) To verify (b'), choose a longest possible route that does not pass through any $\sim$-block more than once; then each edge in this route is a covering pair, and at least one member of a covering pair has a loop (if any of them belongs to a non-singleton $\sim$-block, then condition (a), otherwise condition (b) provides a loop).

Conversely, let us assume that (a') and (b') hold for α, and let us denote the reflexive closure of α by $\lesssim$. Condition (a') implies that α is transitive, hence $\lesssim$ is also transitive, thus it is a quasi-order. Transitivity of α also implies that (a) holds. To

[1]We use the term *route* for a sequence of connecting edges (with possible repetitions). The usual terminology would be *walk*, but we would like to avoid the uncanny image of a walking truck. . .

verify (b), consider a covering pair $x \prec y$. If the $\sim$-block of x or y is not a singleton, then condition (a) shows that there is a loop at x or y. Otherwise, by the definition of covering, no route from x to y passes through any vertex other than x and y. Therefore, the parking lot guaranteed by (b') must be at x or at y, and this proves (b).

We conclude this subsection by stating the characterization of idempotent binary relations given by B. Schein [20]. For the reader's convenience, we provide a proof for this important result using the description of pseudo-orders given in Remark 2.4.

Theorem 2.5. [20] *A matrix over* $\mathbf{2}$ *is idempotent if and only if the corresponding binary relation is a pseudo-order.*

Proof. Let α be the binary relation on X corresponding to the matrix $A \in \mathbf{M}_n(\mathbf{2})$. As a preliminary observation, let us note that α is transitive if and only if $\alpha \circ \alpha \subseteq \alpha$, which in turn is equivalent to $A^2 \leq A$.

Assume first that A is idempotent. Then $A^2 \leq A$, so α is transitive, hence condition (a') of Remark 2.4 holds. Idempotence of A implies $A = A^2 = A^3 = \dots$, thus whenever there is a route from x to y, there are arbitrarily long routes from x to y. A long enough route must include a directed cycle, and every vertex of such a cycle has a loop, by transitivity. This proves (b'), therefore α is a pseudo-order.

Now let us suppose that α is a pseudo-order. Then α is transitive by condition (a'), hence $A^2 \leq A$. Multiplying this inequality by A^{m-1}, we get $A^{m+1} \leq A^m$ for every positive integer m, thus the powers of A form a decreasing sequence: $A \geq A^2 \geq A^3 \geq \cdots$. Since $\mathbf{M}_n(\mathbf{2})$ is a finite set, this sequence cannot be strictly decreasing, i.e., there is a positive integer ℓ such that

$$A \geq A^2 \geq A^3 \geq \cdots \geq A^{\ell} = A^{\ell+1} = A^{\ell+2} = \cdots = \lim_{m\to\infty} A^m. \tag{1}$$

Here the limit is understood in the discrete topology on $\mathbf{M}_n(\mathbf{2})$, but this is not very important, as an ultimately constant sequence converges in every topology. For every edge $(x, y) \in \alpha$, condition (b') provides a route from x to y with a parking lot on the way. We can park there as long as we wish, before continuing our trip to y, thus there are arbitrarily long routes from x to y. This means that $A \leq \lim_{m\to\infty} A^m$, which together with the inequalities of (1) implies that $A = A^2 = A^3 = \dots$, hence A is idempotent. □

2.2. **Green's relations.** Green's equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$ can be defined in any semigroup, but we write out the definition only for the semigroup $\mathbf{M}_n(L)$, where L is a distributive lattice. Two elements $A, B \in \mathbf{M}_n(L)$ are in $\mathcal{L}$ relation if they generate the same principal left ideal, that is, if and only if there exist $C, D \in \mathbf{M}_n(L)$ such that $CA = B$ and $DB = A$. Similarly, the relation $\mathcal{R}$ can be defined by $(A, B) \in \mathcal{R}$ if and only if there exist $C, D \in \mathbf{M}_n(L)$ such that $AC = B$ and $BD = A$. The relation $\mathcal{L} \cap \mathcal{R}$ is denoted by $\mathcal{H}$, and the join $\mathcal{L} \vee \mathcal{R}$ is denoted by $\mathcal{D}$. It is known that $\mathcal{L}$ and $\mathcal{R}$ commute in every semigroup, thus we have $\mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R}$. For further background on semigroup theory, and in particular on Green's relations, see [8].

Green's relations in $\mathbf{M}_n(\mathbf{2})$ can be described in terms of in- and out-neighborhoods in the graphs corresponding to matrices over $\mathbf{2}$. We introduce the following notation for any relation $\alpha \subseteq X^2$:

- $\alpha^+(x) = \{z \mid (x, z) \in \alpha\} \subseteq X$ is the out-neighborhood of $x \in X$,
- $\alpha^+(Y) = \{z \mid (y, z) \in \alpha \text{ for some } y \in Y\} = \bigcup_{y\in Y} \alpha^+(y)$ is the out-neighborhood of a set $Y \subseteq X$, and
- $\alpha^+ = \{\alpha^+(Y) \mid Y \subseteq X\}$ is the set of all out-neighborhoods.

The in-neighborhoods $\alpha^-(x)$ and $\alpha^-(Y)$ and the set α^- of all in-neighborhoods are defined dually.

Note that α^+ and α^- form lattices under inclusion. The bottom element of both lattices is $\alpha^+(\emptyset) = \alpha^-(\emptyset) = \emptyset$, but the top elements of the two lattices might be different. The join operation in α^+ and in α^- is just the union, i.e., $\alpha^+(Y) \vee \alpha^+(Z) = \alpha^+(Y) \cup \alpha^+(Z) = \alpha^+(Y \cup Z)$. However the meet operation need not be the intersection.

The following description of Green's relations on $\mathbf{M}_n(\mathbf{2})$ follows from results obtained by K. A. Zaretskii in [26] (see also [18]).

Proposition 2.6. [18, 26] *Let $A, B \in \mathbf{M}_n(\mathbf{2})$ and let $\alpha, \beta \subseteq X^2$ be the corresponding binary relations. Then the following hold:*

(1) $(A, B) \in \mathcal{L}$ *if and only if* $\alpha^+ = \beta^+$*;*

(2) $(A, B) \in \mathcal{R}$ *if and only if* $\alpha^- = \beta^-$*;*

(3) $(A, B) \in \mathcal{H}$ *if and only if* $\alpha^+ = \beta^+$ *and* $\alpha^- = \beta^-$*;*

(4) $(A, B) \in \mathcal{D}$ *if and only if the lattices* α^+ *and* β^+ *are isomorphic.*

Remark 2.7. Let $A \in \mathbf{M}_n(\mathbf{2})$ be an idempotent matrix and let $\alpha \subseteq X^2$ be the corresponding pseudo-order relation. Let T be a complete system of representatives of the blocks of the equivalence relation $\alpha \cap \alpha^{-1} = \sim \cap\, Q_\alpha^2$ (cf. Remark 2.3). Then the relation $\tilde{\alpha} := \alpha \cap T^2$ is a partial order on the set $T \subseteq X$ (the elements of $X \setminus T$ are isolated points in $\tilde{\alpha}$). Relations of this form, i.e., partial orders on subsets of X, are called *reduced idempotents.* This notion was introduced by J. S. Montague and R. J. Plemmons, and it was proved in [15] that if a $\mathcal{D}$-class contains an idempotent (these are called *regular* $\mathcal{D}$-classes), then it also contains a reduced idempotent. The structure of the poset $(T; \tilde{\alpha})$ is independent of the choice of T; let us denote (the isomorphism type of) this poset by $T(\alpha)$.

2.3. **Maximal subgroups.** According to Green's Theorem, if E is an idempotent matrix in $\mathbf{M}_n(L)$, then there is a maximal subgroup "around" E, having E as its identity element, and this maximal subgroup is nothing else but the $\mathcal{H}$-class H_E of E. Moreover, if two idempotents E, F belong to the same $\mathcal{D}$-class, then the groups H_E and H_F are isomorphic. In this subsection we recall the description of these maximal subgroups of $\mathbf{M}_n(\mathbf{2})$ [4, 15, 17, 18, 19, 26].

For any permutation $\pi \in S_n$, we define the *permutation matrix* corresponding to π as the matrix $P_\pi = (p_{ij})_{i,j=1}^n \in \mathbf{M}_n(L)$ given by

$$p_{ij} = \begin{cases} 1, & \text{if } j = \pi(i); \\ 0, & \text{otherwise.} \end{cases}$$

Remark 2.8. Just as over commutative rings, the matrix $P_\pi A$ is obtained from A by permuting its rows according to the permutation π; similarly, AP_π is obtained from A by permuting its columns according to the permutation π^{-1}. In particular, we have $P_\pi P_\sigma = P_{\pi\sigma}$ for all $\pi, \sigma \in S_n$, and the (unique) inverse of P_π is $P_{\pi^{-1}}$.

Theorem 2.9. [15, 18] *Let $A \in \mathbf{M}_n(\mathbf{2})$ be an idempotent matrix with the corresponding pseudo-order $\alpha \subseteq X^2$. Assume that A is a reduced idempotent, i.e., α is a partial order on the set $T := Q_\alpha \subseteq X$. Then a matrix $B \in \mathbf{M}_n(\mathbf{2})$ belongs to the $\mathcal{H}$-class of A if and only if it can be written as $B = P_f A$, where f is a permutation on X such that $f(T) = T$ and the restriction of f to T is an automorphism of the poset $(T; \alpha)$.*

Theorem 2.9 immediately yields the following corollary (see also [4, 17, 19]).

Corollary 2.10. [15] *Let $A \in \mathbf{M}_n(\mathbf{2})$ be an idempotent matrix with the corresponding pseudo-order $\alpha \subseteq X^2$. Then the $\mathcal{H}$-class containing A is isomorphic to the automorphism group of the poset $T(\alpha)$*

3. Idempotent and nilpotent matrices

Throughout this section $L = (L; +, \cdot)$ is assumed to be a bounded distributive lattice with least element 0 and greatest element 1. By Birkhoff's representation theorem, L can be embedded into the lattice $\mathcal{P}(\Omega)$ of subsets of a set Ω in such a way that 0 is mapped to $\emptyset$ and 1 is mapped to Ω. Identifying L with its embedded image, we can actually assume that L is a sublattice of $\mathcal{P}(\Omega)$ with $0 = \emptyset$ and $1 = \Omega$. This allows us to define a homomorphism Γ_ω from L to $\mathbf{2} = \{0, 1\}$ for each $\omega \in \Omega$ by

$$\Gamma_\omega(a) = \begin{cases} 1, & \text{if } \omega \in a; \\ 0, & \text{if } \omega \notin a. \end{cases}$$

We call $\Gamma_\omega(a)$ the *cut* of the element a (at ω) [28] (also called ω^{th} *constituent* [21] and *section* or *zero pattern* [10]). Since $a \subseteq \Omega$ is exactly the set of those elements $\omega \in \Omega$ for which $\Gamma_\omega(a) = 1$, every element of L is uniquely determined by its cuts. Extending Γ_ω to matrices entrywise, we get cut homomorphisms $\Gamma_\omega \colon \mathbf{M}_n(L) \to \mathbf{M}_n(\mathbf{2})$ for all $\omega \in \Omega$, and matrices are also uniquely determined by their cuts:

$$\forall A, B \in \mathbf{M}_n(L)\colon A = B \iff [\forall \omega \in \Omega \colon \Gamma_\omega(A) = \Gamma_\omega(B)]. \tag{2}$$

Remark 3.1. Let us give an interpretation of matrices over L in the spirit of Remark 2.1. As before, we regard the elements of $X = \{1, \ldots, n\}$ as sites numbered from 1 to n, and we think of the elements of Ω as different types of vehicles that can travel between these sites. The entry $a_{ij} \subseteq \Omega$ of the matrix $A \in \mathbf{M}_n(L)$ determines which vehicles can (or are allowed to) pass through the road from i to j (the diagonal entry a_{ii} is the set of vehicles that can park at site i). In other words, we have a complete directed graph on n vertices, and each edge (i, j) has a "capacity" $a_{ij} \subseteq \Omega$. (In reality, the graph is rarely complete; we can take non-existing connections into account by assigning capacity $\emptyset$.) Given a route $i = v_0 \to v_1 \to \cdots \to v_\ell = j$ of length ℓ, the set of vehicles that can travel all the way along this route from i to j is the intersection (product) of the capacities of the edges involved in the route, i.e., $a_{iv_1} \cdot \ldots \cdot a_{v_{\ell-1}j}$. We will call this element of L the capacity of the route. The set of vehicles that can go from i to j on some route of length ℓ can be computed as the join (sum) of the capacities of the routes of length ℓ from i to j, which is nothing else but the (i, j)-entry of A^ℓ.

In the following we study idempotent and nilpotent elements of the semigroup $\mathbf{M}_n(L)$. For a review of results about powers of matrices over distributive lattices, we refer the reader to the survey paper [1] and to the references therein.

3.1. **Idempotent matrices.** From (2) and from the fact that each Γ_ω is a homomorphism, it follows that a matrix is idempotent if and only if all of its cuts are idempotent [2, 3, 10, 21]:

$$\begin{aligned} A = AA &\iff \forall \omega \in \Omega \colon \Gamma_\omega(A) = \Gamma_\omega(AA) \\ &\iff \forall \omega \in \Omega \colon \Gamma_\omega(A) = \Gamma_\omega(A)\Gamma_\omega(A). \end{aligned}$$

Combining this observation with Theorem 2.5, we get the following description of idempotent matrices over distributive lattices.

Proposition 3.2. *A matrix $A \in \mathbf{M}_n(L)$ over a distributive lattice $L \leq \mathcal{P}(\Omega)$ is idempotent if and only if the binary relation $\alpha_\omega \subseteq X^2$ corresponding to the cut matrix $\Gamma_\omega(A)$ is a pseudo-order for each $\omega \in \Omega$.*

Although Proposition 3.2 certainly characterizes idempotent matrices, this characterization does not give a complete picture about the idempotent elements of the semigroup $\mathbf{M}_n(L)$, since it does not tell us which systems of pseudo-orders α_ω $(\omega \in \Omega)$ can arise as cuts of idempotent matrices. In full generality perhaps one cannot expect a feasible solution for this problem, but for chains we can give a simple criterion. We represent the m-element chain in the power set of $\Omega = \{1, \ldots, m-1\}$ as

$$\emptyset \subset \{1\} \subset \{1, 2\} \subset \cdots \subset \{1, 2, \ldots, m-1\}, \tag{3}$$

so that the cut homomorphisms are $\Gamma_1, \ldots, \Gamma_{m-1}$.

Theorem 3.3. *If L is the m-element chain, then a matrix $A \in \mathbf{M}_n(L)$ is idempotent if and only if the binary relations corresponding to the cut matrices $\Gamma_k(A)$ $(k = 1, \ldots, m-1)$ form a system of nested pseudo-orders $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$.*

Proof. Since we represent L by the chain of sets (3), we have the implication $k \in a \implies k-1 \in a$ for all $a \in L$ and $k \in \{2, \ldots, m-1\}$. This implies the inequalities $\Gamma_1(A) \geq \cdots \geq \Gamma_{m-1}(A)$ for every matrix $A \in \mathbf{M}_n(L)$ (idempotent or not), and these inequalities translate to the containments $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$ of the corresponding relations. This together with Proposition 3.2 proves the necessity of the condition formulated in the proposition.

For sufficiency, assume that we have a nested sequence of pseudo-orders $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$ on X. Define the matrix $A = (a_{ij})_{i,j=1}^n \in \mathbf{M}_n(L)$ by

$$a_{ij} = \big\{k \in \{1, \ldots, m-1\} : (i,j) \in \alpha_k\big\}.$$

Observe that the assumed containments of the relations α_k guarantee that a_{ij} is an element of L. Thus A is indeed a matrix over L, and the binary relations corresponding to the cuts of A are exactly the relations $\alpha_1, \ldots, \alpha_{m-1}$. Since these are all pseudo-orders, each cut of A is idempotent by Theorem 2.5, and then idempotence of A follows from Proposition 3.2. □

3.2. **Nilpotent matrices.** First we recall a simple criterion for the nilpotency of a matrix in terms of the underlying directed graph, and then we use it to explicitly describe nilpotent matrices over bounded distributive lattices with a meet-irreducible bottom element.

Lemma 3.4. [6, 24, 27] *A matrix $A \in \mathbf{M}_n(L)$ over a bounded distributive lattice L is nilpotent if and only if every cycle in the directed graph corresponding to A has capacity 0. Moreover A is nilpotent if and only if $A^n = \mathbf{0}$.*

Remark 3.5. Several other characterizations have been given for nilpotent matrices; see, e.g., [11, 16, 23, 25]. It is obvious that the determinant of a nilpotent matrix is zero (the determinant of a lattice matrix can be defined in a similarly way as for matrices over rings). In [16] it is claimed that the converse is also true. However, as the following counterexample shows, this is not the case. Let us consider the matrix $A \in \mathbf{M}_2(L)$ over an arbitrary bounded distributive lattice L:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

This matrix has zero determinant, but it is not nilpotent; in fact, it is easy to see that A is idempotent, hence $A^n = A$ for all natural numbers n.

By a strictly upper triangular matrix we mean a matrix $A \in \mathbf{M}_n(L)$ that has zeros below its main diagonal as well as on the main diagonal, i.e., $a_{ij} \neq 0 \implies i < j$.

Theorem 3.6. *Let L be a bounded distributive lattice in which 0 is meet-irreducible. Then a matrix $A \in \mathbf{M}_n(L)$ is nilpotent if and only if it is conjugate to a strictly upper triangular matrix, i.e., there exists a strictly upper triangular matrix U and an invertible matrix C such that $A = C^{-1}UC$.*

Proof. If U is a strictly upper triangular matrix, then we have $U \leq V$, where V is the matrix having ones above the diagonal and zeros on and below the diagonal:

$$V = \begin{pmatrix} 0 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{4}$$

In the directed graph corresponding to V, we have an edge from i to j if and only if $i < j$. This means that it is impossible to make a route of length n, hence $V^n = 0$. Since $U \leq V$, it follows that $U^n = 0$, which implies that $(C^{-1}UC)^n = 0$ for every invertible matrix C.

Conversely, let us assume that $A \in \mathbf{M}_n(L)$ is a nilpotent matrix. Consider the relation $\alpha \subseteq X^2$ defined by $\alpha := \{(i,j) : a_{ij} \neq 0\}$. (If L is finite, then meet-irreducibility of 0 implies that 0 has a unique upper cover. If ω is any element of

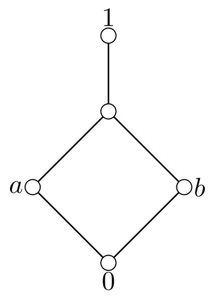

Figure 1. The lattice $(\mathbf{2} \times \mathbf{2}) \oplus \mathbf{1}$

the upper cover of 0, then α is the relation corresponding to the matrix $\Gamma_\omega(A)$, i.e., $(i, j) \in \alpha$ iff trucks of type ω are allowed to travel on the edge from i to j.) By Lemma 3.4, every cycle in the directed graph corresponding to A has zero capacity, hence at least one edge of each cycle has capacity 0, as 0 is meet-irreducible. This means that α contains no directed cycles. Therefore, the reflexive transitive closure of α is a partial order on X, and this partial order can be extended to a linear order $\sqsubseteq$. Since $\sqsubseteq$ is an extension of α, we have $a_{ij} \neq 0 \implies i \sqsubset j$ for all $i, j \in X$.

Let π be the permutation of X given by $\pi(1) \sqsubset \cdots \sqsubset \pi(n)$, and let $C = P_\pi$. We claim that the matrix $U := CAC^{-1}$ is strictly upper triangular. By the definition of the matrix C, we have $u_{ij} = a_{\pi(i)\pi(j)}$, hence

$$u_{ij} \neq 0 \implies a_{\pi(i)\pi(j)} \neq 0 \implies \pi(i) \sqsubset \pi(j) \implies i < j.$$

(The last implication is justified by the definition of π.) Thus U is indeed strictly upper triangular, and this completes the proof, as $A = C^{-1}UC$. ☐

Remark 3.7. We have seen in Lemma 3.4 that A is nilpotent if and only if $A^n = \mathbf{0}$. This cannot be sharpened: the matrix V given in (4) is nilpotent, but $V^{n-1} \neq 0$.

Example 3.8. Theorem 3.6 does not necessarily remain true without the assumption on the irreducibility of 0. Consider the matrix $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ over the lattice $(\mathbf{2} \times \mathbf{2}) \oplus \mathbf{1}$ shown in Figure 1. It is easy to verify that $A^2 = 0$, but A is not a conjugate of a strictly upper triangular matrix. Indeed, we will show later in Theorem 5.7 that if 1 is join-irreducible in a lattice L, then the only invertible matrices in $\mathbf{M}_n(L)$ are the permutation matrices. This is true in particular for the lattice $L = (\mathbf{2} \times \mathbf{2}) \oplus \mathbf{1}$, hence the only conjugates of A are itself and the matrix

$$P_{(12)}^{-1} \cdot A \cdot P_{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix},$$

and neither of them is upper triangular.

3.3. **Fixed point iteration.**

Our results on nilpotent matrices have some implications on a problem about fuzzy relations raised in [9]. The interpretation of a matrix $A \in \mathbf{M}_n(L)$ as a directed graph with a capacity assigned to each edge (see Remark 3.1), is almost the same as a fuzzy relation; we only need to regard the entries a_{ij} as membership values instead of capacities. The inequality $\mathbf{x}A \leq \mathbf{x}$ and the equation $\mathbf{x}A = \mathbf{x}$ were studied in [9] from the viewpoint of fuzzy control. We refer the reader to that paper for more details about fuzzy relations and their applications, and here we focus only on the proposed fixed-point iteration method to find solutions of the equation $\mathbf{x}A = \mathbf{x}$.

The solutions of $\mathbf{x}A = \mathbf{x}$ are exactly the fixed points of the "linear transformation" $\mathbf{x} \mapsto \mathbf{x}A$, hence we can hope that the standard fixed-point iteration method can be used to find solutions [9]. Thus we start with an arbitrary $\mathbf{x} \in L^n$, and we form the sequence

$$\mathbf{x},\ \mathbf{x}A,\ \mathbf{x}A^2,\ \mathbf{x}A^3, \ldots, \mathbf{x}A^k, \ldots. \tag{5}$$

Even if L is an infinite lattice, each entry of each tuple in our sequence belongs to the sublattice generated by the $n+n^2$ elements x_i, a_{ij} $(i,j=1,\dots,n)$, which is finite if L is distributive. Therefore, $\mathbf{x}A^k$ becomes eventually periodic, but the period can be longer than 1 (consider a permutation matrix, for example), hence the sequence might fail to converge. However, if $\lim_{k\to\infty}(\mathbf{x}A^k)$ exists, then it is easy to see that this limit will be a solution of $\mathbf{x}A=\mathbf{x}$. It may happen that (5) converges to the trivial solution $\mathbf{0}=(0,\dots,0)$, hence it is natural to start with the largest possible initial value, namely $\mathbf{x}=\mathbf{1}=(1,\dots,1)$. In this case (5) is a monotone sequence, and this together with periodicity implies that the sequence is ultimately constant (hence convergent). This observation yields that $\lim_{k\to\infty}(\mathbf{1}A^k)$ is the greatest solution of the fixed-point equation $\mathbf{x}A=\mathbf{x}$ (see [22]).

Remark 3.9. The interpretation of the greatest solution of $\mathbf{x}A=\mathbf{x}$ in the transportation network setting of Remark 3.1 is more natural if we work with column vectors instead of row vectors (or we transpose A). If $\lim_{k\to\infty}(A^k\mathbf{1})=(z_1,\dots,z_n)$, then $z_i\in L$ is the set of vehicles that can start arbitrarily long trips at $i\in X$. In other words, z_i is the set of vehicles that can reach a directed cycle from i.

From the considerations above it follows immediately that the fixed-point equation $\mathbf{x}A=\mathbf{x}$ has a nonzero solution if and only if the matrix A is not nilpotent. This was observed for Boolean algebras in [11], and later for bounded distributive lattices in [23]. If the bottom element of L is irreducible, then combining this result with Theorem 3.6 and Theorem 5.7, we obtain the following corollary.

Corollary 3.10. *Let L be a bounded distributive lattice in which* 0 *is meet-irreducible. Then the following are equivalent for any matrix $A\in\mathbf{M}_n(L)$:*

(i) *the only solution of the fixed-point equation $\mathbf{x}A=\mathbf{x}$ is $\mathbf{x}=\mathbf{0}$;*

(ii) $\lim_{k\to\infty}(\mathbf{1}A^k)=\mathbf{0}$*;*

(iii) *A is nilpotent;*

(iv) $A^n=0$*;*

(v) *A is conjugate to a strictly upper triangular matrix, i.e., there exists a strictly upper triangular matrix U and an invertible matrix C such that $A=C^{-1}UC$;*

(vi) *one can rearrange the rows and columns of A so that it becomes a strictly upper triangular matrix, i.e., there exists a permutation $\pi\in S_n$ such that $P_\pi^{-1}AP_\pi$ is strictly upper triangular.*

Corollary 3.10 applies in particular to chains (which is the most relevant case for fuzzy relations) and it shows that the fixed-point equation $\mathbf{x}A=\mathbf{x}$ has a nontrivial solution except for only a few matrices of a very restricted form.

4. Green's relations and maximal subgroups

In Section 2 we have seen that every idempotent of the semigroup $\mathbf{M}_n(\mathbf{2})$ is $\mathcal{D}$-related to a reduced idempotent and that the maximal subgroups can be given in terms of automorphisms of the posets corresponding to these reduced idempotents. Even though descriptions of Green's relations of the semigroups $\mathbf{M}_n(L)$ are available (see [10], where Green's relations are described with the help of row and column spaces of matrices over commutative semirings), we do not a have a clear picture about the maximal subgroups of $\mathbf{M}_n(L)$ for an arbitrary bounded distributive lattice L.

A possible plan to attack this problem is the following.

1. Find a necessary and sufficient condition for two idempotents to be $\mathcal{D}$-related, and then use this to determine a set of "nicest" idempotents that forms a transversal for the regular $\mathcal{D}$-classes.
2. Describe the structure of the $\mathcal{H}$-classes of these nicest idempotents.

For the two-element lattice, a simple solution to the first item can be given using the results of [15].

Theorem 4.1. *Let $A, B \in \mathbf{M}_n(\mathbf{2})$ be idempotent matrices and let $\alpha, \beta \subseteq X^2$ be the corresponding pseudo-order relations. We have $(A, B) \in \mathcal{D}$ if and only if the posets $T(\alpha)$ and $T(\beta)$ are isomorphic.*

Proof. Let us recall from Remark 2.7 that the $\mathcal{D}$-class of α contains a reduced idempotent $\tilde{\alpha}$, which is a partial order on a subset $T \subseteq X$, and $T(\alpha)$ denotes this poset (it is determined uniquely up to isomorphism). If we use the usual symbol $\leq$ for this partial order instead of $\tilde{\alpha}$, then the out- and in-neighborhood of $x \in T$ can be written as:

- $\tilde{\alpha}^+(x) = \{y \in T : y \geq x\} =: {\uparrow} x$;
- $\tilde{\alpha}^-(x) = \{y \in T : y \leq x\} =: {\downarrow} x$.

The elements of $\tilde{\alpha}^+$ (i.e., unions of sets of the form ${\uparrow} x$) are called upper closed sets, or simply *upsets*. Thus $U \subseteq T$ is an upset if and only if $x \in U$ and $y \geq x$ imply $y \in U$ for all $x, y \in T$. Dually, the members of $\tilde{\alpha}^-$ are called *downsets*. It follows from Proposition 2.6 that the lattices α^+ and $\tilde{\alpha}^+$ are isomorphic, and the latter is nothing but the lattice of upsets of $T(\alpha)$.

Thus, by Proposition 2.6, we only need to prove that the posets $T(\alpha)$ and $T(\beta)$ are isomorphic if and only if their upset lattices are isomorphic. The "only if" part is trivial, and the "if" part follows from the observation that for any finite poset P, the join-irreducible elements of the upset lattice are exactly the sets of the form ${\uparrow} x$ $(x \in P)$, and these sets form a poset that is dually isomorphic to P. □

Theorem 4.1 gives a solution to the first item in the "plan of attack": reduced idempotents, i.e., partial orders on subsets of X represent every regular $\mathcal{D}$-class essentially uniquely (up to isomorphism of the corresponding posets). Generalizing this to arbitrary bounded distributive lattices is a topic for further research, but in Theorem 4.3 below we provide a partial result towards the second item of the plan for finite chains, which are the most frequently used lattices in applications. As a preparation, we need a simple auxiliary observation.

Lemma 4.2. *If $(T; \leq)$ is a finite poset and f is a permutation of T such that $f(x) \geq x$ for all $x \in T$, then $f = \mathrm{id}_T$.*

Proof. If x is a maximal element, then $f(x) = x$ follows immediately from the assumption $f(x) \geq x$. From here we can proceed downwards, proving by induction on the size of ${\uparrow} x = \{y \in X : y \geq x\}$ that $f(x) = x$ for all $x \in T$. □

We have seen in the previous section that a matrix is idempotent if and only if all of its cuts are idempotent. The reason behind this observation is that the definition of idempotence is simply an equality; it does not ask for the existence of certain elements. For "existentially quantified" notions the situation is more complicated. As an example, let us recall the definition of the $\mathcal{R}$ relation:

$$(A, B) \in \mathcal{R} \iff \exists C, D \in \mathbf{M}_n(L)\colon AC = B \text{ and } BD = A.$$

Since the cut maps are homomorphisms, $AC = B$ and $BD = A$ imply $\Gamma_\omega(A)\Gamma_\omega(C) = \Gamma_\omega(B)$ and $\Gamma_\omega(B)\Gamma_\omega(D) = \Gamma_\omega(A)$, hence $\Gamma_\omega(A)$ and $\Gamma_\omega(B)$ are $\mathcal{R}$-related in the semigroup $\mathbf{M}_n(\mathbf{2})$ for all $\omega \in \Omega$. However, the converse is not necessarily true: given matrices $C_\omega, D_\omega \in \mathbf{M}_n(\mathbf{2})$ such that $\Gamma_\omega(A)C_\omega = \Gamma_\omega(B)$ and $\Gamma_\omega(B)D_\omega = \Gamma_\omega(A)$ for all $\omega \in \Omega$, it is not guaranteed that there exist matrices $C, D \in \mathbf{M}_n(L)$ whose cuts are C_ω and D_ω, respectively. In fact, an example of $\mathcal{R}$-inequivalent matrices $A, B \in \mathbf{M}_2(L)$ over the three-element chain were presented in [28] such that both of their cuts are $\mathcal{R}$-related.

Nevertheless, as illustrated by the following theorem, in some special cases we can recover information about matrices from their cuts.

Theorem 4.3. *Let L be the m-element chain, and let $A \in \mathbf{M}_n(L)$ be an idempotent matrix such that the binary relations $\alpha_k \subseteq X^2$ corresponding to the cut matrices $\Gamma_k(A)$ $(k = 1, \ldots, m-1)$ are all partial orders. Then a matrix $B \in \mathbf{M}_n(L)$ belongs to* *the $\mathcal{H}$-class of A if and only if it can be written as $B = P_f A$, where f is a permutation on X that is a common automorphism of the posets $(X;\alpha_k)\,(k = 1,\dots,m-1)$.*

Proof. Assume first that f is an automorphism of each of the posets $(X;\alpha_k)$. Regarding f as a binary relation, this fact can be expressed as $f\circ\alpha_k\circ f^{-1} = \alpha_k$, which is in turn equivalent to $f\circ\alpha_k = \alpha_k\circ f$. The latter condition can be formulated in terms of matrices as $P_f\Gamma_k(A) = \Gamma_k(A)P_f$. Since every entry of P_f is 0 or 1, we have $\Gamma_k(P_f) = P_f$, hence we can conclude that

$$\Gamma_k(P_f A) = \Gamma_k(P_f)\Gamma_k(A) = P_f\Gamma_k(A) = \Gamma_k(A)P_f = \Gamma_k(A)\Gamma_k(P_f) = \Gamma_k(AP_f).$$

According to (2), this holds for every k if and only if $P_f A = AP_f$, and the latter implies that the matrix $B = P_f A$ belongs to the $\mathcal{H}$-class of A.

Conversely, assume that $B \in \mathbf{M}_n(L)$ is $\mathcal{H}$-related to A. Since each Γ_k is a homomorphism, $(\Gamma_k(A),\Gamma_k(B)) \in \mathcal{H}$ holds in the semigroup $\mathbf{M}_n(\mathbf{2})$ for all $k \in \{1,\dots,m-1\}$. By Theorem 2.9, for each k there exists an automorphism f_k of the poset $(X;\alpha_k)$ such that $\Gamma_k(B) = P_{f_k}\Gamma_k(A)$. We are going to prove that $f_1 = \cdots = f_{m-1}$.

Denoting by β_k the binary relation corresponding to the matrix $\Gamma_k(B) \in \mathbf{M}_n(\mathbf{2})$, the equality $\Gamma_k(B) = P_{f_k}\Gamma_k(A)$ is equivalent to

$$\forall x,y\in X\colon (f_k(x),y)\in\alpha_k \iff (x,y)\in\beta_k \quad (k=1,\dots,m-1). \tag{6}$$

Since L is a chain, the relations β_k form a nested sequence (cf. the beginning of the proof of Theorem 3.3):

$$\beta_1 \supseteq \cdots \supseteq \beta_{m-1}. \tag{7}$$

For every $k \in \{1,\dots,m-1\}$ and $x \in X$, we have $(f_k(x), f_k(x)) \in \alpha_k$, as α_k was assumed to be a partial order. Using (6), this implies that $(x, f_k(x)) \in \beta_k$, and then $(x, f_k(x)) \in \beta_1$, by (7). Applying (6) with $k = 1$, we can conclude that $(f_1(x), f_k(x)) \in \alpha_1$. We can rewrite this as $(y, f_k(f_1^{-1}(y))) \in \alpha_1$ with the notation $y = f_1(x)$. This holds for every $y \in X$, therefore $f_1 = f_k$ follows from Lemma 4.2.

We have proved that $f := f_1 = \cdots = f_{m-1}$ is a common automorphism of the posets $(X;\alpha_k)\,(k = 1,\dots,m-1)$. It remains to prove that $B = P_f A$. By (2), it suffices to show that the cuts of B and $P_f A$ coincide:

$$\Gamma_k(B) = P_{f_k}\Gamma_k(A) = P_f\Gamma_k(A) = \Gamma_k(P_f)\Gamma_k(A) = \Gamma_k(P_f A).$$

(We used again the fact that cuts preserve 0 and 1, hence each cut of the permutation matrix P_f is itself.) □

Corollary 4.4. *Let L be the m-element chain, and let $A \in \mathbf{M}_n(L)$ be an idempotent matrix such that the binary relations $\alpha_k \subseteq X^2$ corresponding to the cut matrices $\Gamma_k(A)\,(k = 1,\dots,m-1)$ are all partial orders. Then the $\mathcal{H}$-class containing A is isomorphic to the group of common automorphisms of the posets $(X;\alpha_k)\,(k = 1,\dots,m-1)$.*

5. Matrices over arbitrary lattices

5.1. **Antiassociativity of matrix multiplication.** First we characterize lattices with associative matrix multiplication.

Proposition 5.1. *Multiplication of matrices over a lattice L is associative if and only if L is a distributive lattice.*

Proof. If L is distributive, then one can prove associativity of matrix multiplication in the same way as it is proved for matrices over commutative rings. (In fact, if L is bounded, then L is a semiring, hence $\mathbf{M}_n(L)$ is also a semiring [7]).

If L is not distributive, then M_3 or N_5 embeds into L (see Figure 2), so it suffices to prove nonassociativity of matrix multiplication over these two lattices. Let us consider the following three matrices from $\mathbf{M}_2(M_3)$ or from $\mathbf{M}_2(N_5)$:

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}.$$

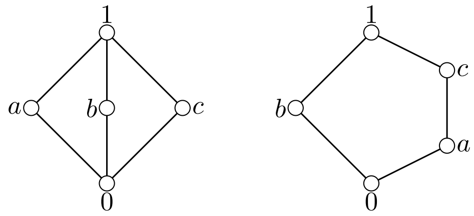

FIGURE 2. The lattices M_3 and N_5

Then it is easy to verify that $(AB)\,C \neq A\,(BC)$. For any $n \geq 2$, we can construct matrices attesting the nonassociativity of multiplication in $\mathbf{M}_n(L)$ by inserting A, B and C into the top left 2×2 corner of an $n \times n$ matrix and filling all the remaining entries with 0. □

We can strengthen Proposition 5.1; if L is not distributive, then matrix multiplication over L is not merely nonassociative: it is antiassociative! We derive this as a corollary of the following general proposition (this has been independently proved by E. Lehtonen [12]).

Proposition 5.2. *If a binary operation has an identity element, then it is either associative (i.e., the associative spectrum is constant* 1*) or it is antiassociative (i.e., the associative spectrum consists of the Catalan numbers).*

Proof. Let $(G;\cdot)$ be a groupoid with an identity element 1, and assume that $(ab)c \neq a(bc)$ for some $a, b, c \in G$. We prove by induction on n that any two bracketings $p \neq q$ of size n induce different term operations on G. For $n = 1, 2$ this claim is void, and for $n = 3$ it holds by the nonassociativity of the multiplication of G. Assume now that $n \geq 4$ and different bracketings of size less than n induce different term functions, and let p, q be two distinct bracketings of size n.

First we consider the case when the "outermost" multiplication of p and q is at the same place: $p = p_1(x_1, \ldots, x_k) \cdot p_2(x_{k+1}, \ldots, x_n)$ and $q = q_1(x_1, \ldots, x_k) \cdot q_2(x_{k+1}, \ldots, x_n)$. Since p and q are not the same term, we have $p_1 \neq q_1$ or $p_2 \neq q_2$ (perhaps both). If $p_1 \neq q_1$, then, by the induction hypothesis, there exist elements $a_1, \ldots, a_k \in G$ such that $p_1(a_1, \ldots, a_k) \neq q_1(a_1, \ldots, a_k)$. This implies

$$\begin{aligned} p(a_1, \ldots, a_k, 1, \ldots, 1) &= p_1(a_1, \ldots, a_k) \cdot p_2(1, \ldots, 1) \\ &= p_1(a_1, \ldots, a_k) \cdot 1 = p_1(a_1, \ldots, a_k) \\ &\neq q_1(a_1, \ldots, a_k) = q_1(a_1, \ldots, a_k) \cdot 1 \\ &= q_1(a_1, \ldots, a_k) \cdot q_2(1, \ldots, 1) \\ &= q(a_1, \ldots, a_k, 1, \ldots, 1), \end{aligned}$$

thus the term functions corresponding to p and q are indeed different. If $p_2 \neq q_2$, then a similar argument can be used, assigning the value 1 to the variables $x_1, \ldots, x_k$.

Now assume that the outermost multiplications in p and q are not at the same place: $p = p_1(x_1, \ldots, x_k) \cdot p_2(x_{k+1}, \ldots, x_n)$ and $q = q_1(x_1, \ldots, x_\ell) \cdot q_2(x_{\ell+1}, \ldots, x_n)$, where $k \neq \ell$. We may suppose without loss of generality that $k < \ell$. Let us put $x_1 = a$, $x_{k+1} = b$, $x_{\ell+1} = c$, and assign the value 1 to all the remaining variables. Then p evaluates to

$$p_1(a, 1, \ldots, 1) \cdot p_2(b, 1, \ldots, 1, c, 1, \ldots, 1) = a(bc),$$

while q gives the value

$$q_1(a, 1, \ldots, 1, b, 1, \ldots, 1) \cdot q_2(c, 1, \ldots, 1) = (ab)c,$$

proving that p and q induce different term functions, as claimed. □

Corollary 5.3. *If the lattice L is not distributive, then the multiplication of matrices over L is antiassociative.*

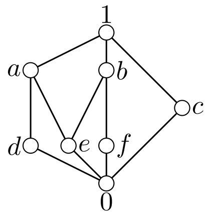

FIGURE 3. The lattice K.

Proof. Since L is not distributive, it has a sublattice L_1 that is isomorphic to M_3 or to N_5. The lattice L_1 is bounded, hence $\mathbf{M}_n(L_1)$ has an identity element, thus its multiplication is antiassociative by propositions 5.1 and 5.2. This implies antiassociativity of the multiplication of $\mathbf{M}_n(L)$, as it contains $\mathbf{M}_n(L_1)$ as a subgroupoid. □

The following example shows that for nondistributive lattices even the definition of a power of a matrix and the notion of nilpotence can be problematic.

Example 5.4. Let A be the following 5×5 matrix over M_3:

$$A = \begin{pmatrix} 0 & a & 0 & 0 & 0 \\ 0 & 0 & b & c & 0 \\ 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have $(AA)A = 0 \neq A(AA)$. Thus A has two different "cubes"; one of them is zero, the other one is not.

5.2. **Invertible matrices.** As another illustration of the unpleasant consequences of nonassociativity, we present an example of a matrix having several inverses.

Example 5.5. Consider the following two matrices over N_5:

$$A = \begin{pmatrix} c & b \\ b & c \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then we have $AA = AB = BA = I$, thus A and B are both inverses of A.

It was proved in [14] that a matrix A over a Boolean algebra is invertible if and only if it is orthogonal, i.e., $AA^T = A^TA = I$. This result was generalized to bounded distributive lattices in [6] (see also [21, 25]). However, if L is not distributive, then there might exist invertible matrices over L that are not orthogonal.

Example 5.6. Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ over the lattice K shown in Figure 3. This lattice is nondistributive, moreover the equations $AB = BA = I$ hold for the matrix $B = \begin{pmatrix} d & f \\ f & c \end{pmatrix}$ over the lattice K. Thus A is invertible, but $AA^T = A^2 \neq I$, so A is not orthogonal.

In the next theorem we determine the invertible elements of $\mathbf{M}_n(L)$ where L is a bounded lattice and at least one of 0 and 1 is irreducible, but we omit the assumption on distributivity of the lattice L. Recall that 0 is meet-irreducible if $ab = 0$ holds only if $a = 0$ or $b = 0$, and similarly, 1 is join-irreducible if $a + b = 1$ implies that $a = 1$ or $b = 1$.

Theorem 5.7. *Let L be a bounded lattice in which 0 is meet-irreducible or 1 is join-irreducible. Then for all matrices $A, B \in \mathbf{M}_n(L)$, we have $AB = I$ if and only if $A = P_\pi$ and $B = P_{\pi^{-1}}$ for some permutation $\pi \in S_n$. Consequently, the invertible elements of the groupoid $\mathbf{M}_n(L)$ form a group that is isomorphic to S_n.*

Proof. The "if" part is clear (see Remark 2.8); so we only prove the "only if" part. Moreover, it suffices to prove that $A = P_\pi$; then $B = P_{\pi^{-1}}$ follows by Remark 2.8. First we make some general observations, assuming only that L is a bounded lattice.

Let $A, B \in \mathbf{M}_n(L)$ with $AB = I$. Considering the diagonal entries of $AB = I$, we have $\sum_{j=1}^n a_{ij} b_{ji} = 1$ for all $i = 1, \dots, n$. This implies that for each i there is at least one j such that $a_{ij} b_{ji} \neq 0$. Denoting such an index j by $\pi(i)$, we get a map $\pi\colon \{1, \dots, n\} \to \{1, \dots, n\}$ such that

$$a_{i\pi(i)} \neq 0 \text{ and } b_{\pi(i)i} \neq 0 \text{ for all } i \in \{1, \dots, n\}. \tag{8}$$

The off-diagonal entries of $AB = I$ yield $\sum_{j=1}^n a_{ij} b_{jk} = 0$ whenever $i \neq k$, hence

$$a_{ij} b_{jk} = 0 \text{ for all } i, j, k \in \{1, \dots, n\} \text{ with } i \neq k. \tag{9}$$

Assume first that 1 is join-irreducible. Then at least one of the summands in $\sum_{j=1}^n a_{ij} b_{ji} = 1$ must be 1, hence we can replace (8) by the following stronger condition:

$$a_{i\pi(i)} = b_{\pi(i)i} = 1 \text{ for all } i \in \{1, \dots, n\}. \tag{8'}$$

Now we can see that π is injective: if we had $\pi(i) = \pi(k) =: j$ for some $i \neq k$, then (8') would imply that $a_{ij} = b_{jk} = 1$, contradicting (9). In order to prove that A is a permutation matrix, let us consider an entry a_{ij} in A with $j \neq \pi(i)$. Letting $k = \pi^{-1}(j)$, we have $b_{jk} = 1$ by (8'); on the other hand, (9) implies $a_{ij} b_{jk} = 0$, as $i \neq k$. Thus $a_{ij} = 0$ whenever $j \neq \pi(i)$, and this together with (8') proves that $A = P_\pi$.

Suppose next that 0 is meet-irreducible. Then (9) takes the following form:

$$a_{ij} = 0 \text{ or } b_{jk} = 0 \text{ for all } i, j, k \in \{1, \dots, n\} \text{ with } i \neq k. \tag{9'}$$

Again, π is injective: if we had $\pi(i) = \pi(k) =: j$ for some $i \neq k$, then (8) would imply that $a_{ij} \neq 0$ and $b_{jk} \neq 0$, contradicting (9'). Just as in the previous case, we can prove that $a_{ij} = 0$ whenever $j \neq \pi(i)$. Indeed, for $k = \pi^{-1}(j)$ we have $b_{jk} \neq 0$ by (8), and then (9') implies $a_{ij} = 0$. To show that $A = P_\pi$, it only remains to prove that $a_{i\pi(i)} = 1$ for every i. This follows from the following inequality:

$$1 = \sum_{j=1}^n a_{ij} b_{ji} = a_{i\pi(i)} b_{\pi(i)i} \leq a_{i\pi(i)}.$$

□

Remark 5.8. As a consequence of Theorem 5.7, we have that $AB = I$ implies $BA = I$ for all matrices $A, B \in \mathbf{M}_n(L)$ if L satisfies the irreducibility condition of the theorem. For monoids (and also for rings), the property $AB = I \implies BA = I$ is called *Dedekind-finiteness*.

Example 5.9. Theorem 5.7 is not necessarily valid if neither 0 nor 1 is irreducible. As an example, let $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ over the lattice $\mathbf{2} \times \mathbf{2}$ shown in Figure 4. This lattice is distributive, hence $\mathbf{M}_2(L)$ is a semigroup and inverses are unique. It is easy to verify that A has an inverse (in fact, we have $A^{-1} = A$), even though A is not a permutation matrix.

Remark 5.10. For chains, Theorem 5.7 is a special case of Theorem 4.3. Indeed, if $A = I$, then each α_k is the equality relation on X, hence the group of automorphisms is S_n.

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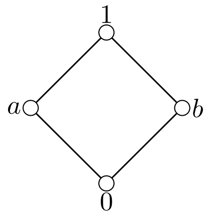

FIGURE 4. The lattice $\mathbf{2} \times \mathbf{2}$.

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