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REFLECTION-CLOSED VARIETIES OF MULTISORTED ALGEBRAS AND MINOR IDENTITIES

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Abstract. The notion of reflection is considered in the setting of multisorted algebras. The Galois connection induced by the satisfaction relation between multisorted algebras and minor identities provides a characterization of reflection-closed varieties: a variety of multisorted algebras is reflection-closed if and only if it is definable by minor identities. Minor-equational theories of multisorted algebras are described by explicit closure conditions. It is also observed that nontrivial varieties of multisorted algebras of a non-composable type are reflection-closed.

1. Introduction

Motivated by considerations of the complexity of constraint satisfaction problems, Barto, Opršal and Pinsker [2] introduced an algebraic construction called reflection. Given an algebra $\mathbf{A} = (A, F^{\mathbf{A}})$ of type τ, a set B, and maps $h_1 \colon B \to A$ and $h_2 \colon A \to B$, we can define an algebra $\mathbf{B} = (B, F^{\mathbf{B}})$ of type τ in which the operations are given by the rule

$$f^{\mathbf{B}}(x_1, \dots, x_n) := h_2(f^{\mathbf{A}}(h_1(x_1), \dots, h_1(x_n))). \tag{1.1}$$

The algebra $\mathbf{B}$ is called a *reflection* of $\mathbf{A}$. Reflections are a common generalization of subalgebras and homomorphic images. It was shown in [2, Corollary 5.4] that the classes of algebras closed under reflections and products are precisely the classes defined by height-1 identities.

Multisorted (or heterogeneous) algebras generalize the notion of an algebra so as to include functions that take arguments and values from possibly different sets. Much of the general theory of usual one-sorted (also called homogeneous) algebras applies to multisorted algebras, and the basics of the theory of multisorted algebras were established as early as in the 1960's and 1970's. In particular, subalgebras, morphisms, congruences, direct products, and free algebras were defined in the setting of multisorted algebras in the papers by Higgins [7] and Birkhoff and Lipson [3]. Furthermore, Higgins [7] defined varieties of multisorted algebras and proved Birkhoff's HSP theorem for multisorted algebras. Further considerations on varieties are included, e.g., in the paper by Taylor [9].

The defining equality (1.1) of reflections allows an immediate generalization from algebras to multisorted algebras in which the carrier comprises two sets A and B and the operations are functions $f \colon A^n \to B$ of several arguments from A to B ("2-algebras"; see Example 2.13(5)). With a little modification of the definition, the notion of reflection can be further generalized to arbitrary multisorted algebras (see Section 4).

In this paper, we consider reflections of multisorted algebras and ask for a characterization of reflection-closed varieties. As it turns out, the right notion for such

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a characterization are the so-called minor identities (also known as height-1 identities or primitive identities), i.e., identities of a special form, where all terms have exactly one occurrence of a function symbol. We thus set out to investigate the Galois connection Mod–mId induced by the relation of satisfaction between multisorted algebras and minor identities. Analogously to the first Birkhoff theorem, the Galois closures of this Galois connection are precisely the reflection-closed varieties of multisorted algebras, i.e., $\operatorname{Mod}\operatorname{mId}\mathcal{K} = \mathsf{RP}\mathcal{K}$. (For usual one-sorted algebras, this was proved by Barto, Opršal and Pinsker [2].) We also characterize by explicit closure conditions the minor-equational theories of multisorted algebras, i.e., the closed sets of minor identities of the Galois connection Mod–mId. (For usual one-sorted algebras, this was done by Čupona and Markovski [5].)

We also discuss how reflection-closed varieties and usual varieties of multisorted algebras are related to each other. These notions can be quite different in general, but for varieties of multisorted algebras of a so-called non-composable type, the only varieties that are not reflection-closed are in a certain sense trivial.

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2. Multisorted algebras

We will start with recalling the definitions of basic concepts in the theory of multisorted sets and multisorted algebras. We will mainly follow the notation and terminology used in the book by Wechler [10].

Definition 2.1. We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_+$ the set of positive integers. For $n \in \mathbb{N}$, let $[n] := \{1, \dots, n\}$. Note that $[0] = \emptyset$.

Definition 2.2. We write tuples $(a_1, a_2, \dots, a_n)$ interchangeably as words $a_1 a_2 \dots a_n$. The set of all words over a set S is denoted by $W(S)$. The empty word is denoted by ε. The *length* of a word $w \in W(S)$ is the number of letters in w and it is denoted by $|w|$. Thus, $|w_1 w_2 \dots w_n| = n$. For $s \in S$, the number of occurrences of s in w is denoted by $|w|_s$.

Definition 2.3. Let S be a set of elements called *sorts.* An S-indexed family of sets is called an *S-sorted set.* Given S-sorted sets $A = (A_s)_{s \in S}$ and $B = (B_s)_{s \in S}$, we say that A is an *(S-sorted) subset* of B and we write $A \subseteq B$ if $A_s \subseteq B_s$ for all $s \in S$. The *union* and *intersection* of S-sorted sets A and B are defined componentwise: $A \cup B := (A_s \cup B_s)_{s \in S}$ and $A \cap B := (A_s \cap B_s)_{s \in S}$. For any subset $S' \subseteq S$, we denote by $A|_{S'}$ the S-sorted subset of A given by

$$(A|_{S'})_s := \begin{cases} A_s, & \text{if } s \in S', \\ \emptyset, & \text{if } s \notin S'. \end{cases}$$

When we make statements such as "let A be an S-sorted set", it is understood that the member of the family A indexed by $s \in S$ is denoted by A_s.

Definition 2.4. Let A be an S-sorted set. If $A_s \neq \emptyset$, then we say that sort s is *essential* in A; otherwise sort s is *inessential* in A. Let $S_A := \{s \in S \mid A_s \neq \emptyset\}$ be the set of essential sorts in A. It follows immediately from the definitions that $A|_{S_A} = A$ and $S_{A|_{S'}} \subseteq S'$ for any $S' \subseteq S$.

Definition 2.5. Let A and B be S-sorted sets. An *S-sorted mapping* f from A to B, denoted by $f\colon A \to B$, is a family $(f_s)_{s \in S}$ of maps $f_s\colon A_s \to B_s$. If $x \in A_s$ and there is no risk of confusion about the sort s, we may write $f(x)$ instead of $f_s(x)$.

Definition 2.6. For an S-sorted set $A = (A_s)_{s \in S}$ and $w = w_1 w_2 \dots w_n \in W(S)$, let $A_w := A_{w_1} \times A_{w_2} \times \dots \times A_{w_n}$. Note that $A_\varepsilon = \{\emptyset\}$.

Definition 2.7. A pair $(w, s) \in W(S) \times S$ is called a *declaration* over S. Let A be an S-sorted set. A declaration (w, s) with $w = w_1 \dots w_n$ is *reasonable* in A if $A_s = \emptyset$ implies $A_{w_i} = \emptyset$ for some i, or, equivalently, if $A_w \neq \emptyset$ implies $A_s \neq \emptyset$. Note that the declaration (ε, s) is reasonable in A if and only if $A_s \neq \emptyset$.

An *S-sorted operation* on A is any function $f\colon A_w \to A_s$ for some declaration (w, s) that is reasonable in A. Note that it is possible that $A_w = \emptyset$, in which case f is just the empty function $\emptyset \to A_s$. The word w is called the *arity* of f and the element s is called the (*output*) *sort* of f. The elements of S occurring in the word w are called the *input sorts* of f. We denote the declaration, arity, sort, and the set of input sorts of f by $\mathrm{dec}(f)$, $\mathrm{ar}(f)$, $\mathrm{sort}(f)$, and $\mathrm{inp}(f)$, respectively. If $|w| = n$, then we also say that f has *numerical arity* n, or that f is *n-ary.*

Definition 2.8. A (*multisorted similarity*) *type* is a triple $\tau = (S, \Sigma, \mathrm{dec})$, where S is a set of sorts, Σ is a set of *function symbols,* and dec is a mapping $\mathrm{dec}\colon \Sigma \to W(S)\times S$. If $f \in \Sigma$ and $\mathrm{dec}(f) = (w, s)$, we say that f has *arity* w and *sort* s. Using the same notation as for functions, we denote the arity, sort and the set of input sorts of a function symbol f by $\mathrm{ar}(f)$, $\mathrm{sort}(f)$, and $\mathrm{inp}(f)$, respectively. For $w \in W(S)$, $s \in S$, we write $\Sigma_{(w,s)} := \{f \in \Sigma \mid \mathrm{dec}(f) = (w, s)\}$, $\Sigma_s := \{f \in \Sigma \mid \mathrm{sort}(f) = s\}$.

A (*multisorted*) *algebra of type* τ is a system $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$, where $A = (A_s)_{s\in S}$ is an S-sorted set, called the *carrier* (or *universe*) of $\mathbf{A}$, and $\Sigma^{\mathbf{A}} = (f^{\mathbf{A}})_{f\in\Sigma}$ is a family of S-sorted operations on A, each $f^{\mathbf{A}}$ of declaration $\mathrm{dec}(f)$. It is implicit in the definition that $\mathrm{dec}(f)$ is reasonable in A for every $f \in \Sigma$. Denote by $\mathrm{Alg}(\tau)$ the class of all multisorted algebras of type τ.

Remark 2.9. One can find in the literature different definitions of multisorted algebras that differ in whether or not the sets A_s in the carrier $(A_s)_{s\in S}$ of an algebra may be empty. Following the approach taken by Higgins [7], we allow carriers with empty components.

Definition 2.10. Let $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ and $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ be multisorted algebras of type $\tau = (S, \Sigma, \mathrm{dec})$. We say that $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and for every $f \in \Sigma_{(w,s)}$, the operation $f^{\mathbf{B}}$ equals the restriction of $f^{\mathbf{A}}$ to B_w. Running the risk of being a bit sloppy, we may designate subalgebras of $\mathbf{A}$ simply by their carrier sets and make statements such as "B is a subalgebra of $\mathbf{A}$" when we mean that B is the carrier of a subalgebra of $\mathbf{A}$.

It is possible that some components B_s of the carrier of $\mathbf{B}$ are empty. However, if $\Sigma^{\mathbf{A}}$ contains a nullary operation which selects an element $a \in A_s$, then we require that $a \in B_s$.

If C is an S-sorted subset of A, then the subalgebra of $\mathbf{A}$ *generated* by C, denoted by $\langle C\rangle_{\mathbf{A}}$, is the smallest subalgebra $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ of $\mathbf{A}$ such that $C \subseteq B$.

Definition 2.11. Let $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ and $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ be algebras of type $\tau = (S, \Sigma, \mathrm{dec})$. A *homomorphism* of $\mathbf{A}$ to $\mathbf{B}$ is an S-sorted mapping $\varphi\colon A \to B$ such that for every $f \in \Sigma_{(w,s)}$ with $w = w_1 \dots w_n$, it holds that

$$f^{\mathbf{B}}(\varphi_{w_1}(a_1), \dots, \varphi_{w_n}(a_n)) = \varphi_s(f^{\mathbf{A}}(a_1, \dots, a_n)),$$

for all $(a_1, \dots, a_n) \in A_w$. If every φ_s is a surjective map onto B_s, then $\mathbf{B}$ is referred to as a *homomorphic image* of $\mathbf{A}$.

Definition 2.12. Let Γ be an index set of a family of multisorted algebras $\mathbf{A}_\gamma = ((A_{\gamma,s})_{s\in S}, (f^{\mathbf{A}_\gamma})_{f\in\Sigma})$ of type $\tau = (S, \Sigma, \mathrm{dec})$, $\gamma \in \Gamma$. The *direct product* $\prod_{\gamma\in\Gamma} \mathbf{A}_\gamma$ is the algebra $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ of type τ, where $B_s = \prod_{\gamma\in\Gamma} A_{\gamma,s}$ and

$$f^{\mathbf{B}}((a_{\gamma,1})_{\gamma\in\Gamma}, \dots, (a_{\gamma,n})_{\gamma\in\Gamma}) = (f^{\mathbf{A}_\gamma}(a_{\gamma,1}, \dots, a_{\gamma,n}))_{\gamma\in\Gamma}.$$

If $\mathbf{A}_\gamma = \mathbf{A}$ for all $\gamma \in \Gamma$, then we speak of the Γ-th *direct power* of $\mathbf{A}$, and we write $\mathbf{A}^\Gamma$ for $\prod_{\gamma\in\Gamma} \mathbf{A}$.

Note that the *empty product* $\prod_{\gamma\in\emptyset}\mathbf{A}_\gamma$ is the algebra $(B_s, \Sigma^{\mathbf{B}})$ where $B_s = \{\emptyset\}$ for all $s \in S$. We will denote the empty product by $\prod\emptyset$.

Example 2.13. Examples of multisorted algebras include the following.

(1) If the set S of sorts is a singleton, then S-sorted sets, mappings, operations, algebras, etc., correspond to the usual ones. Every algebra in the usual sense can be viewed as a multisorted algebra of type $\tau = (S, \Sigma, \mathrm{dec})$ where $|S| = 1$. Such algebras are called *one-sorted* (or *homogeneous*).

(2) Given a multisorted similarity type $\tau = (S, \Sigma, \mathrm{dec})$, we can construct the *canonical trivial algebra* $\mathbf{S} = (\tilde{S}, \Sigma^{\mathbf{S}})$ *of type* τ, in which the carrier $\tilde{S} = (\tilde{S}_s)_{s\in S}$ consists of one-element sets only, $\tilde{S}_s := \{s\}$ for every $s \in S$, and for any $f \in \Sigma_{(w,s)}$, the operation $f^{\mathbf{S}}$ is trivially defined as the constant map $w \mapsto s$.

(3) Let $\tau = (S, \Sigma, \mathrm{dec})$ be a multisorted similarity type, and let $Y = (Y_s)_{s\in S}$ be an S-sorted set in which the components are pairwise disjoint and also disjoint from the function symbols Σ. The elements of Y are referred to as *variables.* We often encounter the *S-sorted standard set of variables,* namely, $X = (X_s)_{s\in S}$ where $X_s = \{x_i^s \mid i \in \mathbb{N}\}$.

The S-sorted set $T_\tau(Y) = (T_\tau^s(Y))_{s\in S}$ of *terms of type* τ *over* Y is defined as follows. Each set $T_\tau^s(Y)$ of terms of type τ over Y of sort s is the least set of words over $\Sigma \cup Y$ such that $Y_s \subseteq T_\tau^s(Y)$ and for all function symbols $f \in \Sigma_{(w,s)}$ and for all $(t_1, \dots, t_n) \in (T_\tau(Y))_w$, the word $ft_1 \dots t_n$ belongs to $T_\tau^s(Y)$. For better readability, we may add some punctuation marks and write $f(t_1, \dots, t_n)$ instead of $ft_1 \dots t_n$.

The terms of type τ over Y carry a multisorted algebra $\mathbf{T}_\tau(Y) = (T_\tau(Y), \Sigma^{\mathbf{T}_\tau(Y)})$ of type τ in which the operations are defined as follows. For each $f \in \Sigma_{(w,s)}$, let $f^{\mathbf{T}_\tau(Y)}(t_1, \dots, t_n) := ft_1 \dots t_n$ for all $(t_1, \dots, t_n) \in (T_\tau(Y))_w$. The algebra $\mathbf{T}_\tau(Y)$ is called the *term algebra of type* τ *over* Y.

(4) An *operation* on a set A (an ordinary set, not an S-sorted set) is a map $f\colon A^n \to A$ for some $n \in \mathbb{N}_+$, called the *arity* of f. The i-th n-ary *projection* on A is the operation $\mathrm{pr}_i^n\colon A^n \to A$, $(a_1, \dots, a_n) \mapsto a_i$. The *composition* of $f\colon A^n \to A$ with $g_1, \dots, g_n\colon A^m \to A$ is the operation $f(g_1, \dots, g_n)\colon A^m \to A$ given by the rule

$$f(g_1, \dots, g_n)(\mathbf{a}) := f(g_1(\mathbf{a}), \dots, g_n(\mathbf{a})),$$

for all $\mathbf{a} \in A^m$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and $\mathcal{O}_A := \bigcup \mathcal{O}_A^{(n)}$. A *clone* on a set A is a set $\mathcal{C} \subseteq \mathcal{O}_A$ of operations on A that contains all projections and is closed under composition.

Clones on A are sometimes viewed as multisorted algebras. Namely, let

$$\begin{aligned}
S &:= \mathbb{N}_+, \\
\Sigma &:= \{C_{n,m} \mid n, m \in \mathbb{N}_+\} \cup \{e_{n,i} \mid n, i \in \mathbb{N}_+, 1 \leq i \leq n\}, \\
\mathrm{dec}(C_{n,m}) &:= (n\underbrace{m \dots m}_{n}, m), \\
\mathrm{dec}(e_{n,i}) &:= (\varepsilon, n),
\end{aligned}$$

and define the algebra $\mathbf{F} = (F, \Sigma^{\mathbf{F}})$ of type $\tau = (S, \Sigma, \mathrm{dec})$, where $F = (F_n)_{n\in\mathbb{N}_+}$ with $F_n := \mathcal{O}_A^{(n)}$, and

$$\begin{aligned}
C_{n,m}^{\mathbf{F}}(f, g_1, \dots, g_n) &:= f(g_1, \dots, g_n), \\
e_{n,i}^{\mathbf{F}} &:= \mathrm{pr}_i^n\,.
\end{aligned}$$

The subalgebras of $\mathbf{F}$ are then in one-to-one correspondence with the clones on A in an obvious way.

(5) A 2-*algebra* is a multisorted algebra of type $\tau = (S, \Sigma, \mathrm{dec})$, where $S = \{1, 2\}$ and for every $f \in \Sigma$, $\mathrm{dec}(f) = (\underbrace{1 \dots 1}_{n}, 2)$ for some $n \in \mathbb{N}$. In other words, the carrier of a 2-algebra comprises two sets A and B, and the operations are functions $f\colon A^n \to B$ of several arguments from A to B.

Let us make a simple but very useful observation about the subalgebras of the canonical trivial algebra $\mathbf{S} = (\tilde{S}, \Sigma^{\mathbf{S}})$ of type $\tau = (S, \Sigma, \mathrm{dec})$ (see Example 2.13(2)). We first introduce the shorthand $\tilde{S}' := \tilde{S}|_{S'}$, for any subset $S' \subseteq S$ (for notation, see Definition 2.3), i.e., $\tilde{S}'$ is the S-sorted set with $\tilde{S}'_s = \{s\}$ if $s \in S'$ and $\tilde{S}'_s = \emptyset$ if $s \notin S'$. Obviously, for subsets S' and S'' of S, the set inclusion $S' \subseteq S''$ holds if and only if $\tilde{S}' \subseteq \tilde{S}''$ holds. In the sequel, we will often slightly abuse the notation and write $\langle S' \rangle_{\mathbf{S}}$ to mean the unique set $S'' \subseteq S$ such that $\langle \tilde{S}' \rangle_{\mathbf{S}} = \tilde{S}''$. We will keep the formally correct notation in the following lemma and its proof.

Lemma 2.14. *For a multisorted algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ of type $\tau = (S, \Sigma, \mathrm{dec})$, $\widetilde{S_A}$ is a subalgebra of the canonical trivial algebra $\mathbf{S} = (\tilde{S}, \Sigma^{\mathbf{S}})$ of type τ, i.e., $\langle \widetilde{S_A} \rangle_{\mathbf{S}} = \widetilde{S_A}$.*

Proof. Since $\mathbf{A}$ is an algebra, the declaration of every $f \in \Sigma$ is reasonable in A. Clearly $S_A = S_{\widetilde{S_A}}$, so the declaration of every $f \in \Sigma$ is reasonable in $\widetilde{S_A}$, too. Moreover, for each $f \in \Sigma_{(w,s)}$, the uniquely determined operation $(\widetilde{S_A})_w \to (\widetilde{S_A})_s$ coincides with the restriction of $f^{\mathbf{S}}$ to $\widetilde{S_A}$. Therefore $\widetilde{S_A}$ is a subalgebra of $\mathbf{S}$. $\square$

3. Minor terms and minor identities

As we have seen in Example 2.13(3), terms can be defined in the multisorted setting in the expected way: the output sorts of the terms $t_1, \dots, t_n$ must match with the input sorts of the function symbol f when a complex term $f(t_1, \dots, t_n)$ is formed. However, the notion of identity (or equation) requires a bit of care. It is not sufficient to define an identity simply as a pair of terms. One also has to specify the variables that are to be valuated when one decides whether an identity is satisfied by an algebra. This sounds superfluous, and it is indeed so in the case of one-sorted algebras, but for multisorted algebras this makes a difference. Namely, an identity would be trivially satisfied by an algebra $\mathbf{A} = (A, F)$ if there is a variable of sort s to which a value must be assigned but the set A_s is empty. If there is no such variable, then the identity may or may not be satisfied by $\mathbf{A}$, depending on whether the two terms of the identity get the same value in all assignments of values to variables. For further discussion and examples on this, see Wechler [10, Section 4.1.1].

As explained above, for a reasonable definition of an identity in the multisorted setting, it is necessary to specify the variables to which values are assigned. What really matters here are the sorts of such variables. For this reason, we have chosen to indicate only the sorts of the variables that are valuated, not the variables themselves. Consequently, our definition of an identity is slightly, but not in any essential way different from what is commonly seen in the literature (e.g., Adámek, Rosický, Vitale [1], Manca, Salibra [8, Definition 1.9], Wechler [10, Section 4.1.1]).

Definition 3.1. Let $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ be an algebra of type $\tau = (S, \Sigma, \mathrm{dec})$ and let Y be an S-sorted set of variables. A *valuation* of Y in A is an S-sorted mapping $\beta\colon Y \to A$. (Note that valuations $\beta\colon Y \to A$ exist if and only if $S_Y \subseteq S_A$.) The map β admits a unique homomorphic extension $\beta^{\#}\colon \mathbf{T}_\tau(Y) \to \mathbf{A}$ (see Example 2.13(3)). For a term $t \in T_\tau(Y)$, we call $\beta^{\#}(t)$ the *value* of t in $\mathbf{A}$ under β.

Definition 3.2. Let $\tau = (S, \Sigma, \mathrm{dec})$ be a multisorted similarity type, and let Y be an S-sorted set of variables. An *identity* of sort s of type τ over variables Y is a triple (S', t_1, t_2), where $S' \subseteq S$ and $t_1, t_2 \in T^s_\tau(Y|_{S'})$. We will use a more suggestive notation for identities and write $t_1 \approx_{S'} t_2$ for (S', t_1, t_2). We say that $t_1 \approx_{S'} t_2$ is *valuated* on sorts S'. We denote the set of all identities of sort s of type τ over Y by $\mathit{ID}^s_\tau(Y)$, and we set $\mathit{ID}_\tau(Y) := \bigcup_{s \in S} \mathit{ID}^s_\tau(Y)$.

An algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ of type τ is said to *satisfy* the identity $t_1 \approx_{S'} t_2$ if $\beta^\#(t_1) = \beta^\#(t_2)$ for all valuation maps $\beta\colon Y|_{S'} \to A$. In this case we also write $\mathbf{A} \models t_1 \approx_{S'} t_2$. Note that $\mathbf{A}$ satisfies $t_1 \approx_{S'} t_2$ vacuously if $Y_s \neq \emptyset$ and $A_s = \emptyset$ for some $s \in S'$.

Remark 3.3. In the literature (e.g., [1, 8, 10]), identities are often written as $\forall Y'(t_1 = t_2)$, where Y' is a subset of the S-sorted set Y of variables and t_1 and t_2 are terms. Using this notation, an identity $t_1 \approx_{S'} t_2$ (according to our Definition 3.2, where a set S' is given instead of a set of variables) would be written as $\forall Y|_{S'}(t_1 = t_2)$.

Lemma 3.4. *Let $\mathbf{A} \in \mathrm{Alg}(\tau)$, and let $t_1 \approx_{S'} t_2 \in \mathit{ID}^s_\tau(Y)$. If $\mathbf{A} \models t_1 \approx_{S'} t_2$, then $\mathbf{A} \models t_1 \approx_{S''} t_2$ for all S'' with $S' \subseteq S'' \subseteq S$.*

Proof. For every valuation $\beta\colon Y|_{S''} \to A$, we have

$$\beta^\#(t_1) = (\beta|_{S'})^\#(t_1) = (\beta|_{S'})^\#(t_2) = \beta^\#(t_2).$$

Thus, if there exists a valuation $\beta\colon Y|_{S''} \to A$, then $\mathbf{A} \models t_1 \approx_{S'} t_2$ implies that $\mathbf{A} \models t_1 \approx_{S''} t_2$. If there is no such valuation, then $\mathbf{A} \models t_1 \approx_{S''} t_2$ vacuously. □

Remark 3.5. A valuation $\beta\colon Y|_{S''} \to A$ exists if and only if $Y_s \neq \emptyset \implies A_s \neq \emptyset$ for every $s \in S''$. If this is the case, then the converse of Lemma 3.4 is also true (i.e., $\mathbf{A} \models t_1 \approx_{S'} t_2$ if and only if $\mathbf{A} \models t_1 \approx_{S''} t_2$).

Definition 3.6. Terms containing exactly one function symbol are called *minor terms.* We denote by $\mathit{MT}^s_\tau(Y)$ the set of all minor terms of sort s of type $\tau = (S, \Sigma, \mathrm{dec})$ over Y, and we set $\mathit{MT}_\tau(Y) := \bigcup_{s \in S} \mathit{MT}^s_\tau(Y)$.

In other words, a general minor term $t \in \mathit{MT}^s_\tau(Y)$ is of the form $f(\sigma(1), \dots, \sigma(n))$, where $f \in \Sigma$ with $\mathrm{dec}(f) = (w, s)$, $w = w_1 \dots w_n$, and $\sigma\colon [n] \to Y$ is a map respecting the sorts, i.e., satisfying $\sigma(i) \in Y_{w_i}$ for all $i \in [n]$. We denote this term by f_σ. The value of f_σ in $\mathbf{A}$ under a valuation $\beta\colon Y \to A$ is $\beta^\#(f_\sigma) = f^{\mathbf{A}}(\beta(\sigma(1)), \dots, \beta(\sigma(n))) = f^{\mathbf{A}}(\beta \circ \sigma)$.

Note that constants $f \in \Sigma$ are also minor terms, corresponding to the case $n = 0$. Recall that $[0] = \emptyset$, so $f_\sigma = f$ for every $\sigma\colon [0] \to Y$ and for any valuation $\beta\colon Y \to A$ we have $\beta^\#(f) = f^{\mathbf{A}}$.

An identity $t_1 \approx_{S'} t_2$ is called a *minor identity* if both t_1 and t_2 are minor terms. Minor identities are also known as *height*-1 *identities* (see Barto, Opršal, Pinsker [2]) or *primitive identities* (see Čupona, Markovski [4, 5] and Čupona, Markovski, Popeska [6]). We denote by $\mathit{MID}^s_\tau(Y)$ the set of all minor identities of sort of type τ over Y, and we set $\mathit{MID}_\tau(Y) := \bigcup_{s \in S} \mathit{MID}^s_\tau(Y)$.

Definition 3.7. The satisfaction relation induces a Galois connection between multisorted algebras and identities of type τ. For a class $\mathcal{K} \subseteq \mathrm{Alg}(\tau)$ of algebras of type τ and for a set $\mathcal{J} \subseteq \mathit{ID}_\tau(Y)$ of identities of type τ, let

$$\begin{aligned}
\mathrm{Id}_Y \mathcal{K} &:= \{t_1 \approx_{S'} t_2 \in \mathit{ID}_\tau(Y) \mid \forall \mathbf{A} \in \mathcal{K}\colon \mathbf{A} \models t_1 \approx_{S'} t_2\},\\
\mathrm{mId}_Y \mathcal{K} &:= \{t_1 \approx_{S'} t_2 \in \mathit{MID}_\tau(Y) \mid \forall \mathbf{A} \in \mathcal{K}\colon \mathbf{A} \models t_1 \approx_{S'} t_2\},\\
\mathrm{Mod}\, \mathcal{J} &:= \{\mathbf{A} \in \mathrm{Alg}(\tau) \mid \forall t_1 \approx_{S'} t_2 \in \mathcal{J}\colon \mathbf{A} \models t_1 \approx_{S'} t_2\}.
\end{aligned}$$

When Y is the standard set of variables X, then we write $\mathrm{Id}\, \mathcal{K}$ and $\mathrm{mId}\, \mathcal{K}$ for $\mathrm{Id}_X \mathcal{K}$ and $\mathrm{mId}_X \mathcal{K}$, respectively.

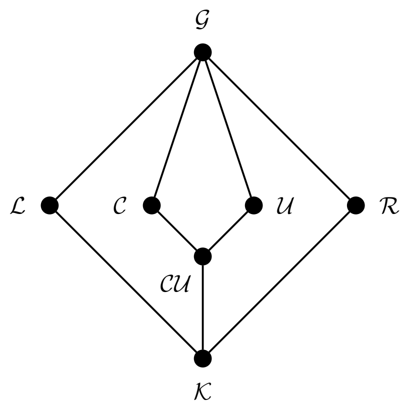


FIGURE 1. Minor varieties of groupoids.

By Birkhoff's theorem for (multisorted) algebras, $\operatorname{Mod} \mathcal{J}$ is a variety for any set $\mathcal{J} \subseteq ID_\tau(Y)$ of identities. We call a variety $\mathcal{V}$ a *minor variety* if $\mathcal{V} = \operatorname{Mod} \mathcal{J}$ for some set $\mathcal{J} \subseteq MID_\tau(Y)$ of minor identities. Minor varieties of one-sorted algebras were investigated by Čupona and Markovski [4, 5] and Čupona, Markovski and Popeska [6].

Example 3.8. As a concrete example of minor varieties, we present here the minor varieties of groupoids (one-sorted algebras with a single binary operation), which were determined by Čupona, Markovski and Popeska [6]. It is easy to verify that every minor identity in the language of groupoids is equivalent to one of the following:

$$xy \approx xy, \quad xy \approx yx, \quad xx \approx yy, \quad xy \approx xz, \quad xy \approx zy, \quad xy \approx zu, \tag{3.1}$$

where we have written identities as is usual in the classical framework and the binary operation as juxtaposition. Therefore, there are six varieties defined by a single minor identity:

$$\begin{aligned}
\mathcal{G} &:= \operatorname{Mod}\{xy \approx xy\} && \text{(all groupoids)}, \\
\mathcal{C} &:= \operatorname{Mod}\{xy \approx yx\} && \text{(commutative groupoids)}, \\
\mathcal{U} &:= \operatorname{Mod}\{xx \approx yy\} && \text{(unipotent groupoids)}, \\
\mathcal{L} &:= \operatorname{Mod}\{xy \approx xz\} && \text{(left unars)}, \\
\mathcal{R} &:= \operatorname{Mod}\{xy \approx zy\} && \text{(right unars)}, \\
\mathcal{K} &:= \operatorname{Mod}\{xy \approx zu\} && \text{(constant groupoids)}.
\end{aligned}$$

The only new variety that can be formed as the intersection of any of the above is $\mathcal{CU} := \operatorname{Mod}\{xy \approx yx, xx \approx yy\}$ (commutative unipotent groupoids). The lattice of minor varieties of groupoids is represented by the Hasse diagram shown in Figure 1.

Example 3.9. As another example, we determine the minor variety generated by the variety of groups. Recall that a group is a one-sorted algebra $(A; \cdot, ^{-1}, e)$ of type $(2, 1, 0)$ satisfying the identities

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \quad e \cdot x \approx x, \quad x \cdot e \approx x, \quad x \cdot x^{-1} \approx e, \quad x^{-1} \cdot x \approx e.$$

Every minor identity in the language of groups is equivalent to one of the groupoid identities listed in (3.1) or one of the following:

$$\begin{aligned}
&xy \approx z^{-1}, \quad xy \approx x^{-1}, \quad xy \approx y^{-1}, \quad xx \approx y^{-1}, \quad xx \approx x^{-1}, \quad xy \approx e, \\
&xx \approx e, \quad x^{-1} \approx y^{-1}, \quad x^{-1} \approx x^{-1}, \quad x^{-1} \approx e, \quad e \approx e.
\end{aligned} \tag{3.2}$$

As trivial identities, $x^{-1} \approx x^{-1}$ and $e \approx e$ are equivalent to $xy \approx xy$. It is an easy exercise to find, for each one of the nontrivial identities listed in (3.1) and (3.2), an example of a group that does not satisfy that identity. Hence the only minor identities satisfied by the variety of groups are the trivial ones, and we conclude that the minor variety generated by the variety of groups is the variety of all algebras of type $(2, 1, 0)$.

Example 3.10. Our last example involves multisorted algebras that are not one-sorted, and it aims at illustrating the role of empty components in carriers of algebras, as well as the importance of specifying the sorts on which identities are valuated. Consider the algebraic similarity type $\tau = (S, \Sigma, \mathrm{dec})$ with $S = \{s, t\}$, $\Sigma = \{\cdot, *\}$ and $\mathrm{dec}(\cdot) = (ss, s)$, $\mathrm{dec}(*) = (st, t)$. Algebras of type τ satisfying the identity $x * (y * u) \approx_S (x \cdot y) * u$ are called *groupoid actions.* Groupoid actions that additionally satisfy the identity $x \cdot (y \cdot z) \approx_{\{s\}} (x \cdot y) \cdot z$ are called *semigroup actions.* (Note that the defining identities of groupoid actions and semigroup actions are not minor identities.)

Let us determine the minor varieties of algebras of type τ. To this end, we introduce some notation. For a variety $\mathcal{V}$ of groupoids, as in Example 3.8, let us denote by $\mathcal{V}^*$ the set of all algebras $\mathbf{A}$ of type τ such that $(A_s, \cdot)$ is in $\mathcal{V}$. Let $\mathcal{T}$ be the class of all algebras $\mathbf{A}$ of type τ such that $A_t = \emptyset$.

Let $\mathcal{J}$ be a set of identities in the language of groupoids (usual one-sorted), and let $\mathcal{J}' \subseteq \mathcal{J}$. Define

$$\mathcal{I} := \{t_1 \approx_{\{s\}} t_2 \mid t_1 \approx t_2 \in \mathcal{J}'\} \cup \{t_1 \approx_{\{s,t\}} t_2 \mid t_1 \approx t_2 \in \mathcal{J} \setminus \mathcal{J}'\}.$$

An algebra $\mathbf{A}$ of type τ satisfies the set $\mathcal{I}$ of identities if and only if $(A_s, \cdot) \models t_1 \approx t_2$ for every $t_1 \approx t_2 \in \mathcal{J}'$ and $(A_s, \cdot) \models t_1 \approx t_2$ or $A_t = \emptyset$ for every $t_1 \approx t_2 \in \mathcal{J} \setminus \mathcal{J}'$. This condition is equivalent to the following: $(A_s, \cdot) \models t_1 \approx t_2$ for every $t_1 \approx t_2 \in \mathcal{J}$, or $A_t = \emptyset$ and $(A_s, \cdot) \models t_1 \approx t_2$ for every $t_1 \approx t_2 \in \mathcal{J}'$. In other words, $\mathbf{A} \models \mathcal{I}$ if and only if $\mathbf{A} \in \mathcal{V}^* \cup (\mathcal{V}'^* \cap \mathcal{T})$, where $\mathcal{V} := \operatorname{Mod} \mathcal{J}$ and $\mathcal{V}' := \operatorname{Mod} \mathcal{J}'$ are varieties of groupoids.

Consequently, the varieties of algebras of type τ defined by identities of sort s are of the form $\lfloor \mathcal{V}, \mathcal{V}' \rceil := \mathcal{V}^* \cup (\mathcal{V}'^* \cap \mathcal{T})$, where $\mathcal{V}$, $\mathcal{V}'$ are varieties of groupoids such that $\mathcal{V} \subseteq \mathcal{V}'$. We can deduce from Figure 1 that there are 20 varieties of type τ that are defined by minor identities of sort s, and they are shown in Figure 2.

It is easy to verify that every minor identity of type τ of sort t is equivalent to one of the following:

$$x * u \approx_S x * u, \qquad x * u \approx_S x * v, \qquad x * u \approx_S y * u, \qquad x * u \approx_S y * v.$$

Therefore, there are four varieties defined by a single minor identity of sort t:

$$\mathcal{M} := \operatorname{Mod}(x * u \approx_S x * u), \qquad \mathcal{N} := \operatorname{Mod}(x * u \approx_S x * v),$$
$$\mathcal{O} := \operatorname{Mod}(x * u \approx_S y * u), \qquad \mathcal{P} := \operatorname{Mod}(x * u \approx_S y * v).$$

Intersections of these four varieties do not yield any new varieties. These varieties are shown in Figure 3.

We conclude that the minor varieties of type τ are of the form $\mathcal{X} \cap \mathcal{Y}$, where $\mathcal{X}$ is a variety defined by minor identities of sort s (see Figure 2) and $\mathcal{Y}$ is a variety defined by minor identities of sort t (see Figure 3). Consequently, the total number of minor varieties of type τ is $20 \cdot 4 = 80$, and the lattice of minor varieties is isomorphic to the direct product of the lattices shown in Figures 2 and 3.

4. Reflections

We are now going to generalize the notion of reflection (see Barto, Opršal and Pinsker [2]) to the multisorted setting.

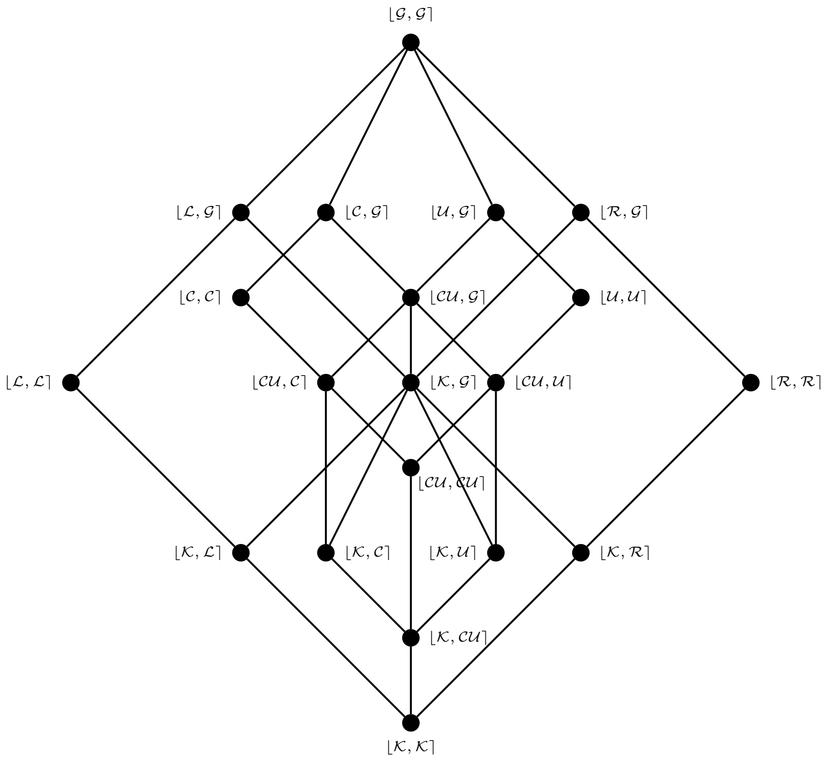


FIGURE 2. Minor varieties of the groupoid action type defined by identities of sort s.

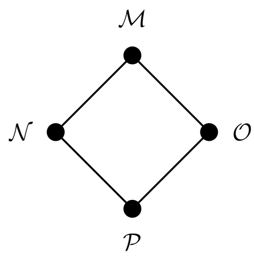


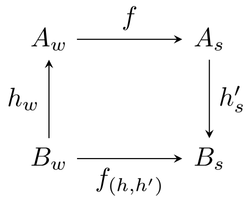
FIGURE 3. Minor varieties of the groupoid action type defined by identities of sort t.

Definition 4.1. Let A and B be S-sorted sets. A *reflection* of A into B is a pair (h, h') of S_B-sorted mappings $h = (h_s)_{s \in S_B}$, $h' = (h'_s)_{s \in S_B}$, $h_s \colon B_s \to A_s$, $h'_s \colon A_s \to B_s$. Note that reflections of A into B exist if and only if $S_B \subseteq S_A$. For, if $S_B \subseteq S_A$, then A_s and B_s are nonempty for all $s \in S_B$ and there clearly exist maps $h_s \colon B_s \to A_s$ and $h'_s \colon A_s \to B_s$. If $S_B \nsubseteq S_A$, then there is $s \in S_B \setminus S_A$, whence $A_s = \emptyset$ and $B_s \neq \emptyset$, so there is no map $h_s \colon B_s \to A_s$.

Assume that A and B are S-sorted sets with $S_B \subseteq S_A$ and (h, h') is a reflection of A into B. If $(w, s) \in W(S) \times S$ is a declaration that is reasonable in both A and B and $f \colon A_w \to A_s$, then we can define the (h, h')-*reflection* of f to be the map $f_{(h,h')} \colon B_w \to B_s$ given by the rule

$$f_{(h,h')}(b_1, \dots, b_n) = h'_s(f(h_{w_1}(b_1), \dots, h_{w_n}(b_n)))$$

for all $(b_1, \dots, b_n) \in B_w$, which we write simply as $f_{(h,h')}(\mathbf{b}) = h'_s(f(h_w(\mathbf{b})))$ for all $\mathbf{b} \in B_w$. This is illustrated by the commutative diagram shown below. Note that if $B_{w_i} = \emptyset$ for some $i \in \{1, \dots, n\}$, then $f_{(h,h')} = \emptyset$.



Let $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ and $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ be algebras of type $\tau = (S, \Sigma, \mathrm{dec})$. If (h, h') is a reflection of A into B and for all $f \in \Sigma$ it holds that $f^{\mathbf{B}} = (f^{\mathbf{A}})_{(h,h')}$, then $\mathbf{B}$ is called the (h, h')-*reflection* of $\mathbf{A}$. (Note that for every $f \in \Sigma$, the declaration $\mathrm{dec}(f)$ of $f^{\mathbf{A}}$ and $f^{\mathbf{B}}$ is reasonable in both A and B, because $\mathbf{A}$ and $\mathbf{B}$ are algebras.) We say that $\mathbf{B}$ is a *reflection* of $\mathbf{A}$ if $\mathbf{B}$ is an (h, h')-reflection of $\mathbf{A}$ for some reflection (h, h') of A into B.

For a class $\mathcal{K}$ of multisorted algebras of type τ, let $\mathsf{R}\mathcal{K}$, $\mathsf{H}\mathcal{K}$, $\mathsf{S}\mathcal{K}$ and $\mathsf{P}\mathcal{K}$ denote the classes of all reflections, homomorphic images, subalgebras and products of algebras of $\mathcal{K}$, respectively.

Lemma 4.2 (cf. [2, Lemma 4.4])**.** *Let $\mathcal{K}$ be a class of multisorted algebras of type τ. Then the following statements hold.*

(1) $\mathsf{H}\mathcal{K} \subseteq \mathsf{R}\mathcal{K}$ *and* $\mathsf{S}\mathcal{K} \subseteq \mathsf{R}\mathcal{K}$.
(2) $\mathsf{RR}\mathcal{K} \subseteq \mathsf{R}\mathcal{K}$.
(3) $\mathsf{PR}\mathcal{K} \subseteq \mathsf{RP}\mathcal{K}$.

Proof. (1) Assume that $\mathbf{B} = (B, \Sigma^{\mathbf{B}}) \in \mathsf{H}\mathcal{K}$. Then there exists an algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}}) \in \mathcal{K}$ such that $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$. Let φ be a homomorphism of $\mathbf{A}$ to $\mathbf{B}$, with each $\varphi_s \colon A_s \to B_s$ surjective. Then there exist mappings $h_s \colon B_s \to A_s$ such that $\varphi_s \circ h_s = \mathrm{id}_{B_s}$. Then for each $f \in \Sigma_{(w,s)}$ with $w = w_1 \dots w_n$ and for all $(b_1, \dots, b_n) \in B_w$, we have

$$\begin{aligned} f^{\mathbf{B}}(b_1, \dots, b_n) &= f^{\mathbf{B}}(\varphi_{w_1}(h_{w_1}(b_1)), \dots, \varphi_{w_n}(h_{w_n}(b_n))) \\ &= \varphi_s(f^{\mathbf{A}}(h_{w_1}(b_1), \dots, h_{w_n}(b_n))). \end{aligned}$$

We clearly have $S_B = S_A$ because the homomorphism φ is surjective. Setting $h = (h_s)_{s \in S_B}$ and $h' = (\varphi_s)_{s \in S_B}$, we conclude that $\mathbf{B}$ is an (h, h')-reflection of $\mathbf{A}$. Thus $\mathsf{H}\mathcal{K} \subseteq \mathsf{R}\mathcal{K}$.

Assume then that $\mathbf{B} = (B, \Sigma^{\mathbf{B}}) \in \mathsf{S}\mathcal{K}$. Then there exists an algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}}) \in \mathcal{K}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. Then clearly $S_B \subseteq S_A$. Let $h = (h_s)_{s \in S_B}$ and $h' = (h'_s)_{s \in S_B}$ where each $h_s \colon B_s \to A_s$ is the inclusion map of B_s into A_s and each $h'_s \colon A_s \to B_s$ is an arbitrary extension of the identity map on B_s. Then for each $f \in \Sigma_{(w,s)}$ with $w = w_1 \dots w_n$, and for all $(b_1, \dots, b_n) \in B_w$, we clearly have $f^{\mathbf{B}}(b_1, \dots, b_n) = h'_s(f^{\mathbf{A}}(h_{w_1}(b_1), \dots, h_{w_n}(b_n)))$, so $\mathbf{B}$ is an (h, h')-reflection of $\mathbf{A}$. Thus $\mathsf{S}\mathcal{K} \subseteq \mathsf{R}\mathcal{K}$.

(2) Assume that $\mathbf{C} \in \mathsf{RR}\mathcal{K}$. Then there exist algebras $\mathbf{B} \in \mathsf{R}\mathcal{K}$ and $\mathbf{A} \in \mathcal{K}$ such that $\mathbf{C}$ is a reflection of $\mathbf{B}$, witnessed by $(h, h') = ((h_s)_{s \in S_C}, (h'_s)_{s \in S_C})$ where $h_s \colon C_s \to B_s$ and $h'_s \colon B_s \to C_s$, and $\mathbf{B}$ is a reflection of $\mathbf{A}$, witnessed by $(k, k') = ((k_s)_{s \in S_B}, (k'_s)_{s \in S_B})$ where $k_s \colon B_s \to A_s$ and $k'_s \colon A_s \to B_s$. Then $S_C \subseteq S_B \subseteq S_A$, so we can define a reflection (ℓ, ℓ') of A into C using the S_C-sorted maps $\ell = (\ell_s)_{s \in S_C}$ where each $\ell_s \colon C_s \to A_s$ is given by $\ell_s := k_s \circ h_s$ and $\ell' = (\ell'_s)_{s \in S_C}$ where each $\ell'_s \colon A_s \to C_s$ is given by $\ell'_s := h'_s \circ k'_s$. Furthermore, for every $f \in \Sigma_{(w,s)}$ with

$w = w_1 \dots w_n$, we have

$$\begin{aligned} f^{\mathbf{C}}(c_1, \dots, c_n) &= h'_s(f^{\mathbf{B}}(h_{w_1}(c_1), \dots, h_{w_n}(c_n))) \\ &= h'_s(k'_s(f^{\mathbf{A}}(k_{w_1}(h_{w_1}(c_1)), \dots, k_{w_n}(h_{w_n}(c_n))))) \\ &= \ell'_s(f^{\mathbf{A}}(\ell_{w_1}(c_1), \dots, \ell_{w_n}(c_n))) \end{aligned}$$

for all $(c_1, \dots, c_n) \in C_w$. We conclude that $\mathbf{C}$ is a reflection of $\mathbf{A}$. Thus $\mathsf{RR}\mathcal{K} \subseteq \mathsf{R}\mathcal{K}$.

(3) Assume that $\mathbf{C} \in \mathsf{PR}\mathcal{K}$. Then $\mathbf{C} = \prod_{\gamma\in\Gamma} \mathbf{B}_\gamma$ for some algebras $\mathbf{B}_\gamma = (B_\gamma, \Sigma^{\mathbf{B}_\gamma})$, and each $\mathbf{B}_\gamma$ is a reflection of some $\mathbf{A}_\gamma = (A_\gamma, \Sigma^{\mathbf{A}_\gamma}) \in \mathcal{K}$, witnessed by $((h_{\gamma,s})_{s\in S_{B_\gamma}}, (h'_{\gamma,s})_{s\in S_{B_\gamma}})$, where $h_{\gamma,s} \colon B_{\gamma,s} \to A_{\gamma,s}$, $h'_{\gamma,s} \colon A_{\gamma,s} \to B_{\gamma,s}$.

Observe that if $C_s = \prod_{\gamma\in\Gamma} B_{\gamma,s} \neq \emptyset$, then $B_{\gamma,s} \neq \emptyset$ for every $\gamma \in \Gamma$. Therefore $S_C \subseteq S_{B_\gamma} \subseteq S_{A_\gamma}$ for every $\gamma \in \Gamma$. Define the S_C-sorted maps $h = (h_s)_{s\in S_C}$ and $h' = (h'_s)_{s\in S_C}$, where $h_s \colon \prod_{\gamma\in\Gamma} B_{\gamma,s} \to \prod_{\gamma\in\Gamma} A_{\gamma,s}$ and $h'_s \colon \prod_{\gamma\in\Gamma} A_{\gamma,s} \to \prod_{\gamma\in\Gamma} B_{\gamma,s}$ are defined componentwise in terms of the $h_{\gamma,s}$ and $h'_{\gamma,s}$ as $h_s((b_\gamma)_{\gamma\in\Gamma}) = (h_{\gamma,s}(b_\gamma))_{\gamma\in\Gamma}$ and $h'_s((a_\gamma)_{\gamma\in\Gamma}) = (h'_{\gamma,s}(a_\gamma))_{\gamma\in\Gamma}$. Then, for every operation $f \in \Sigma_{(w,s)}$ with $w = w_1 \dots w_n$ and for all tuples $((b_{1,\gamma})_{\gamma\in\Gamma}, \dots, (b_{n,\gamma})_{\gamma\in\Gamma}) \in (\prod_{\gamma\in\Gamma} B_\gamma)_w$, we have

$$\begin{aligned} & f^{\prod \mathbf{B}_\gamma}((b_{1,\gamma})_{\gamma\in\Gamma}, \dots, (b_{n,\gamma})_{\gamma\in\Gamma}) = (f^{\mathbf{B}_\gamma}(b_{1,\gamma}, \dots, b_{n,\gamma}))_{\gamma\in\Gamma} \\ &= (h'_{\gamma,s}(f^{\mathbf{A}_\gamma}(h_{\gamma,w_1}(b_{1,\gamma}), \dots, h_{\gamma,w_n}(b_{n,\gamma}))))_{\gamma\in\Gamma} \\ &= h'_s((f^{\mathbf{A}_\gamma}(h_{\gamma,w_1}(b_{1,\gamma}), \dots, h_{\gamma,w_n}(b_{n,\gamma})))_{\gamma\in\Gamma}) \\ &= h'_s(f^{\prod \mathbf{A}_\gamma}((h_{\gamma,w_1}(b_{1,\gamma}))_{\gamma\in\Gamma}, \dots, (h_{\gamma,w_n}(b_{n,\gamma}))_{\gamma\in\Gamma})) \\ &= h'_s(f^{\prod \mathbf{A}_\gamma}(h_{w_1}((b_{1,\gamma})_{\gamma\in\Gamma}), \dots, h_{w_n}((b_{n,\gamma})_{\gamma\in\Gamma}))). \end{aligned}$$

This shows that the algebra $\mathbf{C} = \prod_{\gamma\in\Gamma} \mathbf{B}_\gamma$ is the (h, h')-reflection of the product $\prod_{\gamma\in\Gamma} \mathbf{A}_\gamma$. Thus $\mathbf{C} \in \mathsf{RP}\mathcal{K}$, so $\mathsf{PR}\mathcal{K} \subseteq \mathsf{RP}\mathcal{K}$. □

Remark 4.3. Note that the converse of the inclusion of Lemma 4.2(3), namely $\mathsf{RP}\mathcal{K} \subseteq \mathsf{PR}\mathcal{K}$, does not hold in general. For example, take $\mathcal{K} := \emptyset$. Then $\mathsf{P}\emptyset = \{\prod \emptyset\}$. Since for any S-sorted set $A = (A_s)_{s\in S}$, an algebra with carrier A can be obtained as a reflection of $\prod \emptyset$ (the proof of this assertion is essentially included in the proof of Theorem 6.3, implication (1) $\Longrightarrow$ (3)), it follows that $\mathsf{RP}\mathcal{K}$ contains algebras with arbitrary carrier sets. On the other hand, $\mathsf{R}\emptyset = \emptyset$, whence $\mathsf{PR}\emptyset = \{\prod \emptyset\}$. Thus $\mathsf{RP}\emptyset \nsubseteq \mathsf{PR}\emptyset$.

In order to give also a nonempty counterexample, let $\mathbf{A} = (\{0, 1\}; f^{\mathbf{A}})$ and $\mathbf{B} = (\{a, b, c\}; f^{\mathbf{B}})$ where $f^{\mathbf{A}}$ and $f^{\mathbf{B}}$ are the identity functions on the corresponding sets. We define maps h and h' as follows:

$$\begin{aligned} & h \colon B \to A^2, \quad a \mapsto (0, 0),\ b \mapsto (0, 1),\ c \mapsto (1, 0); \\ & h' \colon A^2 \to B, \quad (0, 0) \mapsto a,\ (0, 1) \mapsto b,\ (1, 0) \mapsto c,\ (1, 1) \mapsto c. \end{aligned}$$

Then $\mathbf{B}$ is the (h, h')-reflection of $\mathbf{A}^2$, hence $\mathbf{B} \in \mathsf{RP}\{\mathbf{A}\}$. On the other hand, if $\mathbf{B}$ were in $\mathsf{PR}\{\mathbf{A}\}$, then $\mathbf{B}$ would be a reflection of $\mathbf{A}$, since, having a prime number of elements, it cannot be a proper product. However, $\mathbf{B}$ cannot be a reflection of $\mathbf{A}$, because the range of $f^{\mathbf{B}}$ is larger than the range of $f^{\mathbf{A}}$. We can conclude that $\mathbf{B} \notin \mathsf{PR}\{\mathbf{A}\}$, thus $\mathsf{RP}\{\mathbf{A}\} \nsubseteq \mathsf{PR}\{\mathbf{A}\}$.

The following proposition shows that it would, in principle, be sufficient to consider multisorted algebras with carriers in which the same set is associated to every essential sort (i.e., $A_s = A_t$ for all $s, t \in S_A$). Every S-sorted algebra is reflection-equivalent to such an algebra with a single carrier set. This comes, however, at the cost of the carrier sets becoming possibly much larger than in the given algebra.

Proposition 4.4. *Let $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ be an algebra of type $\tau = (S, \Sigma, \mathrm{dec})$. Then there exists an algebra $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ of type τ such that $S_A = S_B$, $B_i = B_j$ for all $i, j \in S_B$, and $\mathbf{A}$ and $\mathbf{B}$ are reflections of each other.*

Proof. Let C be a set of cardinality greater than or equal to the cardinality of each of the sets A_i, $i \in S$ (for example, we may choose $C := \bigcup_{i \in S} A_i$). For $i \in S$, let $B_i := C$ if $i \in S_A$ and let $B_i := \emptyset$ if $i \notin S_A$. Then clearly $S_A = S_B$. For each $i \in S_A$, let $h'_i \colon A_i \to C$ be an injection, and let $h_i \colon C \to A_i$ be a pseudoinverse of h'_i, i.e., a map such that $h_i(h'_i(a)) = a$ for all $a \in A_i$. Such maps h'_i and h_i exist because $|A_i| \leq |C|$. Let $h := (h_s)_{s \in S_B}$, $h' := (h'_s)_{s \in S_B}$. Let $\mathbf{B} := (B, \Sigma^{\mathbf{B}})$, with $f^{\mathbf{B}} := (f^{\mathbf{A}})_{(h,h')}$ for each $f \in \Sigma$ (for notation, see Definition 4.1). Then $\mathbf{B}$ is an (h, h')-reflection of $\mathbf{A}$ by definition. Furthermore, for each $f \in \Sigma$, say, of declaration $(w_1 \dots w_n, s)$, it holds that

$$\begin{aligned}(f^{\mathbf{B}})_{(h',h)}(a_1, \dots, a_n) &= h_s(h'_s(f^{\mathbf{A}}(h_{w_1}(h'_{w_1}(a_1)), \dots, h_{w_n}(h'_{w_n}(a_n))))) \\ &= f^{\mathbf{A}}(a_1, \dots, a_n),\end{aligned}$$

that is, $f^{\mathbf{A}} = (f^{\mathbf{B}})_{(h',h)}$ for every $f \in \Sigma$. In other words, $\mathbf{A}$ is an (h', h)-reflection of $\mathbf{B}$. □

5. The Galois connection mId–Mod

It is known from the classical Birkhoff theorem for (multisorted) algebras that HSP-closed classes are equational classes. By Lemma 4.2, RP-closed classes are also HSP-closed and therefore must be characterizable by identities. In this section we prove that the "right" kind of identities for this setting are the minor identities: $\mathrm{Mod}\,\mathrm{mId}\,\mathcal{K} = \mathsf{RP}\mathcal{K}$ for every class $\mathcal{K} \subseteq \mathrm{Alg}(\tau)$ of multisorted algebras. For the proof we need the following technical lemma, which essentially states that (under some reasonable assumptions) the validity of an identity does not change if we rename the variables and extend the set of variables.

For a term t, denote by $\mathrm{var}(t)$ the S-sorted set of variables occurring in t, i.e., $\mathrm{var}(t) = (v_s)_{s \in S}$ where v_s is the set of variables of sort s occurring in t.

Lemma 5.1. *Let $\tau = (S, \Sigma, \mathrm{dec})$ be a multisorted similarity type, and let Y be an S-sorted set of variables. Let $\mu := t_1 \approx_{S'} t_2 \in MID^s_\tau(Y)$, and assume that $S' \subseteq S_Y$. Let $Y' := \mathrm{var}(t_1) \cup \mathrm{var}(t_2)$. Let Z be an S-sorted set of variables such that $S' \subseteq S_Z$ and there exists an injective S-sorted map $\delta \colon Y' \to Z$. Let t'_1 and t'_2 be terms in $MT^s_\tau(Z)$ that are obtained from t_1 and t_2, respectively, by replacing each occurrence of a variable symbol $y \in Y'$ by $\delta(y)$, and let $\mu' := t'_1 \approx_{S'} t'_2 \in MID^s_\tau(Z)$. Then for every algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ of type τ, it holds that $\mathbf{A} \models \mu$ if and only if $\mathbf{A} \models \mu'$.*

Proof. Assume first that $\mathbf{A} \models t_1 \approx_{S'} t_2$. We want to show that $\mathbf{A} \models t'_1 \approx_{S'} t'_2$. If $S' \nsubseteq S_A$, then $\mathbf{A} \models t'_1 \approx_{S'} t'_2$ holds vacuously (note that $S' \subseteq S_Z$), so we may assume that $S' \subseteq S_A$. Let $\beta \colon Z|_{S'} \to A$ be a valuation map, and define $\gamma \colon Y|_{S'} \to A$ by the rule

$$\gamma_s(x) = \begin{cases} \beta_s(\delta_s(x)), & \text{if } x \in Y'_s \\ a_s, & \text{if } x \in Y_s \setminus Y'_s, \end{cases}$$

where a_s is an arbitrary fixed element of A_s. It is clear that $\beta^\#(t'_1) = \gamma^\#(t_1)$ and $\beta^\#(t'_2) = \gamma^\#(t_2)$. Since $\mathbf{A} \models t_1 \approx_{S'} t_2$, we have $\gamma^\#(t_1) = \gamma^\#(t_2)$. Consequently, $\beta^\#(t'_1) = \beta^\#(t'_2)$, and we conclude that $\mathbf{A} \models t'_1 \approx_{S'} t'_2$.

The proof of the converse implication is very similar. Assume that $\mathbf{A} \models t'_1 \approx_{S'} t'_2$. We want to show that $\mathbf{A} \models t_1 \approx_{S'} t_2$. We may assume that $S' \subseteq S_A$, for otherwise $\mathbf{A} \models t_1 \approx_{S'} t_2$ holds vacuously (note that $S' \subseteq S_Y$). Let $\gamma \colon Y|_{S'} \to A$ be a valuation map, and define $\beta\colon Z|_{S'} \to A$ by the rule

$$\beta_s(x) = \begin{cases} \gamma(y), & \text{if } x = \delta_s(y) \text{ for } y \in Y'_s, \\ a_s, & \text{otherwise,} \end{cases}$$

where a_s is an arbitrary fixed element of A_s. Again, it is clear that $\beta^\#(t'_1) = \gamma^\#(t_1)$ and $\beta^\#(t_2) = \gamma^\#(t_2)$. Since $\mathbf{A} \models t'_1 \approx_{S'} t'_2$, we have $\beta^\#(t'_1) = \beta^\#(t'_2)$. Consequently, $\gamma^\#(t_1) = \gamma^\#(t_2)$, and we conclude that $\mathbf{A} \models t_1 \approx_{S'} t_2$. □

Theorem 5.2. *Let* $\mathcal{K} \subseteq \mathrm{Alg}(\tau)$. *Then* $\mathrm{Mod}\,\mathrm{mId}\,\mathcal{K} = \mathsf{RP}\mathcal{K}$.

Proof. For any set $\mathcal{J}$ of minor identities, the inclusion $\mathsf{P}(\mathrm{Mod}\,\mathcal{J}) \subseteq \mathrm{Mod}\,\mathcal{J}$ holds by the classical (multisorted) Birkhoff theorem. In order to show that $\mathsf{R}(\mathrm{Mod}\,\mathcal{J}) \subseteq \mathrm{Mod}\,\mathcal{J}$, let $\mathbf{B} \in \mathsf{R}(\mathrm{Mod}\,\mathcal{J})$; then $\mathbf{B}$ is an (h, h')-reflection of some $\mathbf{A} \in \mathrm{Mod}\,\mathcal{J}$ for some $h\colon B \to A$ and $h'\colon A \to B$. We need to show that $\mathbf{B} \models f_\sigma \approx_{S'} g_\pi$ for every $f_\sigma \approx_{S'} g_\pi \in \mathcal{J}$. For every $\beta\colon X|_{S'} \to B$, we have

$$\begin{aligned} \beta^\#(f_\sigma) &= f^{\mathbf{B}}(\beta \circ \sigma) = h'(f^{\mathbf{A}}(h \circ \beta \circ \sigma)) = h'(f_\sigma^{\mathbf{A}}(h \circ \beta)) \\ &= h'(g_\pi^{\mathbf{A}}(h \circ \beta)) = h'(g^{\mathbf{A}}(h \circ \beta \circ \pi)) = g^{\mathbf{B}}(\beta \circ \pi) = \beta^\#(g_\pi), \end{aligned}$$

where the fourth equality holds because $\mathbf{A} \models f_\sigma \approx_{S'} g_\pi$, whence $f_\sigma^{\mathbf{A}}(h \circ \beta) = (h \circ \beta)^\#(f_\sigma) = (h \circ \beta)^\#(g_\pi) = g_\pi^{\mathbf{A}}(h \circ \beta)$. We have proved the inclusion $\mathsf{RP}\mathcal{K} \subseteq \mathrm{Mod}\,\mathrm{mId}\,\mathcal{K}$.

It remains to show $\mathrm{Mod}\,\mathrm{mId}\,\mathcal{K} \subseteq \mathsf{RP}\mathcal{K}$. Assume that $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ is an algebra of type $\tau = (S, \Sigma, \mathrm{dec})$ satisfying every minor identity that holds in $\mathcal{K}$. We want to show that $\mathbf{B} \in \mathsf{RP}\mathcal{K}$.

Let $Y = (Y_s)_{s \in S}$ be the S-sorted set of variables with $Y_s := B_s \times \{s\}$ for all $s \in S$ (i.e., we take the variable symbols to be the disjoint union of the sets B_s), and let

$$\mathcal{N} := \{t_1 \approx_{S_Y} t_2 \in \mathit{MID}_\tau(Y) \mid \mathcal{K} \not\models t_1 \approx_{S_Y} t_2\}$$

be the set of minor identities over Y valuated on the set S_Y that do not hold in $\mathcal{K}$.

We first consider the case $\mathcal{N} \neq \emptyset$. Then for each $\nu \in \mathcal{N}$, say $\nu = f_\sigma \approx_{S_Y} g_\pi$ with $f \in \Sigma_{(w,s)}$, $\sigma\colon [n] \to Y$ with $|w| = n$, $g \in \Sigma_{(u,s)}$, $\pi\colon [m] \to Y$ with $|u| = m$, there exists a counterexample $\mathbf{A}_\nu = (A_\nu, \Sigma^{\mathbf{A}_\nu}) \in \mathcal{K}$ that does not satisfy ν. This means that there exists a valuation map $\beta_\nu\colon Y|_{S_Y} \to A_\nu$ such that $f^{\mathbf{A}_\nu}(\beta_\nu \circ \sigma) \neq g^{\mathbf{A}_\nu}(\beta_\nu \circ \pi)$; hence $S_Y \subseteq S_{A_\nu}$. Now let $\mathbf{P} := \prod_{\nu \in \mathcal{N}} \mathbf{A}_\nu$ be the product of all the counterexamples. Then $\mathbf{P} = (P, \Sigma^{\mathbf{P}})$ and $S_P = \bigcap_{\nu \in \mathcal{N}} S_{A_\nu} \supseteq S_Y$. Note that $\mathbf{P} \in \mathsf{P}\mathcal{K}$.

For every $y \in Y_s$ with $s \in S_Y$, the tuple $\overline{y} := (\beta_\nu(y))_{\nu \in \mathcal{N}}$ is an element of P_s. Let $h = (h_s)_{s \in S_Y}$ where each $h_s\colon B_s \to P_s$ is the map $b \mapsto \overline{(b, s)}$ (note that $(b, s) \in Y_s$). For each $s \in S_Y$, let $Z_s := \{f^{\mathbf{P}}(\overline{y}_1, \dots, \overline{y}_n) \mid f \in \Sigma_{(w,s)}, (y_1, \dots, y_n) \in Y_w\} \subseteq P_s$. Now we shall define maps $h'_s\colon P_s \to B_s$, for $s \in S_B$ and we set $h' = (h'_s)_{s \in S_B}$ such that $\mathbf{B}$ is an (h, h')-reflection of $\mathbf{P}$. For any $z \in P_s \setminus Z_s$, the value $h'_s(z)$ can be chosen arbitrarily in B_s. For an element $f^{\mathbf{P}}(\overline{y}_1, \dots, \overline{y}_n) \in Z_s$, where $y_i := (b_i, w_i)$ with $b_i \in B_{w_i}$ $(i = 1, \dots, n)$, define $h'_s(f^{\mathbf{P}}(\overline{y}_1, \dots, \overline{y}_n)) := f^{\mathbf{B}}(b_1, \dots, b_n)$ according to the reflection property (cf. Definition 4.1).

We have to verify that h'_s is well defined. Suppose, to the contrary, that $f^{\mathbf{P}}(\overline{y}_1, \dots, \overline{y}_n) = g^{\mathbf{P}}(\overline{z}_1, \dots, \overline{z}_m)$ but $f^{\mathbf{B}}(b_1, \dots, b_n) \neq g^{\mathbf{B}}(c_1, \dots, c_m)$ for some $f \in \Sigma_{(w,s)}$, $g \in \Sigma_{(u,s)}$, $(\overline{y}_1, \dots, \overline{y}_n) \in P_w$, $(\overline{z}_1, \dots, \overline{z}_m) \in P_u$, where $y_i := (b_i, w_i)$ for $b_i \in B_{w_i}$ $(i = 1, \dots, n)$ and $z_i := (c_i, u_i)$ for $c_i \in B_{u_i}$ $(i = 1, \dots, m)$. From the latter it follows that $\mathbf{B}$ does not satisfy the minor identity $\mu := f_\sigma \approx_{S_Y} g_\pi \in \mathit{MID}^s_\tau(Y)$, where $\sigma\colon [n] \to Y$, $i \mapsto y_i$ and $\pi\colon [m] \to Y$, $i \mapsto z_i$. Write $Y' := \mathrm{var}(f_\sigma) \cup \mathrm{var}(g_\pi)$, and let $\delta\colon Y' \to X$ be an injective map to the set X of standard variables. Let $\mu' := f'_\sigma \approx_{S_Y} g'_\pi \in \mathit{MID}^s_\tau(X)$, where f'_σ and g'_π are the minor terms in

$MT^s_\tau(X)$ that are obtained from f_σ and g_π by replacing each occurrence of a variable symbol $y \in Y'$ by $\delta(y)$. Since $\mathbf{B} \not\models \mu$, it follows from Lemma 5.1 that $\mathbf{B} \not\models \mu'$. Since $\mathbf{B} \in \operatorname{Mod} \operatorname{mId} \mathcal{K}$, this implies $\mathcal{K} \not\models \mu'$, whence $\mathcal{K} \not\models \mu$ by Lemma 5.1. Therefore $\mu \in \mathcal{N}$. Then, by the definition of $\mathbf{A}_\mu$ and β_μ, we have $f^{\mathbf{A}_\mu}(\beta_\mu(y_1), \dots \beta_\mu(y_n)) \neq g^{\mathbf{A}_\mu}(\beta_\mu(z_1), \dots, \beta_\mu(z_m))$. This means that the μ-th coordinates of the tuples $f^{\mathbf{P}}(\overline{y}_1, \dots, \overline{y}_n)$ and $g^{\mathbf{P}}(\overline{z}_1, \dots, \overline{z}_m)$ are different, contradicting our assumption. We conclude that $\mathbf{B} \in \mathsf{R}\mathbf{P} \subseteq \mathsf{RP}\mathcal{K}$.

Finally, we must consider the case $\mathcal{N} = \emptyset$, i.e., $\mathcal{K}$ satisfies every minor identity $f_\sigma \approx_{S_Y} g_\pi$ where $f_\sigma, g_\pi \in MT^s_\tau(Y)$ for some $s \in S$. Let $f \in \Sigma_{(w,s)}$, $g \in \Sigma_{(u,s)}$, $(a_1, \dots, a_n) \in B_w$, $(b_1, \dots, b_m) \in B_u$, and define $\sigma\colon [n] \to Y$, $\sigma(i) = (a_i, w_i)$ and $\pi\colon [m] \to Y$, $\pi(i) = (b_i, u_i)$. Let $\beta\colon Y|_{S_Y} \to B$, $\beta((x, s)) = x$. Since $\mathbf{B} \models f_\sigma \approx_{S_Y} g_\pi$, we have $\beta^\#(f_\sigma) = \beta^\#(g_\pi)$. Therefore,

$$f^{\mathbf{B}}(a_1, \dots, a_n) = f^{\mathbf{B}}(\beta \circ \sigma) = \beta^\#(f_\sigma) = \beta^\#(g_\pi) = g^{\mathbf{B}}(\beta \circ \pi) = g^{\mathbf{B}}(b_1, \dots, b_n).$$

Since the choice of f, g and the a_i and b_i was arbitrary, it follows that there exist constants $c_s \in B_s$ ($s \in S_B$) such that every function $f^{\mathbf{B}}$ of sort s is constant c_s. Let $\mathbf{D} = (D, \Sigma^{\mathbf{D}}) := \prod \emptyset$ be the empty product of algebras in $\mathcal{K}$. Then $\mathbf{D} \in \mathsf{P}\mathcal{K}$. As noted in Definition 2.12, D_s equals the singleton $\{\emptyset\}$ for every $s \in S$. Define $h = (h_s)_{s \in S_B}$ and $h' = (h'_s)_{s \in S_B}$ where $h_s\colon B_s \to D_s$, $b \mapsto \emptyset$ for all $b \in B_s$ and $h'_s\colon D_s \to B_s$, $\emptyset \mapsto c_s$. Then for each $f \in \Sigma_{(w,s)}$ with $w = w_1 \dots w_n$, we have $f^{\mathbf{B}}(a_1, \dots, a_n) = c_s = h'_s(f^{\mathbf{D}}(h_{w_1}(a_1), \dots, h_{w_n}(a_n)))$. Therefore $\mathbf{B}$ is an (h, h')-reflection of $\mathbf{D}$. Thus $\mathbf{B} \in \mathsf{RP}\mathcal{K}$. □

Theorem 5.2 characterizes the closed classes of algebras corresponding to the Galois connection mId–Mod as the classes that are closed under reflections and direct products. Next we describe the Galois closed classes of minor identities in terms of closure conditions, which are analogous to the classical characterization of equational theories as fully invariant congruences of free algebras. In order to state the result, we make use of the canonical trivial algebra $\mathbf{S}$ of type τ defined in Example 2.13(2). Recall also Lemma 2.14 and the notational shorthand $\langle S' \rangle_{\mathbf{S}}$ involving subalgebras of $\mathbf{S}$ introduced in the paragraph preceding Lemma 2.14.

Theorem 5.3. *Let $\mathcal{J} \subseteq MID_\tau(X)$ be a set of minor identities of type $\tau = (S, \Sigma, \mathrm{dec})$ over X. Then $\mathcal{J} = \operatorname{mId} \operatorname{Mod} \mathcal{J}$ if and only if $\mathcal{J}$ satisfies the following conditions:*

(1) *For every $S' \subseteq S$ and $s \in S$, the set*
$$\mathcal{J}^{(S')}_s := \{(f_\sigma, g_\pi) \mid f_\sigma \approx_{S'} g_\pi \in \mathcal{J},\ \operatorname{sort}(f) = \operatorname{sort}(g) = s\}$$
is an equivalence relation on $MT^s_\tau(X|_{S'})$.

(2) *If $t_1 \approx_{S'} t_2 \in \mathcal{J}$ and $S' \subseteq S''$, then $t_1 \approx_{S''} t_2 \in \mathcal{J}$ ("sort expansion").*

(3) *If $t_1 \approx_{S'} t_2 \in MID_\tau(X)$ and $t_1 \approx_{\langle S' \rangle_{\mathbf{S}}} t_2 \in \mathcal{J}$, then $t_1 \approx_{S'} t_2 \in \mathcal{J}$ ("sort contraction").*

(4) *If $f_\sigma \approx_{S'} g_\pi \in \mathcal{J}$, then $f_{\lambda \circ \sigma} \approx_{S'} g_{\lambda \circ \pi} \in \mathcal{J}$ for all $\lambda\colon X|_{S'} \to X|_{S'}$ ("closure under minors").*

Proof. We will prove the equivalent statement that a set $\mathcal{J} \subseteq MID_\tau(X)$ of minor identities is of the form $\mathcal{J} = \operatorname{mId} \mathcal{K}$ for some set $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$ of algebras if and only if $\mathcal{J}$ satisfies conditions (1)–(4).

Assume first that $\mathcal{J} = \operatorname{mId} \mathcal{K}$ for some $\mathcal{K} \subseteq \operatorname{Alg}(\tau)$. It is easy to verify that condition (1) holds. Condition (2) holds by Lemma 3.4.

Let $t_1 \approx_{S'} t_2 \in MID_\tau(X)$ and assume that $t_1 \approx_{\langle S' \rangle_{\mathbf{S}}} t_2 \in \mathcal{J}$. Suppose, to the contrary, that $t_1 \approx_{S'} t_2 \notin \mathcal{J}$. Then there exists an algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}}) \in \mathcal{K}$ such that $\mathbf{A} \not\models t_1 \approx_{S'} t_2$, i.e., there exists a valuation $\beta\colon X|_{S'} \to A$ such that $\beta^\#(t_1) \neq \beta^\#(t_2)$. This is possible only if $S' \subseteq S_A$, which implies $\langle S' \rangle_{\mathbf{S}} \subseteq \langle S_A \rangle_{\mathbf{S}} = S_A$ by

Lemma 2.14. Consequently, there exist maps $X|_{\langle S'\rangle_{\mathbf{S}}} \to A$, and for any extension $\gamma\colon X|_{\langle S'\rangle_{\mathbf{S}}} \to A$ of β, it holds that $\gamma^{\#}(t_1) = \beta^{\#}(t_1) \neq \beta^{\#}(t_2) = \gamma^{\#}(t_2)$. Therefore $\mathbf{A} \not\models t_1 \approx_{\langle S'\rangle_{\mathbf{S}}} t_2$, so $t_1 \approx_{\langle S'\rangle_{\mathbf{S}}} t_2 \notin \mathcal{J}$, a contradiction. We conclude that condition (3) holds.

Let then $(f_\sigma, g_\pi) \in \mathcal{J}_s^{(S')}$ and $\lambda\colon X|_{S'} \to X|_{S'}$. Then for every valuation map $\beta\colon X|_{S'} \to A$, we have $\beta^{\#}(f_{\lambda\circ\sigma}) = f(\beta\circ\lambda\circ\sigma) = (\beta\circ\lambda)^{\#}(f_\sigma) = (\beta\circ\lambda)^{\#}(g_\pi) = g(\beta\circ\lambda\circ\pi) = \beta^{\#}(g_{\lambda\circ\pi})$. Consequently, $(f_{\lambda\circ\sigma}, g_{\lambda\circ\pi}) \in \mathcal{J}_s^{(S')}$, that is, condition (4) holds.

For the converse implication, assume that $\mathcal{J}$ satisfies conditions (1)–(4). For each $\delta = f_\sigma \approx_{S'} g_\pi \in \mathit{MID}_\tau(X) \setminus \mathcal{J}$, we will construct an algebra $\mathbf{F}_\delta \in \mathrm{Alg}(\tau)$ such that $\mathbf{F}_\delta \models \mathcal{J}$ but $\mathbf{F}_\delta \not\models \delta$. Taking $\mathcal{K}$ to be the set of all such "separating" algebras $\mathbf{F}_\delta$, for every $\delta \in \mathit{MID}_\tau(X) \setminus \mathcal{J}$, we have $\mathcal{J} = \mathrm{mId}\,\mathcal{K}$.

Let $\delta = f_\sigma \approx_{S'} g_\pi \in \mathit{MID}_\tau(X) \setminus \mathcal{J}$. Let $S'' := \langle S'\rangle_{\mathbf{S}}$. Define the algebra $\mathbf{F}_\delta = (F, \Sigma^{\mathbf{F}_\delta})$ of type τ as follows. Let $q := \mathrm{sort}(f) = \mathrm{sort}(g)$. For $s \in S$, let

$$F_s := \begin{cases} \emptyset, & \text{if } s \in S \setminus S'', \\ X_s, & \text{if } s \in S'' \setminus \{q\}, \\ X_q \cup MT_\tau^q(X|_{S''})/\mathcal{J}_q^{(S'')}, & \text{if } s = q. \end{cases}$$

Note that the quotient $MT_\tau^q(X|_{S''})/\mathcal{J}_q^{(S'')}$ appearing in the definition of F_q is a well-defined object, because $\mathcal{J}_q^{(S'')}$ is an equivalence relation on $MT_\tau^q(X|_{S''})$ by condition (1). We will denote the $\mathcal{J}_q^{(S'')}$-equivalence class of a term $t \in MT_\tau^q(X|_{S''})$ by $[t]$. For $d \in \Sigma_{(w,s)}$, the operation $d^{\mathbf{F}_\delta}\colon F_w \to F_s$ is defined by the following rules (for notation, see Definition 2.7). If $\mathrm{inp}(d) \nsubseteq S''$, then $d^{\mathbf{F}_\delta} = \emptyset$. If $\mathrm{inp}(d) \subseteq S''$ and $s \neq q$, then $d^{\mathbf{F}_\delta}(\alpha) = x_1^s$ for all $\alpha \in F_w$. If $\mathrm{inp}(d) \subseteq S''$ and $s = q$, then $d^{\mathbf{F}_\delta}(\alpha) = [d_{\varphi\circ\alpha}]$, where $\varphi\colon F \to X$ is given by $x_i^s \mapsto x_i^s$ for any $x_i^s \in X|_{S''}$ and $[t] \mapsto x_1^q$ for any $[t] \in MT_\tau^q(X|_{S''})/\mathcal{J}_q^{(S'')}$. Note that $S_F = S'' = \langle S'\rangle_{\mathbf{S}}$, from which it follows by Lemma 2.14 that the declaration of every $d \in \Sigma$ is reasonable in F, so the algebra $\mathbf{F}_\delta$ is well defined.

We show first that $\mathbf{F}_\delta \not\models \delta$. Let $\beta\colon X|_{S'} \to F$ be the inclusion map $x \mapsto x$. Then $\beta^{\#}(f_\sigma) = f^{\mathbf{F}_\delta}(\beta\circ\sigma) = [f_{\varphi\circ\beta\circ\sigma}] = [f_\sigma]$ and $\beta^{\#}(g_\pi) = g^{\mathbf{F}_\delta}(\beta\circ\pi) = [g_{\varphi\circ\beta\circ\pi}] = [g_\pi]$. Since $f_\sigma \approx_{S''} g_\pi \notin \mathcal{J}$ by condition (3), we have $[f_\sigma] \neq [g_\pi]$, and we conclude that $\mathbf{F}_\delta \not\models \delta$.

Finally we show that $\mathbf{F}_\delta \models \mathcal{J}$. Let $d_\rho \approx_T d'_{\rho'} \in \mathcal{J}$. If $\mathrm{sort}(d) \neq q$, then $\mathbf{F}_\delta$ obviously satisfies the identity $d_\rho \approx_T d'_{\rho'}$. Assume that $\mathrm{sort}(d) = \mathrm{sort}(d') = q$. If $T \nsubseteq S'' = S_F$, then $\mathbf{F}_\delta \models d_\rho \approx_T d'_{\rho'}$ holds vacuously. Thus we may assume that $T \subseteq S''$. Let $\beta\colon X|_T \to F$. By condition (2) we have $d_\rho \approx_{S''} d'_{\rho'} \in \mathcal{J}$, and by condition (4) we have $d_{\varphi\circ\beta\circ\rho} \approx_{S''} d'_{\varphi\circ\beta\circ\rho'} \in \mathcal{J}$. Then

$$\beta^{\#}(d_\rho) = d^{\mathbf{F}_\delta}(\beta\circ\rho) = [d_{\varphi\circ\beta\circ\rho}] = [d'_{\varphi\circ\beta\circ\rho'}] = d'^{\mathbf{F}_\delta}(\beta\circ\rho') = \beta^{\#}(d'_{\rho'}).$$

Thus $\mathbf{F}_\delta$ satisfies $d_\rho \approx_T d'_{\rho'}$. We conclude that $\mathbf{F}_\delta \models \mathcal{J}$. □

Remark 5.4. Theorem 5.2 was proved in the case of usual one-sorted algebras by Barto, Opršal and Pinsker [2, Corollary 5.4]. As for Theorem 5.3, sort expansion and sort contraction play no role when $|S| = 1$, and the theorem reduces to the description of closed sets of minor identities given by Čupona and Markovski [5, Theorem 2.1].

6. How are HSP and RP related?

It is clear from Lemma 4.2 that every RP-closed class is also HSP-closed. The converse is not true, and we would like to describe which HSP-closed classes are not

RP-closed. For similarity types of a special form that does not admit compositions of terms, we can provide a complete description: the HSP-closed classes that are not RP-closed are somewhat "trivial" in this case. For arbitrary similarity types, a characterization eludes us.

Definition 6.1. A multisorted similarity type $\tau = (S, \Sigma, \mathrm{dec})$ is *non-composable* if S can be partitioned into two subsets I and O such that for every $f \in \Sigma$, it holds that $\mathrm{inp}(f) \subseteq I$ and $\mathrm{sort}(f) \in O$. A type is *composable* if it is not non-composable.

Examples of non-composable similarity types include all types of 2-algebras, as introduced in Example 2.13(5).

Definition 6.2. The *height* of a term t, denoted by $h(t)$, is defined inductively as follows:

(1) Variable symbols have height 0, i.e., $h(x) = 0$ for all $x \in Y_s$, $s \in S$.
(2) If $t = f$, where $f \in \Sigma$ and $\mathrm{dec}(f) = (\varepsilon, s)$ (constant), then $h(t) = 1$.
(3) If $t = f(t_1, \dots, t_n)$, where $f \in \Sigma$, $\mathrm{dec}(f) = (w_1 \dots w_n, s)$, $n \geq 1$, and $t_1, \dots, t_n$ are terms, then $h(t) = \max(h(t_1), \dots, h(t_n)) + 1$.

Theorem 6.3. *Let $\tau = (S, \Sigma, \mathrm{dec})$ be a non-composable similarity type, and let $\mathcal{K} \subseteq \mathrm{Alg}(\tau)$ be an* HSP*-closed class of algebras. Then the following are equivalent.*

(1) *$\mathcal{K}$ is* R*-closed.*
(2) *For all $s \in S$, $\mathcal{K} \not\models x_1^s \approx_S x_2^s$.*
(3) *For all $s \in S$, there exists $\mathbf{A} \in \mathcal{K}$ such that $S_A = S$ and $|A_s| \geq 2$.*

Proof. (1) $\Longrightarrow$ (3) Assume that $\mathcal{K}$ is R-closed. Since $\mathsf{HSP}\mathcal{K} = \mathcal{K}$, we have $\mathbf{P} = (P, \Sigma^{\mathbf{P}}) := \prod \emptyset \in \mathcal{K}$. Let $A = (A_s)_{s \in S}$ be an S-sorted set with $|A_s| \geq 2$ for all $s \in S$, let $h \colon A \to P$, $h' \colon P \to A$ be arbitrary maps, and let $\mathbf{A}$ be the (h, h')-reflection of $\mathbf{P}$; hence $\mathbf{A} \in \mathsf{R}\mathcal{K} \subseteq \mathcal{K}$. The required condition is then satisfied by $\mathbf{A}$ for every $s \in S$.

(3) $\Longrightarrow$ (2) An algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ with $S_A = S$ and $|A_s| \geq 2$ clearly does not satisfy the identity $x_1^s \approx_S x_2^s$. Since $\mathcal{K}$ contains such an algebra for every $s \in S$, it follows that $\mathcal{K} \not\models x_1^s \approx_S x_2^s$ for all $s \in S$.

(2) $\Longrightarrow$ (1) Assume that $\mathcal{K} \not\models x_1^s \approx_S x_2^s$ for all $s \in S$. Let $\mathcal{J} := \mathrm{Id}\,\mathcal{K}$; since $\mathcal{K}$ is HSP-closed, we have $\mathcal{K} = \mathrm{Mod}\,\mathcal{J}$. We need to show that every identity in $\mathcal{J}$ is satisfied by all reflections of every algebra in $\mathcal{K}$. Let $\mu := t_1 \approx_{S'} t_2 \in \mathcal{J}$. If $t_1 = t_2$, then μ is satisfied by every algebra in $\mathrm{Alg}(\tau)$. Therefore we may assume that $t_1 \neq t_2$. Since τ is non-composable, the terms t_1 and t_2 have height at most 1. Consider the different possibilities. If $h(t_1) = h(t_2) = 0$, then $\mu = x_i^s \approx_{S'} x_j^s$ with $i \neq j$. It is clear that then $\mathcal{K} \models x_1^s \approx_{S'} x_2^s$, from which it follows by Lemma 3.4 that $\mathcal{K} \models x_1^s \approx_S x_2^s$. This contradicts our assumption and shows that this case is impossible. If $h(t_1) = h(t_2) = 1$, then μ is a minor identity, and $\mathsf{R}\mathcal{K} \models \mu$ holds by Theorem 5.2. Finally, if $h(t_1) \neq h(t_2)$, say $h(t_1) = 1$ and $h(t_2) = 0$, then $\mu = f_\sigma \approx_{S'} x_i^s$ for some $f \in \Sigma_{(w,s)}$. Note that $s \notin \mathrm{inp}(f)$, because τ is non-composable. Then in fact $f_\sigma \approx_{S'} x_j^s \in \mathcal{J}$ for every $j \in \mathbb{N}$. By symmetry and transitivity, we get $x_1^s \approx_{S'} x_2^s \in \mathcal{J}$. As above, this leads to a contradiction and shows that this last case is impossible. We conclude that $\mathsf{R}\mathcal{K} \subseteq \mathcal{K}$. ☐

Remark 6.4. Note that the proofs of the implications (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2) of Theorem 6.3 did not rely on the assumption that τ is non-composable, and it is also easy to see that (2) and (3) are actually equivalent for every type τ (whether it is composable or not). Hence the crucial part is (3) $\Longrightarrow$ (1) (or, equivalently, (2) $\Longrightarrow$ (1)), and we will prove in the next proposition that this implication actually characterizes non-composable types.

Proposition 6.5. *Let τ be a similarity type. If for every* HSP*-closed class $\mathcal{K} \subseteq \mathrm{Alg}(\tau)$, the conditions* (1)–(3) *of Theorem 6.3 are equivalent, then τ is non-composable.*

Proof. We prove the contrapositive. Assume that τ is composable. Then there exist $w = w_1 \dots w_n, u = u_1 \dots u_m \in W(S)$, $s \in S$ and $i \in [n]$ such that $\Sigma_{(w,s)} \neq \emptyset$ and $\Sigma_{(u,w_i)} \neq \emptyset$. Without loss of generality, we may assume that $i = 1$. Let $f \in \Sigma_{(w,s)}$ and $g \in \Sigma_{(u,w_1)}$, let $S' := \mathrm{inp}(f) \cup \mathrm{inp}(g)$, and let

$$\mu := f(g(y_1, \dots, y_m), z_2, \dots, z_n) \approx_{S'} f(g(y'_1, \dots, y'_m), z'_2, \dots, z'_n),$$

where $y_1, \dots, y_m, z_2, \dots, z_n, y'_1, \dots, y'_m, z'_2, \dots, z'_n$ are pairwise distinct variables with $y_i, y'_i \in X_{u_i}$, $z_i, z'_i \in X_{w_i}$, and let $\mathcal{K} := \mathrm{Mod}\,\mu$.

Define an S-sorted algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$ of type τ as follows. The carrier is $A = (A_s)_{s \in S}$ with $A_s := \{0, 1, 2\}$ for all $s \in S$. Define $f^{\mathbf{A}} \colon A_w \to A_s$ and $g^{\mathbf{A}} \colon A_u \to A_{w_1}$ by the rules

$$f^{\mathbf{A}}(a_1, \dots, a_n) := \psi(a_1), \qquad \text{where } \psi\colon\ 0 \mapsto 0,\ 1 \mapsto 0,\ 2 \mapsto 2,$$
$$g^{\mathbf{A}}(a_1, \dots, a_m) := \varphi(a_1), \qquad \text{where } \varphi\colon\ 0 \mapsto 0,\ 1 \mapsto 1,\ 2 \mapsto 0.$$

The other operations in $\Sigma^{\mathbf{A}}$ can be defined in an arbitrary way. Since $\mathrm{Im}\, g = \{0, 1\}$, we have

$$f^{\mathbf{A}}(g^{\mathbf{A}}(a_1, \dots, a_m), c_2, \dots, c_n) = \psi(g^{\mathbf{A}}(a_1, \dots, a_m)) = 0$$

for all $a_1, \dots, a_m, c_2, \dots, c_n \in \{0, 1, 2\}$. Hence $\mathcal{A} \models \mu$, i.e., $\mathbf{A} \in \mathcal{K}$. Thus condition (3) of Theorem 6.3 is satisfied with $\mathbf{A}$ for every $s \in S$.

Let $B := A$, i.e., $B_s := A_s = \{0, 1, 2\}$ $(s \in S)$. Let $\mathbf{B} = (B, \Sigma^{\mathbf{B}})$ be the (h, h')-reflection of $\mathbf{A}$ with $h_s \colon B_s \to A_s$, $0 \mapsto 2$, $1 \mapsto 1$, $2 \mapsto 0$ and $h'_s \colon A_s \to B_s$, $x \mapsto x$. Then

$$f^{\mathbf{B}}(b_1, \dots, b_n) = h'(f^{\mathbf{A}}(h(b_1), \dots, h(b_n))) = \psi(h(b_1)) = \begin{cases} 2, & \text{if } b_1 = 0, \\ 0, & \text{if } b_1 = 1, \\ 0, & \text{if } b_1 = 2, \end{cases}$$

$$g^{\mathbf{B}}(b_1, \dots, b_m) = h'(g^{\mathbf{A}}(h(b_1), \dots, h(b_m))) = \varphi(h(b_1)) = \begin{cases} 0, & \text{if } b_1 = 0, \\ 1, & \text{if } b_1 = 1, \\ 0, & \text{if } b_1 = 2. \end{cases}$$

Consequently,

$$f^{\mathbf{B}}(g^{\mathbf{B}}(b_1, \dots, b_m), c_2, \dots, c_n) = \psi(h(\varphi(h(b_1)))) = \begin{cases} 2, & \text{if } b_1 = 0, \\ 0, & \text{if } b_1 = 1, \\ 2, & \text{if } b_1 = 2. \end{cases}$$

Hence $f^{\mathbf{B}}(g^{\mathbf{B}}(b_1, \dots, b_m), c_2, \dots, c_n) \neq f^{\mathbf{B}}(g^{\mathbf{B}}(b'_1, \dots, b'_m), c'_2, \dots, c'_n)$ if $b_1 = 0$ and $b'_1 = 1$. Therefore $\mathbf{B} \not\models \mu$, i.e., $\mathbf{B} \notin \mathcal{K}$, so $\mathcal{K}$ is not R-closed, that is, condition (1) of Theorem 6.3 does not hold. We conclude that conditions (1)–(3) of Theorem 6.3 are not equivalent for $\mathcal{K}$. $\square$

Remark 6.6. According to Theorem 6.3, the only HSP-varieties of a non-composable type τ that are not RP-varieties are the ones satisfying an identity of the form $x_1^s \approx_S x_2^s$ for some $s \in S$. Using identities of this form, we can express the fact that a sort s is trivial in an algebra $\mathbf{A} = (A, \Sigma^{\mathbf{A}})$, in the sense that A_s is empty or a singleton.

References

[1] Adámek, J., Rosický, J., Vitale, E.M.: Birkhoff's variety theorem in many sorts. Algebra Universalis **68**, 39–42 (2012)

[2] Barto, L., Opršal, J., Pinsker, M.: The wonderland of reflections. Israel J. Math. **223**, 363–398 (2018)

[3] Birkhoff, G., Lipson, J.D.: Heterogeneous algebras. J. Combin. Theory **8**, 115–133 (1970)

[4] Čupona, G., Markovski, S.: Free objects in primitive varieties of n-groupoids. Publ. Inst. Math. (Beograd) (N.S.) **57(71)**, 147–154 (1995)

[5] Čupona, Ǵ., Markovski, S.: Primitive varieties of algebras. Algebra Universalis **38**, 226–234 (1997)

[6] Čupona, G., Markovski, S., Popeska, Ž.: Primitive n-identities: In: Contributions to General Algebra, vol. 9, pp. 107–116. Hölder-Pichler-Tempsky, Vienna (1995)

[7] Higgins, P.J.: Algebras with a scheme of operators. Math. Nachr. **27**, 115–132 (1963)

[8] Manca, V., Salibra, A.: Soundness and completeness of the Birkhoff equational calculus for many-sorted algebras with possibly empty carrier sets. Theoret. Comput. Sci. **94**, 101–124 (1992)

[9] Taylor, W.: Characterizing Mal'cev conditions. Algebra Universalis **3**, 351–397 (1973)

[10] Wechler, W.: Universal Algebra for Computer Scientists. EATCS Monogr. Theoret. Comput. Sci., vol. 25. Springer, Berlin (1992)

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