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ON THE TOPOLOGICAL SEMIGROUP OF EQUATIONAL CLASSES OF FINITE FUNCTIONS UNDER COMPOSITION

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This paper is dedicated to Professor I.G. Rosenberg on the occasion of his 80th birthday.

Abstract. We consider the set of equational classes of finite functions endowed with the operation of class composition. Thus defined, this set gains a semigroup structure. This paper is a contribution to the understanding of this semigroup. We present several interesting properties of this semigroup. In particular, we show that it constitutes a topological semigroup that is profinite and we provide a description of its regular elements in the Boolean case.

Introduction

Throughout this paper, let A be a finite nonempty set. Without loss of generality, we assume that $A = [m] = \{0, \ldots, m-1\}$ for some natural number m. An *n-ary function on A* is a mapping $f\colon A^n \to A$. By a *class* (of functions) on A we simply mean a set of such mappings of possibly different arities. In this paper we shall be particularly interested in classes of functions definable by certain functional equations, namely, with a unique functional symbol that binds each variable and that does not occur as argument of itself. In [7] it was shown that such classes, which we refer to as being equational, are exactly those classes that are closed under identifications and permutations of variables as well as addition and deletion of inessential variables. For further background and variants, see e.g. [5, 7, 9, 10, 15].

If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two classes of functions on A, then their composition $\mathcal{K}_1\mathcal{K}_2$ is defined as the set of all compositions of functions in $\mathcal{K}_1$ with functions in $\mathcal{K}_2$, i.e.,

$$\mathcal{K}_1\mathcal{K}_2 := \{f(g_1, \ldots, g_n) : f \in \mathcal{K}_1,\ g_1, \ldots, g_n \in \mathcal{K}_2\}.$$

When restricted to equational classes of functions on A, class composition is associative, and thus it endows the set of all equational classes on a set A with a (fairly complicated) semigroup structure. As the size of the underlying set A determines this semigroup up to isomorphism, we denote by $\mathbf{E}_m$ the semigroup of all equational classes on an m-element set.

Apart from the theoretical interest, this study is motivated by the many connections to areas pertaining to the multiple valued logic and universal algebra communities. For instance, idempotent elements of $\mathbf{E}_m$ subsume so-called clones (composition-closed classes of functions that contain the projections), which are of key importance in multiple-valued logic. The study of these semigroups may bring additional information and lead to a better understanding of the complex structure of the lattice of clones.

In Section 1 we recall the necessary definitions and preliminary results on equational classes and clones, in particular, clones of Boolean functions and idempotent elements of $\mathbf{E}_2$. We introduce a metric on $\mathbf{E}_m$ in Section 2 and show that $\mathbf{E}_m$ is a compact topological semigroup with respect to the topology induced by this metric. We focus on the semigroup $\mathbf{E}_2$ made of equational classes of Boolean functions in Section 3. In particular, we describe its regular elements and we determine the restriction of the Green relations to the regular $\mathcal{D}$-classes. For general background on Green's relations see, e.g., [18].

1. Preliminaries

The set of all n-ary functions on $A = [m]$ is denoted by $\mathcal{O}_m^{(n)}$, and the set of all functions on $A = [m]$ is $\mathcal{O}_m := \cup_{n \geq 1} \mathcal{O}_m^{(n)}$. For any class $\mathcal{K} \subseteq \mathcal{O}_m$ and any positive integer n, let $\mathcal{K}^{(n)}$ denote the n-ary part of $\mathcal{K}$, i.e., $\mathcal{K}^{(n)} := \mathcal{K} \cap \mathcal{O}_m^{(n)}$.

1.1. The simple minor quasiorder.

We say that the i-th variable of a function $f \in \mathcal{O}_m^{(n)}$ is *essential*, if there exist $a_1, \ldots, a_n, a_i' \in [m]$ such that $f(a_1, \ldots, a_i, \ldots, a_n) \neq f(a_1, \ldots, a_i', \ldots, a_n)$. We denote the set of essential variables of f by $\operatorname{Ess} f$, and we define the *essential arity* of f by $\operatorname{ess} f := |\operatorname{Ess} f|$.

For $f \in \mathcal{O}_m^{(n)}$ and $g \in \mathcal{O}_m^{(k)}$, we say that g is a *simple minor* of f, denoted by $g \preceq f$, if there exists a map $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ such that

$$g(x_1, \ldots, x_k) = f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right).$$

Observe that the operation of taking simple minors subsumes permutation and identification of variables, and addition and deletion of inessential variables.

The simple minor relation gives rise to a quasiorder on $\mathcal{O}_m$ (see [9]). The corresponding *equivalence* is defined by $f \equiv g \iff f \preceq g$ and $g \preceq f$, and it is clear that f and g are equivalent if and only if they differ only in inessential variables and/or in the order of their variables. We will not distinguish between equivalent functions in the sequel. For example, $\{\mathrm{id}\}$ will stand for the set of all *projections* $(x_1, \ldots, x_n) \mapsto x_i$, as these are the functions equivalent to the identity function.

Being a quasiorder, the simple minor relation induces naturally a partial order on $\mathcal{O}_m/{\equiv}$. This poset was studied in more detail in [9] for $m = 2$. Functions on [2] are called *Boolean functions*, and we will use the notation Ω instead of $\mathcal{O}_2$ for the set of all Boolean functions. The bottom of the poset $(\Omega/{\equiv}, \preceq)$ is shown in Figure 1. We can see (and it is easy to prove) that Ω (or equivalently, $\Omega/{\equiv}$) has four connected components, namely $\Omega_{00}, \Omega_{11}, \Omega_{01}, \Omega_{10}$, where

$$\Omega_{ab} = \{f \in \Omega : f(\mathbf{0}) = a,\ f(\mathbf{1}) = b\} \quad (a, b \in \{0, 1\}).$$

Hereinafter, $\mathbf{0}$ and $\mathbf{1}$ denote the tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$, respectively; the length of the tuples is not specified, as it should be always clear from the context. For an arbitrary function class $\mathcal{K}$, we will abbreviate $\mathcal{K} \cap \Omega_{ab}$ by $\mathcal{K}_{ab}$, and we will use the following (hopefully intuitive) notation :

$$\mathcal{K}_{0*} = \mathcal{K}_{00} \cup \mathcal{K}_{01},\ \mathcal{K}_{*1} = \mathcal{K}_{01} \cup \mathcal{K}_{11},\ \mathcal{K}_{=} = \mathcal{K}_{00} \cup \mathcal{K}_{11}.$$

The minimal elements of $(\Omega/{\equiv}; \preceq)$ are the unary functions: $0, 1, \mathrm{id}$ and $\neg$ (negation). On the next level we can see the binary functions $+$ (addition modulo 2), $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction) and the ternary

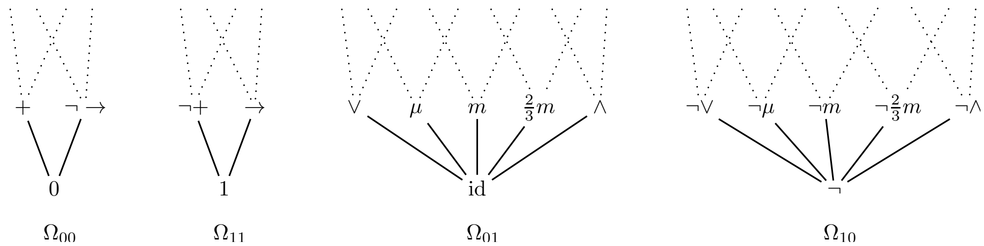

FIGURE 1. The subfunction quasiorder on Boolean functions

functions μ (majority function), m (minority function), $\frac{2}{3}m$ ($\frac{2}{3}$-minority function, see [4]) together with their negations. Here negation is taken "from outside", e.g., $\neg\frac{2}{3}m$ is a shorthand notation for the function $\neg\frac{2}{3}m\,(x,y,z) = 1+\frac{2}{3}m\,(x,y,z) = 1+xy+yz+xz+x+z$.

1.2. **Equational classes and composition.** A class $\mathcal{K} \subseteq \mathcal{O}_m$ is an *equational class* if it is an order ideal in the simple minor quasiorder, i.e., if $f \in \mathcal{K}$ and $g \preceq f$ imply $g \in \mathcal{K}$. This terminology is motivated by the fact that these are exactly the classes that can be defined by certain functional equations [7, 10]. (Note that in universal algebra this term is used for equationally definable classes of algebras, also called varieties.) Two natural examples are the class of monotone (order-preserving) and antimonotone (order-reversing) Boolean functions, which can be defined by the functional equations $f\,(\mathbf{x}\wedge\mathbf{y}) \wedge f\,(\mathbf{x}) = f\,(\mathbf{x}\wedge\mathbf{y})$ and $f\,(\mathbf{x}\wedge\mathbf{y}) \wedge f\,(\mathbf{x}) = f\,(\mathbf{x})$, respectively. Another example is the class $\Omega_{=} \subseteq \Omega$ defined by the equation $f\,(\mathbf{0}) = f\,(\mathbf{1})$. Equational classes can be also defined by relational constraints; we will discuss this approach in more detail in Subsection 1.3. The equational classes on $[m]$ form a lattice $\mathbf{E}_m$ with intersection and union as the lattice operations. This lattice has continuum cardinality already on the two-element set, and its structure is very complicated [9].

For $f \in \mathcal{O}_m^{(n)}$ and $g_1,\dots,g_n \in \mathcal{O}_m^{(k)}$, we define their *composition* as the function $f\,(g_1,\dots,g_n) \in \mathcal{O}_m^{(k)}$ given by

$$f\,(g_1,\dots,g_n)\,(\mathbf{x}) = f\,(g_1\,(\mathbf{x})\,,\dots,g_n\,(\mathbf{x}))\,.$$

We refer to f as the *outer function* and to $g_1,\dots,g_n$ as the *inner functions* of the composition.

As we saw in the introduction, this notion naturally extends to classes. If $\mathcal{K}_1,\mathcal{K}_2 \subseteq \mathcal{O}_m$, then their *composition* is defined by

$$\mathcal{K}_1\mathcal{K}_2 := \{f\,(g_1,\dots,g_n) : f \in \mathcal{K}_1,\ g_1,\dots,g_n \in \mathcal{K}_2\}\,.$$

Let us note that the simple minor relation can be defined very compactly using function class composition:

$$g \preceq f \iff g \in \{f\}\,\{\mathrm{id}\}\,. \tag{1}$$

Hence, equational classes can be characterized as those classes $\mathcal{K}$ that verify the condition $\mathcal{K} = \mathcal{K}\,\{\mathrm{id}\}$.

Now, in general, class composition is not associative. However, it becomes an associative operation when restricted to equational classes.

Associativity Lemma ([7]). *Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \subseteq \mathcal{O}_m$. The following assertions hold:*

(i) $(\mathcal{K}_1\mathcal{K}_2)\mathcal{K}_3 \subseteq \mathcal{K}_1(\mathcal{K}_2\mathcal{K}_3)$;
(ii) *If $\mathcal{K}_2$ is an equational class, then*

$$(\mathcal{K}_1\mathcal{K}_2)\mathcal{K}_3 = \mathcal{K}_1(\mathcal{K}_2\mathcal{K}_3).$$

Hence $\mathbf{E}_m$ endowed with class composition can be regarded as a semigroup. In fact, $\mathbf{E}_m$ is a monoid with identity element $\{\mathrm{id}\}$. In the sequel we will simply write $f\mathcal{K}$ instead of $\{f\}\mathcal{K}$ and $\mathcal{K}f$ instead of $\mathcal{K}\{f\}$.

A class $\mathcal{K}$ is *closed under composition* if $\mathcal{K}\mathcal{K} \subseteq \mathcal{K}$. Clearly, if $\mathcal{K}$ is *idempotent*, i.e., $\mathcal{K}\mathcal{K} = \mathcal{K}$, then $\mathcal{K}$ is closed under composition. It was proved in [21] that the converse also holds for equational classes of Boolean functions. (Let us note that this is a distinguishing feature of Boolean functions: if $m \geq 3$, then we can construct a class $\mathcal{K} \in \mathbf{E}_m$ such that $\mathcal{K}\mathcal{K} \subsetneq \mathcal{K}$.)

Proposition 1.1 ([21]). *For any equational class $\mathcal{K}$ of Boolean functions we have $\mathcal{K}\mathcal{K} \subseteq \mathcal{K}$ if and only if $\mathcal{K}\mathcal{K} = \mathcal{K}$.*

A class $\mathcal{C} \subseteq \mathcal{O}_m$ is a *clone* if it is closed under composition and contains all projections. From (1) it follows that every clone is an equational class. The converse is not true: the class of antimonotone Boolean functions and the class $\Omega_=$ are both equational classes but neither of them is a clone.

Remark 1.2. It is not hard to see that $\Omega_=$ is the largest composition-closed equational class of Boolean functions that is not a clone (see [21]).

The set of clones on $[m]$ constitutes a lattice, which has continuum cardinality for $m \geq 3$ (see [12]) and the description of its structure remains a topic of active research. However, there are only countably many clones on the two-element set, and these have been described by E. L. Post in [16] (see Subsection 1.4). The clone generated by $F \subseteq \mathcal{O}_m$, i.e., the least clone containing F will be denoted by $[F]$. For general background on clones and relations (cf. Subsection 1.3) we refer the reader to the monographs [13] and [17].

1.3. **Relational constraints.** By a *relation* of arity k on $[m]$ we mean a set of k-tuples $P \subseteq [m]^k$. If $T \in [m]^{k\times n}$ is a $k \times n$ matrix such that each column of T belongs to P, then we say that T is a *P-matrix*. Applying an n-ary function f to the rows of T, we obtain the column vector $f(T) \in [m]^k$. A *relational constraint* of arity k is a pair (P, Q), where P and Q are k-ary relations. An n-ary function f *satisfies* the constraint (P, Q) if $f(T) \in Q$ for every P-matrix T of size $k \times n$. Satisfaction of relational constraints gives rise to a Galois connection that defines equational classes of functions. We say that a class $\mathcal{K} \subseteq \mathcal{O}_m$ is *defined by relational constraints* if there is a set $\mathcal{Q}$ of relational constraints such that $\mathcal{K}$ is the class of all functions that satisfy every member of $\mathcal{Q}$.

Theorem 1.3 ([15]). *A class $\mathcal{K} \subseteq \mathcal{O}_m$ is an equational class if and only if $\mathcal{K}$ can be defined by relational constraints.*

As an illustration of this theorem let us consider our three examples from Subsection 1.2: the class of monotone and antimonotone Boolean functions can be defined by the constraints $(\leq, \leq)$ and $(\leq, \geq)$, respectively, and the class $\Omega_=$ can be defined by $(\{(0,1)\}, \{(0,0), (1,1)\})$.

A function f *preserves* the relation P if f satisfies the constraint (P, P). This induces the well-known Pol-Inv Galois connection between clones and relational clones. As for relational constraints, we say that a class $\mathcal{K} \subseteq \mathcal{O}_m$ is *defined by relations* if there is a set $\mathcal{Q}$ of relations such that $\mathcal{K}$ is the class of all functions that preserve every member of $\mathcal{Q}$.

Theorem 1.4 ([3, 11]). *A class $\mathcal{K} \subseteq \mathcal{O}_m$ is a clone if and only if $\mathcal{K}$ can be defined by relations.*

As an example, let us observe that the clone of monotone Boolean functions is defined by the relation $\leq$.

Now we present another Galois connection that characterizes composition-closed equational classes. Let us say that a function f *strongly satisfies* the relational constraint (P, Q), if f satisfies both (P, Q) and (Q, Q) (i.e., f satisfies (P, Q) and preserves Q). As before, we say a class $\mathcal{K} \subseteq \mathcal{O}_m$ is *strongly defined by relational constraints* if there is a set $\mathcal{Q}$ of relational constraints such that $\mathcal{K}$ is the class of all functions that strongly satisfy every member of $\mathcal{Q}$.

Theorem 1.5 ([21]). *A class $\mathcal{K} \subseteq \mathcal{O}_m$ is a composition-closed equational class if and only if $\mathcal{K}$ can be strongly defined by relational constraints.*

Concerning our three examples, let us note that the class of antimonotone functions is not closed under composition, and the class of monotone functions is strongly defined by the constraint $(\leq, \leq)$, as we have already observed. The class $\Omega_=$ is also closed under composition, and it is strongly defined by $(\{(0,1)\}, \{(0,0), (1,1)\})$, since the relation $\{(0,0), (1,1)\}$ is just the equality relation, and it is preserved by every function.

1.4. **The Post lattice.** The *dual* of an n-ary Boolean function f is the function f^d defined by $f^d(x_1, \ldots, x_n) := \neg f(\neg x_1, \ldots, \neg x_n)$. We say that f is *selfdual* if $f^d = f$ and we say that f is *reflexive* if $f = \neg f^d$, i.e., $f(x_1, \ldots, x_n) = f(\neg x_1, \ldots, \neg x_n)$. The set of self-dual functions is denoted by S, and the set of reflexive functions is denoted by N. The dual of a class $\mathcal{K} \subseteq \Omega$ is $\mathcal{K}^d := \left\{f^d : f \in \mathcal{K}\right\}$. Observe that $\mathcal{K}^d$ can be also written as $\neg\mathcal{K}\neg$ using function class composition. (Let us recall that here $\neg$ stands for $\{\neg\}$, which in turn is an abbreviation for the class of all functions that are equivalent to the unary negation function. As we will use composition with this class very often, it will be convenient to use this simplified notation.)

Figure 2 shows the lattice of clones on [2], usually referred to as the *Post lattice*. Only some clones are labelled on the figure; all other Boolean clones can be obtained as intersections of these:

- Ω is the clone of all Boolean functions;
- Ω_{0*} is the clone of 0-preserving functions;
- Ω_{*1} is the clone of 1-preserving functions;
- M is the clone of monotone (order-preserving) functions;
- S is the clone of self-dual functions;
- L is the clone of linear functions, i.e., functions of the form $x_1 + \cdots + x_n + c$ with $n \geq 0, c \in \{0,1\}$;

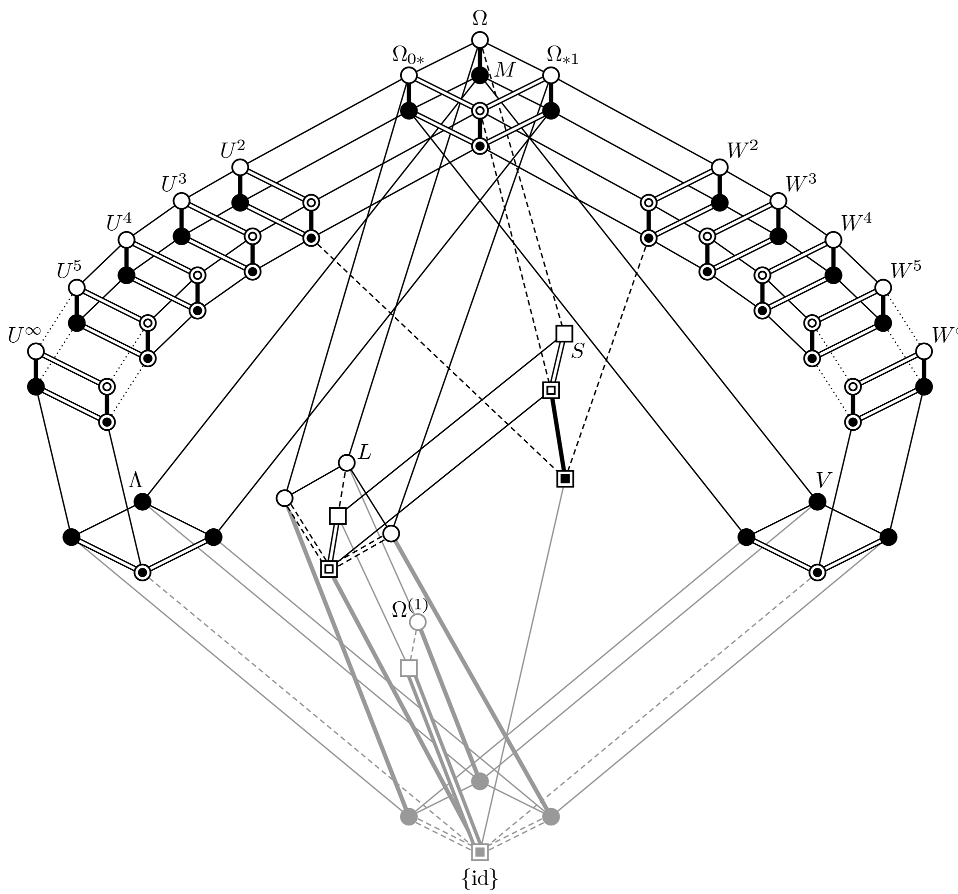

FIGURE 2. The Post lattice

- Λ consists of conjunctions $x_1 \wedge \cdots \wedge x_n$ $(n \geq 1)$ and the two constants $0, 1$;
- V consists of disjunctions $x_1 \vee \cdots \vee x_n$ $(n \geq 1)$ and the two constants $0, 1$;
- $\Omega^{(1)}$ is the clone of essentially at most unary functions;
- $\{\mathrm{id}\}$ is the clone consisting of projections only;
- W^k is the clone of functions preserving the relation $\{0,1\}^k \setminus \{\mathbf{0}\}$;
- $W^\infty = W^2 \cap W^3 \cap \cdots$ is the clone generated by implication;
- U^k is the dual of W^k for $k = 2, 3, \ldots, \infty$.

1.5. **Idempotent equational classes.** The usual notation for the set of idempotents of a semigroup $\mathbf{S}$ is $E(\mathbf{S})$, but in our case this would lead to the somewhat awkward notation $E(\mathbf{E}_m)$, therefore we will simply write $\mathbf{I}_m$ for the set of idempotent equational classes on $[m]$. If $\mathcal{C}$ is a clone, then $\mathcal{C}\mathcal{C} \subseteq \mathcal{C}$, since $\mathcal{C}$ is closed under composition, and $\mathcal{C}\mathcal{C} \supseteq \mathcal{C}\{\mathrm{id}\} = \mathcal{C}$, since $\mathcal{C}$ contains the projections. Therefore, every clone is idempotent, and this means that for $m \geq 3$ it is probably a hopelessly difficult task to describe the idempotents of $\mathbf{E}_m$. However, for $m = 2$ the idempotents have been described in [21]. Here we recall some of these results that we will use in Section 3.

It follows from Proposition 1.1 that $\mathbf{I}_2$ is closed under arbitrary intersections (we allow the empty class), hence it is a complete lattice[1]. For any clone $\mathcal{C} \subseteq \Omega$, we define the set

$$\mathbf{I}(\mathcal{C}) := \{\mathcal{K} \in \mathbf{I}_2 : [\mathcal{K}] = \mathcal{C}\}.$$

Here $[\mathcal{K}]$ stands for the clone generated by $\mathcal{K}$, i.e., the least clone containing $\mathcal{K}$. For each clone $\mathcal{C}$, the set $\mathbf{I}(\mathcal{C})$ turns out to be an interval in the lattice $\mathbf{I}_2$, whose greatest element is $\mathcal{C}$, which is obviously the only clone in $\mathbf{I}(\mathcal{C})$. Clearly, these intervals form a partition of $\mathbf{I}_2$, hence in order to determine all idempotents it suffices to describe $\mathbf{I}(\mathcal{C})$ for every Boolean clone $\mathcal{C}$.

Theorem 1.6 ([21])**.** *Let $\mathcal{C}$ be a Boolean clone. The interval $\mathbf{I}(\mathcal{C})$ is one the following :*

(1) *if $\mathcal{C} = \{\mathrm{id}\}, \{\mathrm{id}, 0\}, \{\mathrm{id}, 1\}, \{\mathrm{id}, 0, 1\}, L, L_{0*}, L_{*1}$, then $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}, \mathcal{C}_=\}$;*
(2) *if $\mathcal{C} = \Omega, \Omega_{0*}, \Omega_{*1}$, then $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}, \mathcal{C}_=, \mathcal{C} \cap N\}$;*[2]
(3) *if $2 \leq k < \infty$, then $k+1 \leq \left|\mathbf{I}\left(U^k\right)\right|, \left|\mathbf{I}\left(W^k\right)\right| < \infty$, whereas $\mathbf{I}(U^\infty)$ and $\mathbf{I}(W^\infty)$ have continuum cardinality;*
(4) *for all other clones, $\mathbf{I}(\mathcal{C})$ only contains $\mathcal{C}$.*

A characterization of the idempotents belonging to $\mathbf{I}\left(U^k\right)$ and $\mathbf{I}\left(W^k\right)$ was also provided in [21]; here we describe only the least elements of these intervals. For every $k \geq 2$, let B^k be the class of functions that strongly satisfy the constraint $\left(\{0,1\}^k \backslash \{\mathbf{1}\}, \{0,1\}^k \backslash \{\mathbf{0}\}\right)$, and let D^k be the dual of B^k. Moreover, let $B^\infty = \bigcap_{k\geq 2} B^k$ and $D^\infty = \bigcap_{k \geq 2} D^k$. Then the least element of $\mathbf{I}\left(W^k\right)$ is B^k and the least element of $\mathbf{I}\left(U^k\right)$ is D^k for $2 \leq k \leq \infty$.

2. Topological properties of $\mathbf{E}_m$

2.1. **The metric on $\mathbf{E}_m$.** For $\mathcal{A} \neq \mathcal{B} \subseteq \mathcal{O}_m$, we define the quantities $m(\mathcal{A}, \mathcal{B})$ and $d(\mathcal{A}, \mathcal{B})$ by

$$m(\mathcal{A}, \mathcal{B}) := \min\{\operatorname{ess} f \mid f \in \mathcal{A} \triangle \mathcal{B}\},$$
$$d(\mathcal{A}, \mathcal{B}) := 2^{-m(\mathcal{A}, \mathcal{B})},$$

and we put $m(\mathcal{A}, \mathcal{A}) = \infty$ and $d(\mathcal{A}, \mathcal{A}) = 0$ for all $\mathcal{A} \subseteq \mathcal{O}_m$. (Here $\mathcal{A} \triangle \mathcal{B}$ denotes the symmetric difference of the sets $\mathcal{A}$ and $\mathcal{B}$.) It is straightforward to verify that d is an ultrametric on $\mathcal{P}(\mathcal{O}_m)$, the power set of $\mathcal{O}_m$ (i.e., a metric satisfying the strong triangle inequality $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ for all $x, y, z \in \mathcal{P}(\mathcal{O}_m)$). Intuitively, two classes are close to each other, if they coincide up to a large essential arity.

Theorem 2.1. *The metric space $(\mathbf{E}_m, d)$ is compact.*

Proof. To prove compactness, we will interpret equational classes as sequences of $\equiv$-classes of functions, and embed $(\mathbf{E}_m, d)$ into a compact product space.

Equivalent functions have the same essential arity, thus we can speak of the essential arity of an $\equiv$-class. Let us denote the set of all essentially n-ary

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[1] We consider the inclusion as the ordering on $\mathbf{I}_2$, and not the natural ordering defined by $\mathcal{K} \leq \mathcal{K}' \iff \mathcal{K} = \mathcal{K}\mathcal{K}' = \mathcal{K}'\mathcal{K}$.

[2] Let us note that the class of reflexive functions was denoted by $\mathcal{R}$ in [21]. In order to avoid confusion with Green's $\mathcal{R}$ relation, we use the notation N here.

equivalence classes by $\mathcal{E}_m^{(n)}$. Clearly, $\mathcal{E}_m^{(n)}$ is a finite set, since its cardinality is bounded by the number of n-ary functions on $[m]$.

Let us say that $\mathcal{K} \subseteq \mathcal{O}_m$ is *saturated*, if it is a union of $\equiv$-classes, and let $\mathbf{S}$ denote the set of all saturated subsets of $\mathcal{O}_m$. Note that every equational class is saturated. A saturated set $\mathcal{K} \subseteq \mathcal{O}_m$ can be naturally identified with a sequence $\{\mathcal{K}_n\}_{n\geq 0}$, where $\mathcal{K}_n \subseteq \mathcal{E}_m^{(n)}$ is the set of essentially n-ary $\equiv$-classes contained in $\mathcal{K}$. This identification gives rise to a bijection between $\mathbf{S}$ and $\prod_{n=0}^{\infty} \mathcal{P}(\mathcal{E}_m^{(n)})$. It is easy to see that this bijection is a homeomorphism between $\mathbf{S}$, equipped with the topology induced by the metric d, and the product space $\prod_{n=0}^{\infty} \mathcal{P}(\mathcal{E}_m^{(n)})$, equipped with the product of the discrete topologies on each $\mathcal{P}(\mathcal{E}_m^{(n)})$. Since each $\mathcal{E}_m^{(n)}$ is finite, this product space is compact by Tychonoff's theorem, hence $\mathbf{S}$ is also compact. Therefore it only remains to prove that $\mathbf{E}_m$ is a closed subset of $\mathbf{S}$. We shall see that, in fact, $\mathbf{E}_m$ is closed in $\mathcal{P}(\mathcal{O}_m)$.

Let $\mathcal{K}$ be any set of functions that is not an equational class. We will prove that there is an open ball around $\mathcal{K}$ that is contained in $\mathcal{P}(\mathcal{O}_m) \setminus \mathbf{E}_m$. Since $\mathcal{K}$ is not an equational class, there exist $f, g \in \mathcal{O}_m$ such that $f \in \mathcal{K}$ and $g \preceq f$ (and thus $\operatorname{ess} g \leq \operatorname{ess} f$), but $g \notin \mathcal{K}$. Let $n = \operatorname{ess} f$, and let us consider the open ball of radius 2^{-n} centered at $\mathcal{K}$. Let $\mathcal{A} \subseteq \mathcal{O}_m$ be an arbitrary class in this ball, that is, such that $d(\mathcal{K}, \mathcal{A}) < 2^{-n}$. Then $m(\mathcal{K}, \mathcal{A}) > n$, i.e., $\mathcal{K}$ and $\mathcal{A}$ coincide up to essential arity n. In particular, we have $f \in \mathcal{A}$ and $g \notin \mathcal{A}$, which implies that $\mathcal{A}$ is not an equational class. Therefore, $\mathcal{P}(\mathcal{O}_m) \setminus \mathbf{E}_m$ is an open set and, hence, $\mathbf{E}_m$ is closed. $\square$

As it turns out, the set of those classes $\mathcal{K} \in \mathbf{E}_m$ that are not closed under composition constitutes an open subset of $(\mathbf{E}_m, d)$. Indeed, if $\mathcal{K} \in \mathbf{E}_m$ is not closed under composition, then there exist $f \in \mathcal{K}^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{K}^{(k)}$ such that $h := f(g_1, \ldots, g_n) \notin \mathcal{K}$. Let $\mathcal{K}'$ be any equational class in the open ball of radius $2^{-\max\{n,k\}}$ centered at $\mathcal{K}$. Then $m(\mathcal{K}, \mathcal{K}') > \max\{n, k\}$, i.e., $\mathcal{K}$ and $\mathcal{K}'$ coincide up to essential arity $\max\{n, k\}$. Therefore, we have $f, g_1, \ldots, g_n \in \mathcal{K}'$ and $h \notin \mathcal{K}'$, and hence $\mathcal{K}'$ is not closed under composition. This shows that the set of all equational classes that are not closed under composition forms an open set, and thus we have the following result.

Proposition 2.2. *The set of composition-closed equational classes is a closed subset of $\mathbf{E}_m$, hence it is compact.*

Remark 2.3. A similar metric (which induces the same topology) was considered in [14] for clones, and it has been shown that the resulting "clone space" is compact. We have seen in Proposition 2.2 that the space of composition-closed equational classes is compact. Clones are just the composition-closed equational classes that contain the projections, hence we also obtain the compactness of the clone space from the above results.

2.2. **Finitely generated equational classes.** For $f \in \mathcal{O}_m$, let $\downarrow f$ denote the principal ideal generated by f in the simple minor quasiorder, i.e., $\downarrow f := \{g \in \mathcal{O}_m : g \preceq f\}$. Observe that $\downarrow f$ is the least equational class containing f. We say that an equational class $\mathcal{K}$ is *finitely generated* if there exist $t \geq 0, f_1, \ldots, f_t \in \mathcal{O}_m$ such that $\mathcal{K} = \downarrow f_1 \cup \cdots \cup \downarrow f_t$. In this case $\mathcal{K}$ is the least equational class containing $\{f_1, \ldots, f_t\}$. For an equational class $\mathcal{K}$, let

$\deg \mathcal{K} = \max\{\operatorname{ess} f : f \in \mathcal{K}\}$, if this maximum exists, and let $\deg \mathcal{K} = \infty$ otherwise. Clearly, $\mathcal{K} \in \mathbf{E}_m$ is finitely generated if and only if $\mathcal{K}$ contains, up to equivalence, only finitely many functions. From this it follows that $\mathcal{K}$ is finitely generated if and only if $\deg \mathcal{K} < \infty$.

As mentioned in Subsection 1.2, a class of functions is an equational class if and only if it is definable by functional equations. It has been proved in [10] that finitely generated equational classes can be defined by finitely many functional equations. The topological counterpart of this notion is that of being isolated: we say that $\mathcal{K} \in \mathbf{E}_m$ is *isolated*, if $\{\mathcal{K}\}$ is an open set in the topological space $\mathbf{E}_m$, i.e., if $\mathcal{K}$ has an open neighborhood in $\mathcal{P}(\mathcal{O}_m)$ that contains no equational class other than $\mathcal{K}$.

Theorem 2.4. *An equational class $\mathcal{K} \subseteq \mathcal{O}_m$ is finitely generated if and only if it is isolated.*

Proof. Let us assume first that $\mathcal{K}$ is finitely generated, i.e., $d := \deg \mathcal{K} < \infty$. We will show that the open ball of radius $2^{-(d+m)}$ around $\mathcal{K}$ contains no other equational class than $\mathcal{K}$. Suppose for the sake of a contradiction that there exists $\mathcal{K}' \in \mathbf{E}_m$ with $m(\mathcal{K}, \mathcal{K}') > d + m$ and $\mathcal{K}' \neq \mathcal{K}$. Then $\mathcal{K}'$ and $\mathcal{K}$ coincide up to essential arity $d+m$, therefore we have $\mathcal{K} \subseteq \mathcal{K}'$ (as all members of $\mathcal{K}$ are essentially at most d-ary). Since $\mathcal{K}' \neq \mathcal{K}$, it follows that $\mathcal{K}' \setminus \mathcal{K} \neq \emptyset$. Let us choose $f \in \mathcal{K}' \setminus \mathcal{K}$ of minimum essential arity. If g is any proper simple minor of f, then $g \in \mathcal{K}$ by the minimality of $\operatorname{ess} f$, and hence $\operatorname{ess} g \leq \deg \mathcal{K} = d$. On the other hand, we have $\operatorname{ess} f > d + m$, and thus the *arity gap* $\operatorname{gap} f := \min\{\operatorname{ess} f - \operatorname{ess} g \colon g \prec f\}$ of f is greater than m. This contradicts the fact that the arity gap of any function of several variables defined on an m-element set is at most m (see [8]).

Now assume that $\mathcal{K}$ is not finitely generated, i.e., $\deg \mathcal{K} = \infty$. For any $n \geq 0$, let $\mathcal{K}_n = \{f \in K \colon \operatorname{ess} f \leq n\}$. From $\deg \mathcal{K} = \infty$, it follows that $\mathcal{K}_n \neq \mathcal{K}$ for all $n \geq 0$. Moreover, it is clear that $d(\mathcal{K}, \mathcal{K}_n) < 2^{-n}$, and thus that $\mathcal{K}$ is not isolated. $\square$

Remark 2.5. It was shown in [14] that if a clone $\mathcal{C}$ is isolated in the clone space, then $\mathcal{C}$ is a finitely generated clone, i.e., there is a finite set $F \subseteq \mathcal{O}_m$ such that $\mathcal{C} = [F]$. Observe that if $\mathcal{C}$ is a finitely generated equational class, then $\mathcal{C}$ is a finitely generated clone, but the converse is not true.

Theorem 2.6. *Finitely generated equational classes constitute a dense subsemigroup of $\mathbf{E}_m$.*

Proof. We have seen in the second part of the proof of Theorem 2.4 that if $\mathcal{K} \in \mathbf{E}_m$ is not finitely generated, then there exists a sequence $\{\mathcal{K}_n\}_{n \geq 0}$ of finitely generated equational classes $\mathcal{K}_n$ such that $\mathcal{K}_n \to \mathcal{K}$. This shows that the set of finitely generated equational classes is dense in $\mathbf{E}_m$.

In order to prove that they form a subsemigroup, let us consider two arbitrary finitely generated equational classes $\mathcal{K}$ and $\mathcal{K}'$ with $\deg \mathcal{K} = r$ and $\deg \mathcal{K}' = s$. Any function $h \in \mathcal{K}\mathcal{K}'$ can be written in the form $h = f(g_1, \ldots, g_n)$ with $f \in \mathcal{K}$ and $g_1, \ldots, g_n \in \mathcal{K}'$. Moreover, we may assume without loss of generality that f depends on all of its variables, i.e., $\operatorname{ess} f = n$. If a variable is inessential in all of the inner functions $g_1, \ldots, g_n$, then it is also inessential in the composite function h. Thus we have $\operatorname{Ess} h \subseteq \operatorname{Ess} g_1 \cup \cdots \cup \operatorname{Ess} g_n$, and this yields the estimate $\operatorname{ess} h \leq \operatorname{ess} g_1 + \cdots + \operatorname{ess} g_n \leq n \cdot s \leq r \cdot s$. This proves that $\deg \mathcal{K}\mathcal{K}' \leq r \cdot s$. Hence, $\mathcal{K}\mathcal{K}'$ is indeed finitely generated. $\square$

In general, the estimate $\deg \mathcal{K}\mathcal{K}' \leq \deg \mathcal{K} \cdot \deg \mathcal{K}'$ that has been established in the above proof is not sharp. As an example, let $\mathcal{K}$ be the clone of term functions of a rectangular band. Then we have $\mathcal{K}\mathcal{K} = \mathcal{K}$ and $\deg \mathcal{K} = 2$, thus $2 = \deg \mathcal{K}\mathcal{K} < \deg \mathcal{K} \cdot \deg \mathcal{K} = 4$. However, it is noteworthy to observe that for Boolean functions (i.e., when $m = 2$) we always have an equality.

Theorem 2.7. *For any equational classes* $\mathcal{K}, \mathcal{K}' \in \mathbf{E}_2$*, we have* $\deg \mathcal{K}\mathcal{K}' = \deg \mathcal{K} \cdot \deg \mathcal{K}'$.

Proof. When dealing with non-finitely generated classes, we use the conventions $\infty \cdot 0 = 0 \cdot \infty = 0$ and $n \cdot \infty = \infty \cdot n = \infty \cdot \infty = \infty$ for all $n \geq 1$. We will also need the following notation: for $\mathbf{x} \in [2]^n$, $1 \leq i \leq n$, and $a \in [2]$, let $\mathbf{x}_i^a$ be the n-tuple whose j-th component is x_j if $j \neq i$, and a if $j = i$.

Let $\mathcal{K}, \mathcal{K}' \in \mathbf{E}_2$ with $\deg \mathcal{K} = n$ and $\deg \mathcal{K}' = k$, where $n, k \in \{0, 1, \ldots, \infty\}$. If either $n = 0$ or $k = 0$, then $\deg \mathcal{K}\mathcal{K}' = 0 = \deg \mathcal{K} \cdot \deg \mathcal{K}'$ holds trivially, so we will assume that $n, k \geq 1$.

Suppose first that $\mathcal{K}$ and $\mathcal{K}'$ are both finitely generated, i.e., $n, k < \infty$. We have seen in the proof of Theorem 2.6 that $\deg \mathcal{K}\mathcal{K}' \leq n \cdot k$. In order to prove that $\deg \mathcal{K}, \mathcal{K}' \geq n \cdot k$, let us fix $f \in \mathcal{K}^{(n)}$ and $g \in \mathcal{K}'^{(k)}$ such that $\operatorname{ess} f = n$ and $\operatorname{ess} g = k$. Since f and g depend on all of their variables, for each $1 \leq i \leq n$ and $1 \leq j \leq k$ there exist $\mathbf{a} \in [2]^n$ and $\mathbf{b} \in [2]^k$ such that $f(\mathbf{a}_i^0) \neq f(\mathbf{a}_i^1)$ and $g(\mathbf{b}_j^0) \neq g(\mathbf{b}_j^1)$. In particular, the range of both f and g is $[2]$.

For $0 \leq i \leq n-1$, let $g_i \colon [2]^{n \cdot k} \to [2]$ be defined by

$$g_i(x_1, \ldots, x_{ik}, x_{ik+1}, \ldots, x_{(i+1)k}, x_{(i+1)k+1}, \ldots, x_{nk}) = g(x_{ik+1}, \ldots, x_{(i+1)k}).$$

Consider the function $h \colon [2]^{n \cdot k} \to [2]$ given by $h = f(g_0, \ldots, g_{n-1})$. Observe that $g_i \equiv g$, hence $g_i \in \mathcal{K}'$ for all $0 \leq i \leq n-1$, therefore $h \in \mathcal{K}\mathcal{K}'$. We show that x_1 is an essential variable of h.

Let $\mathbf{a} = (a_1, \ldots, a_n) \in [2]^n$ and $\mathbf{b} \in [2]^k$ such that $f(\mathbf{a}_1^0) \neq f(\mathbf{a}_1^1)$ and $g(\mathbf{b}_1^u) = 0 \neq 1 = g(\mathbf{b}_1^v)$, for some $u, v \in [2]$. (Note that $\mathbf{b}_1^u$ denotes the k-tuple obtained from $\mathbf{b}$ by substituting its first component by u.) For each $2 \leq i \leq n$, let $\mathbf{b}_i \in [2]^k$ be such that $g(\mathbf{b}_i) = a_i$, and define $\mathbf{c} := (\mathbf{b}, \mathbf{b}_2, \cdots, \mathbf{b}_n)$. By construction, we have

$$h(\mathbf{c}_1^u) = f(\mathbf{a}_1^0) \neq f(\mathbf{a}_1^1) = h(\mathbf{c}_1^v).$$

This shows that x_1 is essential in h. Similarly, it can be shown that the remaining variables x_i, $1 \leq i \leq nk$, are also essential in h, hence $\operatorname{ess} h = n \cdot k$. This proves that $\deg \mathcal{K}\mathcal{K}' \geq n \cdot k$ as claimed.

Now let us assume that $\deg \mathcal{K} = \infty$ and $\deg \mathcal{K}' \in \{1, 2, \ldots, \infty\}$. Since $\deg \mathcal{K} = \infty$, for any $n \geq 0$ we can find a function $f \in \mathcal{K}$ with $\operatorname{ess} f \geq n$. Let us choose $g \in \mathcal{K}$ with $\operatorname{ess} g = k > 0$, and let us construct the function $h \in \mathcal{K}\mathcal{K}'$ as above. We have seen that $\operatorname{ess} h = n \cdot k$ and, hence, $\deg \mathcal{K}\mathcal{K}' \geq n \cdot k$. Since this holds for all $n \geq 0$, we have that $\deg \mathcal{K}\mathcal{K}' = \infty = \deg \mathcal{K} \cdot \deg \mathcal{K}'$. If $\deg \mathcal{K}' = \infty$, then we can proceed similarly, by letting f be any nonconstant function in $\mathcal{K}$ and by choosing functions $g_i \in \mathcal{K}'$ with unbounded essential arities. $\square$

2.3. **Continuity of composition.** In this subsection we will prove that composition of equational classes is continuous with respect to the topology induced by the metric d. By making use of the compactness of $\mathbf{E}_m$ established in Theorem 2.1, we will show that $\mathbf{E}_m$ is a profinite semigroup. The proof will essentially rely on the following two estimates.

Lemma 2.8. *For all $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 \in \mathbf{E}_m$, we have*

$$m(\mathcal{K}\mathcal{K}_1, \mathcal{K}\mathcal{K}_2) \geq m(\mathcal{K}_1, \mathcal{K}_2),$$
$$d(\mathcal{K}\mathcal{K}_1, \mathcal{K}\mathcal{K}_2) \leq d(\mathcal{K}_1, \mathcal{K}_2).$$

Proof. The second inequality is an immediate consequence of the first. Let $u = f(g_1, \ldots, g_n) \in \mathcal{K}\mathcal{K}_1$ be such that $\operatorname{ess} u < m(\mathcal{K}_1, \mathcal{K}_2)$. We have to show that $u \in \mathcal{K}\mathcal{K}_2$. Whenever a variable is inessential in u, we may identify that variable with another variable in every g_i, without changing the value of the function u. In this way we can replace each g_i with some g_i' such that $\operatorname{ess} g_i' \leq \operatorname{ess} u$, $g_i' \preceq g_i$ and $u \equiv f(g_1', \ldots, g_n')$. Since $\mathcal{K}_1$ is an equational class, $g_i' \in \mathcal{K}_1$, and since $\operatorname{ess} g_i' < m(\mathcal{K}_1, \mathcal{K}_2)$, we have $g_i' \in \mathcal{K}_2$ for $i = 1, 2, \ldots, n$. Thus $u \equiv f(g_1', \ldots, g_n') \in \mathcal{K}\mathcal{K}_2$. □

Lemma 2.9. *If $\mathcal{K} = \downarrow h_1 \cup \downarrow h_2 \cup \cdots \cup \downarrow h_t$ and $k = \max\{\operatorname{ess} h_i \mid 1 \leq i \leq t\}$, then for all $\mathcal{K}_1, \mathcal{K}_2 \in \mathbf{E}_m$ we have*

$$m(\mathcal{K}_1\mathcal{K}, \mathcal{K}_2\mathcal{K}) \geq \sqrt[k]{\frac{m(\mathcal{K}_1, \mathcal{K}_2)}{t}},$$
$$d(\mathcal{K}_1\mathcal{K}, \mathcal{K}_2\mathcal{K}) \leq 2^{-\sqrt[k]{\frac{-\log_2 d(\mathcal{K}_1, \mathcal{K}_2)}{t}}}.$$

Proof. We prove only the first inequality, since the second follows from the first one using the definition of d. Let $u = f(g_1, \ldots, g_n) \in \mathcal{K}_1\mathcal{K}$ be such that $l = \operatorname{ess} u < \sqrt[k]{\frac{m(\mathcal{K}_1, \mathcal{K}_2)}{t}}$. We have to show that $u \in \mathcal{K}_2\mathcal{K}$.

Let us suppose that u depends on all of its variables, and let us denote these variables by $x_1, x_2, \ldots, x_l$. Each of the inner functions g_i is a simple minor of one of $h_1, \ldots, h_t$, i.e., they are of the form

$$h_j(z_1, z_2, \ldots, z_r) \text{ where } 1 \leq j \leq t,\ z_1, z_2, \ldots, z_r \in \{x_1, x_2, \ldots, x_l\},\ r \leq k.$$

The number of such functions is at most $t \cdot l^k$, so we can index them by (some of) the numbers $1, 2, \ldots, t \cdot l^k$. Let $v_1, v_2, \ldots$ be the list of these functions. Let s_i be the number corresponding to the function g_i $(i = 1, 2, \ldots, n)$, i.e., $g_i = v_{s_i}$, and let f' be the $(t \cdot l^k)$-ary function defined by

$$f'(x_1, \ldots, x_{t \cdot l^k}) = f(x_{s_1}, x_{s_2}, \ldots, x_{s_n}).$$

Since $f' \preceq f$ and $f \in \mathcal{K}_1$, we have $f' \in \mathcal{K}_1$. Also $\operatorname{ess} f' \leq t \cdot l^k < m(\mathcal{K}_1, \mathcal{K}_2)$, and thus $f' \in \mathcal{K}_2$. Since u is equivalent to $f'(v_1, v_2, \ldots)$ and each v_i belongs to $\mathcal{K}$, we have that $u \in \mathcal{K}_2\mathcal{K}$. □

From Lemma 2.8 we see that multiplication of equational classes is left continuous in the sense that $\mathcal{K}_n \to \mathcal{K}_0$ implies $\mathcal{K}\mathcal{K}_n \to \mathcal{K}\mathcal{K}_0$. In the next proposition we prove the analogous right continuity property using Lemma 2.9.

Proposition 2.10. *Let $(\mathcal{K}_n)_{n \geq 1}$ be a sequence in $\mathbf{E}_m$ converging to $\mathcal{K}_0 \in \mathbf{E}_m$. Then, for any $\mathcal{K} \in \mathbf{E}_m$, the sequence $\mathcal{K}_n\mathcal{K}$ converges to $\mathcal{K}_0\mathcal{K}$.*

Proof. Let us fix an arbitrary $\varepsilon > 0$. By Theorem 2.6, the set of finitely generated equational classes form a dense subset of $\mathbf{E}_m$, and thus there exists a finitely generated equational class $\mathcal{K}'$ such that $d(\mathcal{K}', \mathcal{K}) < \varepsilon$. Using the ultrametric inequality we have that for each $n \geq 1$

$$d(\mathcal{K}_n\mathcal{K}, \mathcal{K}_0\mathcal{K}) \leq \max\{d(\mathcal{K}_n\mathcal{K}, \mathcal{K}_n\mathcal{K}'), d(\mathcal{K}_n\mathcal{K}', \mathcal{K}_0\mathcal{K}'), d(\mathcal{K}_0\mathcal{K}', \mathcal{K}_0\mathcal{K})\}. \tag{2}$$

From Lemma 2.8 it follows that the first and the last terms are at most $d(\mathcal{K}', \mathcal{K})$ and thus less than ε. We can apply the estimate of Lemma 2.9 to the middle term, as $\mathcal{K}'$ is finitely generated. Therefore, we obtain $d(\mathcal{K}_n\mathcal{K}', \mathcal{K}_0\mathcal{K}') \to 0$, since $d(\mathcal{K}_n, \mathcal{K}_0) \to 0$. Thus, there is a natural number N such that for all $n > N$ we have $d(\mathcal{K}_n\mathcal{K}', \mathcal{K}_0\mathcal{K}') < 0$. From (2) we have then $d(\mathcal{K}_n\mathcal{K}, \mathcal{K}_0\mathcal{K}) < \varepsilon$ for all $n > N$, and this proves that $\mathcal{K}_n\mathcal{K} \to \mathcal{K}_0\mathcal{K}$. □

Theorem 2.11. *Composition of equational classes is a continuous operation, i.e., $\mathbf{E}_m$ is a topological semigroup.*

Proof. Let us assume that $\mathcal{K}_n \to \mathcal{K}$ and $\mathcal{K}'_n \to \mathcal{K}'$ in $\mathbf{E}_m$, and let us fix $\varepsilon > 0$. By Proposition 2.10, $\mathcal{K}_n\mathcal{K}' \to \mathcal{K}\mathcal{K}'$. Hence there exists a natural number N such that $d(\mathcal{K}_n\mathcal{K}', \mathcal{K}\mathcal{K}') < \varepsilon$ for all $n > N$. Since $\mathcal{K}'_n \to \mathcal{K}'$, there also exists a natural number N' such that $d(\mathcal{K}'_n, \mathcal{K}') < \varepsilon$ for all $n > N'$.

Using the ultrametric inequality we get the following upper bound:

$$d(\mathcal{K}_n\mathcal{K}'_n, \mathcal{K}\mathcal{K}') \leq \max\{d(\mathcal{K}_n\mathcal{K}'_n, \mathcal{K}_n\mathcal{K}'), d(\mathcal{K}_n\mathcal{K}', \mathcal{K}\mathcal{K}')\}. \tag{3}$$

Let $n > \max\{N, N'\}$. Then $n > N$, and hence $d(\mathcal{K}_n\mathcal{K}', \mathcal{K}\mathcal{K}') < \varepsilon$. By Lemma 2.8, we have

$$d(\mathcal{K}_n\mathcal{K}'_n, \mathcal{K}_n\mathcal{K}') \leq d(\mathcal{K}'_n, \mathcal{K}'),$$

and the latter is less than ε, since $n > N'$. Thus $d(\mathcal{K}_n\mathcal{K}'_n, \mathcal{K}\mathcal{K}') < \varepsilon$ whenever $n > \max\{N, N'\}$, and this shows that $\mathcal{K}_n\mathcal{K}'_n \to \mathcal{K}\mathcal{K}'$. □

Now as a metric space, $\mathbf{E}_m$ is obviously Hausdorff. Moreover, $\mathbf{E}_m$ is also compact by Theorem 2.1. From the ultrametric inequality it follows that each ball of this topological space is clopen, and hence $\mathbf{E}_m$ is a zero-dimensional space. Consequently, we have the following corollary (for further background see, e.g., [2]).

Corollary 2.12. *The topological semigroup $\mathbf{E}_m$ is profinite.*

3. Regular elements of $\mathbf{E}_2$

One of the fundamental tools in the study of a semigroup is the description of *Green's relations*. Recall that two elements of a semigroup are $\mathcal{L}$-related ($\mathcal{R}$-related, $\mathcal{J}$-related) if they generate the same left ideal (right ideal, two-sided ideal, respectively). In particular, equational classes $\mathcal{K}_1, \mathcal{K}_2 \in \mathbf{E}_2$ are $\mathcal{L}$-related if and only if there exist $\mathcal{K}', \mathcal{K}'' \in \mathbf{E}_2$ such that $\mathcal{K}_1 = \mathcal{K}'\mathcal{K}_2$ and $\mathcal{K}_2 = \mathcal{K}''\mathcal{K}_1$. Similarly, $\mathcal{K}_1, \mathcal{K}_2 \in \mathbf{E}_2$ are $\mathcal{R}$-related if and only if there exist $\mathcal{K}', \mathcal{K}'' \in \mathbf{E}_2$ such that $\mathcal{K}_1 = \mathcal{K}_2\mathcal{K}'$ and $\mathcal{K}_2 = \mathcal{K}_1\mathcal{K}''$. Furthermore, we also have the relations $\mathcal{H}$ and $\mathcal{D}$ that are defined by $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. All of these five relations are equivalence relations; we use the notation $\mathcal{L}_\mathcal{K}$ and $\mathcal{R}_\mathcal{K}$ for the $\mathcal{L}$-class and the $\mathcal{R}$-class of $\mathcal{K}$, respectively. We state Green's lemma only for the relation $\mathcal{R}$ for the semigroup $\mathbf{E}_2$, as we shall need this lemma only in this case. For further background see, e.g., [1, 18].

Green's lemma. *If $\mathcal{K}_1, \mathcal{K}_2 \in \mathbf{E}_2$ are $\mathcal{R}$-related, say $\mathcal{K}_1 = \mathcal{K}_2\mathcal{K}'$ and $\mathcal{K}_2 = \mathcal{K}_1\mathcal{K}''$ for some $\mathcal{K}', \mathcal{K}'' \in \mathbf{E}_2$, then $\mathcal{X} \mapsto \mathcal{X}\mathcal{K}'$ and $\mathcal{X} \mapsto \mathcal{X}\mathcal{K}''$ define mutually inverse bijections between $\mathcal{L}_{\mathcal{K}_1}$ and $\mathcal{L}_{\mathcal{K}_2}$ that preserve the $\mathcal{H}$ relation.*

Another important concept in the investigation of the structure of a semigroup is that of a *regular element*, i.e., elements that are $\mathcal{R}$-related (or, equivalently, $\mathcal{L}$-related) to some idempotent element. As mentioned in Subsection 1.5, the description of the idempotent elements of $\mathbf{E}_m$ does not seem feasible for $m > 2$; however, they have been completely described for $m = 2$ in [21]. As we will see, this result constitutes a key step in classifying the regular elements of $\mathbf{E}_2$.

In this section we will justify the latter claim by providing an explicit description of the regular elements of $\mathbf{E}_2$ and, as a by-product, of the structure of the regular $\mathcal{D}$-classes of $\mathbf{E}_2$.[3]

First we consider classes consisting only of constant functions.

Proposition 3.1. *The empty set forms a singleton $\mathcal{D}$-class. The function classes $\{0\}$, $\{1\}$ and $\{0,1\}$ form an $\mathcal{L}$-class with singleton $\mathcal{R}$- and $\mathcal{H}$-classes. Thus the eggbox pictures of these classes are the following (hereinafter, the grey background color indicates an $\mathcal{H}$-class containing an idempotent element):*

$\emptyset$

$\{0\}$
$\{1\}$
$\{0,1\}$

Proof. The empty class is a two-sided zero element, hence it forms a singleton $\mathcal{D}$-class. The classes $\{0\}, \{1\}$ and $\{0,1\}$ are left zero elements in the subsemigroup of nonempty equational classes, hence each of them is a singleton $\mathcal{R}$-class, and they are all $\mathcal{L}$-equivalent. Moreover, for any nonempty $\mathcal{A} \in \mathbf{E}_2$ we have $\mathcal{A}\{0\} \in \{\{0\}, \{1\}, \{0,1\}\}$, therefore the $\mathcal{L}$-class of $\{0\}$ contains only the three classes $\{0\}, \{1\}$ and $\{0,1\}$. $\square$

In the sequel we discard the trivial cases covered by the above proposition, and we only work with equational classes that contain at least one nonconstant function. It follows from Theorem 2.7 that these classes form a subsemigroup of $\mathbf{E}_2$, which we will denote by $\widetilde{\mathbf{E}}$. Similarly, let us use the notation $\widetilde{\mathbf{I}} = \mathbf{I}_2 \cap \widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{I}}(\mathcal{C}) = \mathbf{I}(\mathcal{C}) \cap \widetilde{\mathbf{E}}$. (We drop the subscript 2 as we only work with Boolean functions in this section.) Note that $\widetilde{\mathbf{I}}(\mathcal{C}) = \mathbf{I}(\mathcal{C})$ for almost all clones, the only exceptions being $\mathcal{C} = \{\mathrm{id}\}$, $\{\mathrm{id}, 0\}$, $\{\mathrm{id}, 1\}$ and $\{\mathrm{id}, 0, 1\}$, for which $\widetilde{\mathbf{I}}(\mathcal{C}) = \{\mathcal{C}\}$, whereas $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}, \mathcal{C}_=\}$.

The following result reveals the relation between an equational class $\mathcal{K}$ and the clone $[\mathcal{K}]$ generated by it.

Proposition 3.2 ([21])**.** *Let $\mathcal{K} \in \widetilde{\mathbf{E}}$ be an idempotent, and let $\mathcal{C} = [\mathcal{K}]$. Then we have $\mathcal{C}\mathcal{K} = \mathcal{K}$ and $\mathcal{K}\mathcal{C} = \mathcal{C}$.*

Since every regular $\mathcal{R}$-class contains an idempotent $\mathcal{K}$ and, by Proposition 3.2, such an idempotent $\mathcal{K}$ is $\mathcal{R}$-equivalent to $[\mathcal{K}]$, we obtain the following result.

[3] Note that $\mathcal{D} = \mathcal{J}$, since $\mathbf{E}_2$ is a compact semigroup.

Proposition 3.3. *Each regular $\mathcal{R}$-class of $\widetilde{\mathbf{E}}$ contains a clone.*

In fact, our next lemma shows that each $\mathcal{R}$-class (and also each $\mathcal{L}$-class) of $\widetilde{\mathbf{E}}$ contains at most one clone.

Lemma 3.4. *If $\mathcal{C}_1, \mathcal{C}_2 \in \widetilde{\mathbf{E}}$ are clones, and $\mathcal{C}_1 \mathrel{\mathcal{R}} \mathcal{C}_2$ or $\mathcal{C}_1 \mathrel{\mathcal{L}} \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2$.*

Proof. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are $\mathcal{L}$-related: $\mathcal{C}_1 = \mathcal{K}'\mathcal{C}_2$ and $\mathcal{C}_2 = \mathcal{K}''\mathcal{C}_1$ for some $\mathcal{K}', \mathcal{K}'' \in \mathbf{E}_2$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are idempotents, it follows that $\mathcal{C}_1\mathcal{C}_2 = \mathcal{K}'\mathcal{C}_2\mathcal{C}_2 = \mathcal{K}'\mathcal{C}_2 = \mathcal{C}_1$ and similarly $\mathcal{C}_2\mathcal{C}_1 = \mathcal{K}''\mathcal{C}_1\mathcal{C}_1 = \mathcal{K}''\mathcal{C}_1 = \mathcal{C}_2$. Since $\mathrm{id} \in \mathcal{C}_1$, we have $\mathcal{C}_1 = \mathcal{C}_1\mathcal{C}_2 \supseteq \mathcal{C}_2$. By exchanging the roles of $\mathcal{C}_1$ and $\mathcal{C}_2$, we obtain $\mathcal{C}_2 \supseteq \mathcal{C}_1$, and thus $\mathcal{C}_1 = \mathcal{C}_2$.

An analogous argument shows that $\mathcal{C}_1 \mathrel{\mathcal{R}} \mathcal{C}_2$ implies $\mathcal{C}_1 = \mathcal{C}_2$. □

According to Proposition 3.3, we can find all regular elements by computing the $\mathcal{R}$-classes of clones. We will need the following technical lemma.

Lemma 3.5. *If $\mathcal{K}, \mathcal{K}' \in \widetilde{\mathbf{E}}$ and $\mathrm{id} \in \mathcal{K}\mathcal{K}'$, then $\mathrm{id} \in \mathcal{K}'$ or $\neg \in \mathcal{K}'$.*

Proof. Suppose for contradiction that $\mathrm{id} \in \mathcal{K}\mathcal{K}'$ but neither id nor $\neg$ belong to $\mathcal{K}'$. Then every unary function in $\mathcal{K}'$ is constant, that is, $\mathcal{K}'^{(1)} \subseteq [2]$. Since $\mathrm{id} \in \mathcal{K}\mathcal{K}'$, there exist $f \in \mathcal{K}^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{K}'^{(k)}$ for some $n, k \geq 1$, such that $f(g_1, \ldots, g_n)$ is a projection, i.e.,

$$f\left(g_1\left(x_1, \ldots, x_k\right), \ldots, g_n\left(x_1, \ldots, x_k\right)\right) = x_j$$

holds identically for some $j \in \{1, \ldots, k\}$. By identifying all the variables, we obtain

$$f\left(g_1\left(x, \ldots, x\right), \ldots, g_n\left(x, \ldots, x\right)\right) = x. \tag{4}$$

For each $1 \leq i \leq n$, the unary function $g_i(x, \ldots, x)$ is a simple minor of $g_i \in \mathcal{K}'$, hence it is a member of $\mathcal{K}'^{(1)}$. Since $\mathcal{K}'^{(1)}$ contains only constant functions, this implies that the left hand side of (4) is constant, which yields the desired contradiction. □

Recall that $\mathcal{R}_{\mathcal{C}}$ and $\mathcal{L}_{\mathcal{C}}$ denote the $\mathcal{R}$-class and $\mathcal{L}$-class, respectively, of $\mathcal{C}$.

Proposition 3.6. *If $\mathcal{C} \in \widetilde{\mathbf{E}}$ is a clone, then $\mathcal{R}_{\mathcal{C}} = \widetilde{\mathbf{I}}(\mathcal{C}) \cup \widetilde{\mathbf{I}}(\mathcal{C})\neg$.*

Proof. From Proposition 3.2 it follows that $\widetilde{\mathbf{I}}(\mathcal{C}) \subseteq \mathcal{R}_{\mathcal{C}}$, and then $\widetilde{\mathbf{I}}(\mathcal{C})\neg \subseteq \mathcal{R}_{\mathcal{C}}$ follows since $\{\neg\}$ is a unit. To prove the inclusion $\mathcal{R}_{\mathcal{C}} \subseteq \widetilde{\mathbf{I}}(\mathcal{C}) \cup \widetilde{\mathbf{I}}(\mathcal{C})\neg$, let us choose an arbitrary $\mathcal{K} \in \mathcal{R}_{\mathcal{C}}$. Then $\mathcal{C} = \mathcal{K}\mathcal{K}'$ and $\mathcal{K} = \mathcal{C}\mathcal{K}''$ for some $\mathcal{K}', \mathcal{K}'' \in \widetilde{\mathbf{E}}$. It follows that $\mathcal{K}\mathcal{K}'\mathcal{K} = \mathcal{C}\mathcal{K} = \mathcal{C}\mathcal{C}\mathcal{K}'' = \mathcal{C}\mathcal{K}'' = \mathcal{K}$, as $\mathcal{C}$ is idempotent. Since $\mathrm{id} \in \mathcal{C} = \mathcal{K}\mathcal{K}'$, we have $\mathrm{id} \in \mathcal{K}'$ or $\neg \in \mathcal{K}'$ by Lemma 3.5.

Let us examine these two cases separately. If $\mathrm{id} \in \mathcal{K}'$, then $\mathcal{K}^2 = \mathcal{K}\,\mathrm{id}\,\mathcal{K} \subseteq \mathcal{K}\mathcal{K}'\mathcal{K} = \mathcal{K}$, hence $\mathcal{K}$ is idempotent by Proposition 1.1. Moreover, $\mathcal{C} \mathrel{\mathcal{R}} \mathcal{K}$ since $\mathcal{K} \in \mathcal{R}_{\mathcal{C}}$, and $\mathcal{K} \mathrel{\mathcal{R}} [\mathcal{K}]$ by Proposition 3.2. By transitivity, we have $\mathcal{C} \mathrel{\mathcal{R}} [\mathcal{K}]$, and then $\mathcal{C} = [\mathcal{K}]$ follows from Lemma 3.4. Thus $\mathcal{K}$ is an idempotent that generates the clone $\mathcal{C}$, hence $\mathcal{K} \in \mathbf{I}(\mathcal{C})$.

If $\neg \in \mathcal{K}'$, then let $\mathcal{K}^* = \mathcal{K}\neg$. Similarly to the previous case, we can show that $\mathcal{K}^*$ is idempotent:

$$(\mathcal{K}^*)^2 = \mathcal{K}\neg\mathcal{K}\neg \subseteq \mathcal{K}\mathcal{K}'\mathcal{K}\neg = \mathcal{K}\neg = \mathcal{K}^*.$$

Also, we have the following $\mathcal{R}$-relations:

$$[\mathcal{K}^*] \mathrel{\mathcal{R}} \mathcal{K}^* \mathrel{\mathcal{R}} \mathcal{K} \mathrel{\mathcal{R}} \mathcal{C}.$$

Hence, $[\mathcal{K}^*] \mathrel{\mathcal{R}} \mathcal{C}$ and, by Lemma 3.4, $[\mathcal{K}^*] = \mathcal{C}$. This means that $\mathcal{K}^* \in \mathbf{I}(\mathcal{C})$ and thus $\mathcal{K} = \mathcal{K}^*\neg \in \mathbf{I}(\mathcal{C})\neg$. □

Theorem 3.7. *The set of regular elements of* $\widetilde{\mathbf{E}}$ *is* $\widetilde{\mathbf{I}} \cup \widetilde{\mathbf{I}}\neg$

Proof. Combine Proposition 3.3 and Proposition 3.6. □

Remark 3.8. Informally, we can say that the regular elements of $\widetilde{\mathbf{E}}$ are exactly the idempotents and the negations of idempotents. We do not have to specify whether we mean negation from the left or from the right, since the left (right) negation of the idempotent $\mathcal{K}$ is the same as the right (left) negation of the idempotent $\mathcal{K}^d = \neg\mathcal{K}\neg$. Indeed, we have

$$\neg\mathcal{K} = (\neg\mathcal{K}\neg)\neg \quad \text{and} \quad \mathcal{K}\neg = \neg(\neg\mathcal{K}\neg).$$

Moreover, since $\mathcal{K} \mapsto \neg\mathcal{K}\neg$ is an automorphism of the semigroup $\widetilde{\mathbf{E}}$, we also have that $\mathcal{K}$ is idempotent if and only if $\mathcal{K}^d$ is idempotent.

Proposition 3.9. *If* $\mathcal{C} \in \widetilde{\mathbf{E}}$ *is a clone, then* $\mathcal{L}_{\mathcal{C}} = \{\mathcal{C}, \neg\mathcal{C}\}$.

Proof. Clearly $\mathcal{C}$ and $\neg\mathcal{C}$ are $\mathcal{L}$-equivalent to $\mathcal{C}$. To see that these are in fact the only ones, let us consider an arbitrary $\mathcal{K} \in \mathcal{L}_{\mathcal{C}}$. Then $\mathcal{K} = \mathcal{K}'\mathcal{C}$ and $\mathcal{C} = \mathcal{K}''\mathcal{K}$ for some $\mathcal{K}', \mathcal{K}'' \in \widetilde{\mathbf{E}}$. Since $\mathrm{id} \in \mathcal{C} = \mathcal{K}''\mathcal{K}$, by Lemma 3.5 we have that $\mathrm{id} \in \mathcal{K}$ or $\neg \in \mathcal{K}$. Moreover, since $\mathcal{K}$ is a regular element, either $\mathcal{K}$ or $\mathcal{K}^* := \neg\mathcal{K}$ is idempotent, according to Theorem 3.7 (see also Remark 3.8). Thus we can separate the following four cases:

1. If $\mathcal{K} \in \widetilde{\mathbf{I}}$ and $\mathrm{id} \in \mathcal{K}$, then $\mathcal{K}$ is a clone, and thus $\mathcal{K} = \mathcal{C}$ by Lemma 3.4.
2. If $\mathcal{K} \in \widetilde{\mathbf{I}}$ and $\neg \in \mathcal{K}$, then $\mathrm{id} = \neg\neg \in \mathcal{K}^2 = \mathcal{K}$, hence $\mathcal{K}$ is a clone and $\mathcal{K} = \mathcal{C}$ as in the previous case.
3. If $\mathcal{K}^* \in \widetilde{\mathbf{I}}$ and $\neg \in \mathcal{K}$, then $\mathrm{id} \in \mathcal{K}^*$, hence $\mathcal{K}^*$ is a clone. Moreover, $\mathcal{K}^* \mathrel{\mathcal{L}} \mathcal{K} \mathrel{\mathcal{L}} \mathcal{C}$. By Lemma 3.4, $\mathcal{K}^* = \mathcal{C}$ and thus $\mathcal{K} = \neg\mathcal{C}$.
4. If $\mathcal{K}^* \in \widetilde{\mathbf{I}}$ and $\mathrm{id} \in \mathcal{K}$, then $\neg \in \mathcal{K}^*$, which implies that $\mathrm{id} \in (\mathcal{K}^*)^2 = \mathcal{K}^*$. Therefore $\mathcal{K}^*$ is a clone, and we have $\mathcal{K} = \neg\mathcal{C}$ just like in the previous case. □

Now we are ready to present the description of the structure of the regular $\mathcal{D}$-classes of $\widetilde{\mathbf{E}}$. The contents of the following theorem are illustrated and summarized in Table 1. (Let us recall that in these eggbox pictures, an $\mathcal{H}$-class has a grey background if it contains an idempotent element.)

Theorem 3.10. *The regular* $\mathcal{D}$*-classes of the semigroup* $\widetilde{\mathbf{E}}$ *are the following:*

(1) *a one-element class* $\{\mathcal{C}\}$ *for a clone* $\mathcal{C} = S, SL, \Omega^{(1)}, \{\mathrm{id}, \neg\}$;
(2) *a two-element class* $\{\mathcal{C}, \mathcal{C}\neg\}$ *that consists of a single* $\mathcal{H}$*-class, for a clone* $\mathcal{C} = \Omega_{01}, M, M_{01}, S_{01}, SM, L_{01}, \{\mathrm{id}, 0, 1\}, \{\mathrm{id}\}$;
(3) *a four-element class* $\{\mathcal{C}, \mathcal{C}\neg, \neg\mathcal{C}, \neg\mathcal{C}\neg\}$ *that consists of two* $\mathcal{R}$*-classes with singleton* $\mathcal{H}$*-classes, for a clone* $\mathcal{C} = M_{0*}, U_{01}^k, MU^k, MU_{01}^k, \Lambda, \Lambda_{0*}, \Lambda_{*1}, \Lambda_{01}, \{\mathrm{id}, 0\}$;
(4) *the class* $\{\Omega, \Omega_=, N\}$ *that consists of one* $\mathcal{R}$*-class with singleton* $\mathcal{H}$*-classes;*

(5) *the class* $\{L, L_=\}$ *that consists of one* $\mathcal{R}$*-class with singleton* $\mathcal{H}$*-classes;*
(6) *the class* $\{\Omega_{0*}, \Omega_{00}, N_{00}, \Omega_{*0}, \Omega_{1*}, \Omega_{11}, N_{11}, \Omega_{*1}\}$ *that consists of two* $\mathcal{R}$*-classes with singleton* $\mathcal{H}$*-classes;*
(7) *the class* $\{L_{0*}, L_{00}, L_{*0}, L_{1*}, L_{11}, L_{*1}\}$ *that consists of two* $\mathcal{R}$*-classes with singleton* $\mathcal{H}$*-classes;*
(8) *the class* $\widetilde{\mathbf{I}}\left(W^k\right) \cup \widetilde{\mathbf{I}}\left(W^k\right) \neg \cup \widetilde{\mathbf{I}}\left(U^k\right) \cup \widetilde{\mathbf{I}}\left(U^k\right) \neg$ *that consists of two* $\mathcal{R}$*-classes with singleton* $\mathcal{H}$*-classes, for* $k = 2, 3, \ldots, \infty$.

Proof. By Proposition 3.3, it suffices to determine the $\mathcal{R}$-classes of clones, and we have seen in Proposition 3.6 that $\mathcal{R}_{\mathcal{C}} = \widetilde{\mathbf{I}}(\mathcal{C}) \cup \widetilde{\mathbf{I}}(\mathcal{C})\neg$ for any clone $\mathcal{C} \in \widetilde{\mathbf{E}}$. From Proposition 3.9 and Green's lemma, we obtain $\mathcal{L}_{\mathcal{K}} = \{\mathcal{K}, \neg\mathcal{K}\}$ for all $\mathcal{K} \in \mathcal{R}_{\mathcal{C}}$, and hence the eggbox picture of a regular $\mathcal{D}$-class is the following:

$$
\underbrace{\overbrace{\begin{array}{|c|c|c|c|}\hline \mathcal{C} & \cdots & \mathcal{K} & \cdots \\ \hline \neg\mathcal{C} & \cdots & \neg\mathcal{K} & \cdots \\ \hline \end{array}}^{\widetilde{\mathbf{I}}(\mathcal{C})}}_{\neg\widetilde{\mathbf{I}}(\mathcal{C})}\underbrace{\overbrace{\begin{array}{|c|c|c|c|}\hline \cdots & \mathcal{K}\neg & \cdots & \mathcal{C}\neg \\ \hline \cdots & \neg\mathcal{K}\neg & \cdots & \neg\mathcal{C}\neg \\ \hline \end{array}}^{\widetilde{\mathbf{I}}(\mathcal{C})\neg}}_{\neg\widetilde{\mathbf{I}}(\mathcal{C})\neg}
$$

Observe the symmetries of this picture: reflection to the vertical axis of the rectangle corresponds to negation from the right, reflection to the horizontal axis (i.e., interchanging the two rows) corresponds to negation from the left, and reflection to the center point of the rectangle corresponds to negation from both sides (i.e., dualizing). Note also that $\neg\widetilde{\mathbf{I}}(\mathcal{C})\neg = \widetilde{\mathbf{I}}(\neg\mathcal{C}\neg) = \widetilde{\mathbf{I}}\left(\mathcal{C}^d\right)$. The above picture shows a general situation, but for certain clones, there may be some coincidences, namely, it is possible that there is only one row, or an $\mathcal{H}$-class contains two elements, or $\widetilde{\mathbf{I}}(\mathcal{C})$ and $\widetilde{\mathbf{I}}(\mathcal{C})\neg$ have nonempty intersection. In the following, we describe explicitly these possible coincidences.

First let us determine the cases when $\mathcal{L}_{\mathcal{C}}$ is a singleton. By Proposition 3.9, this holds if and only if $\mathcal{C} = \neg\mathcal{C}$, and it is easily seen to be equivalent to $\neg \in \mathcal{C}$. These clones form the principal filter generated by the clone $\{\mathrm{id}, \neg\}$ (the grey square with a single outline and an empty interior in Figure 2) in the Post lattice.

The $\mathcal{H}$-class of $\mathcal{C}$ has two elements if and only if $\mathcal{C} \neq \neg\mathcal{C} \in \mathcal{R}_{\mathcal{C}} = \widetilde{\mathbf{I}}(\mathcal{C}) \cup \widetilde{\mathbf{I}}(\mathcal{C})\neg$ (cf. Proposition 3.6). If $\neg\mathcal{C} \in \widetilde{\mathbf{I}}(\mathcal{C})$, then $\neg\mathcal{C}$ is an idempotent that contains the negation, hence $\neg\mathcal{C}$ is a clone. However, the only clone in $\widetilde{\mathbf{I}}(\mathcal{C})$ is $\mathcal{C}$ itself, and we have assumed that $\mathcal{C} \neq \neg\mathcal{C}$. If $\neg\mathcal{C} \in \widetilde{\mathbf{I}}(\mathcal{C})\neg$, then $\mathcal{C} \in \neg\widetilde{\mathbf{I}}(\mathcal{C})\neg = \widetilde{\mathbf{I}}\left(\mathcal{C}^d\right)$, and then $\mathcal{C} = \mathcal{C}^d$, as the only clone in $\widetilde{\mathbf{I}}\left(\mathcal{C}^d\right)$ is $\mathcal{C}^d$. Thus we conclude that the $\mathcal{H}$-class of $\mathcal{C}$ has two elements if and only if $\mathcal{C}^d = \mathcal{C}$ and $\neg \notin \mathcal{C}$. Again, these clones can be easily read from the Post lattice.

The above observations and the description of the intervals $\widetilde{\mathbf{I}}(\mathcal{C})$ (cf. Theorem 1.6) prove all the statements of the theorem. However, since the description of the intervals $\widetilde{\mathbf{I}}\left(W^k\right)$ and $\widetilde{\mathbf{I}}\left(U^k\right)$ is not explicit, we still need to determine $\widetilde{\mathbf{I}}\left(W^k\right) \cap \widetilde{\mathbf{I}}\left(W^k\right)\neg$ and $\widetilde{\mathbf{I}}\left(U^k\right) \cap \widetilde{\mathbf{I}}\left(U^k\right)\neg$ to have a complete picture of the $\mathcal{D}$-class $\mathcal{D}_{W^k} = \mathcal{D}_{U^k}$. So let us consider an arbitrary class $\mathcal{K} \in \widetilde{\mathbf{I}}\left(W^k\right) \cap \widetilde{\mathbf{I}}\left(W^k\right)\neg$. Then we have $\mathcal{K}, \mathcal{K}\neg \in \widetilde{\mathbf{I}}\left(W^k\right)$, and hence $\mathcal{K}, \mathcal{K}\neg \subseteq W^k$. In particular, if $f \in \mathcal{K}$,

$\mathcal{C}$

case (1)

$\mathcal{C}, \mathcal{C}\neg$

case (2)

$\mathcal{C}$	$\mathcal{C}\neg$
$\neg\mathcal{C}$	$\neg\mathcal{C}\neg$

case (3)

Ω	$\Omega_{=}$	N

case (4)

L	$L_{=}$

case (5)

Ω_{0*}	Ω_{00}	N_{00}	Ω_{*0}
Ω_{1*}	Ω_{11}	N_{11}	Ω_{*1}

case (6)

L_{0*}	L_{00}	L_{*0}
L_{1*}	L_{11}	L_{*1}

case (7)

W^k	$\cdots$	B^k	$\cdots$	$W^k\neg$
$U^k\neg$	$\cdots$	D^k	$\cdots$	U^k

case (8)

Table 1. The regular $\mathcal{D}$-classes of $\widetilde{\mathbf{E}}$ (cf. Theorem 3.10)

then both f and $f\neg$ preserve the relation $[2]^k \setminus \{\mathbf{0}\}$.[4] It is easy to see that $f\neg$ preserves $[2]^k \setminus \{\mathbf{0}\}$ if and only if f satisfies the constraint $\left([2]^k \setminus \{\mathbf{1}\}, [2]^k \setminus \{\mathbf{0}\}\right)$. Therefore, f strongly satisfies $\left([2]^k \setminus \{\mathbf{1}\}, [2]^k \setminus \{\mathbf{0}\}\right)$ for all $f \in \mathcal{K}$, i.e., $\mathcal{K} \subseteq B^k$. Since $\mathcal{K} \in \widetilde{\mathbf{I}}\left(W^k\right)$ and B^k is the least element of the interval $\widetilde{\mathbf{I}}\left(W^k\right)$, it follows that $\mathcal{K} = B^k$. Thus the only possible common member of $\widetilde{\mathbf{I}}\left(W^k\right)$ and $\widetilde{\mathbf{I}}\left(W^k\right)\neg$ is B^k. Since $B^k = B^k\neg$, it follows that $\widetilde{\mathbf{I}}\left(W^k\right) \cap \widetilde{\mathbf{I}}\left(W^k\right)\neg = \left\{B^k\right\}$. Dually, we have $\widetilde{\mathbf{I}}\left(U^k\right) \cap \widetilde{\mathbf{I}}\left(U^k\right)\neg = \left\{D^k\right\}$. $\square$

Denote by Ab_2 the pseudovariety of all finite (Abelian) groups of exponent 2. For a pseudovariety H of groups, denote by $\overline{\mathsf{H}}$ the pseudovariety consisting of all finite semigroups whose subgroups lie in H. Taking into account [18, Lemma 3.1.14], we obtain the following result as an immediate application of Theorem 3.10.

Corollary 3.11. *The regular $\mathcal{D}$-classes of finite continuous homomorphic images of $\mathbf{E}_2$ are either groups of order two or contain no nontrivial subgroups. In particular, $\mathbf{E}_2$ is a pro-$\overline{\mathsf{Ab}_2}$ semigroup.*

[4]If $k = \infty$, then this is to be understood as preserving $[2]^k \setminus \{\mathbf{0}\}$ for all $k \geq 2$, and similarly in the rest of the proof.

4. Concluding remarks and future work

In this paper we have initiated the study of the semigroup $\mathbf{E}_m$ of equational classes of functions of several variables defined on an m-element set as a means of obtaining a better understanding of the structure of composition-closed systems in m-valued logic. We have introduced a metric on this semigroup such that the resulting topology is compact, and we have used this topology to prove that $\mathbf{E}_m$ is a profinite semigroup. Moreover, we described the regular elements of $\mathbf{E}_2$ and brought light into the understanding of the structure of its Green's relations.

In this, the description of the idempotents of $\mathbf{E}_2$ (given in [21]) played a key role. Sadly, such a description is out of reach for $m > 2$. Nevertheless, maximal idempotents (with respect to inclusion) of $\mathbf{E}_m$ have been described in [22] in the spirit of Rosenberg's theorem on maximal clones [19], and Rosenberg's five-type classification of minimal clones [20] has been also generalized to composition-closed equational classes.

Finally, let us list some problems that seem relevant for further study of the semigroups $\mathbf{E}_m$.

- Decide whether the regular elements and the regular $\mathcal{D}$-classes of $\mathbf{E}_m$ can be described "modulo clones" for $m > 2$ in a similar manner as they have been described for $m = 2$ in Section 3. Does every $\mathcal{R}$-class contain a clone? Is every member of the $\mathcal{L}$-class of a clone $\mathcal{C}$ obtained by multiplying $\mathcal{C}$ with a unit?
- Determine the maximal subgroups of $\mathbf{E}_m$.
- Knowing that $\mathbf{E}_m$ is profinite, the natural question is to ask for the smallest pseudovariety $\mathbf{V}$ of finite semigroups such that $\mathbf{E}_m$ is a pro-$\mathbf{V}$ semigroup. Seeking the description of this pseudovariety, we come to the problem of determining all finite continuous homomorphic images of $\mathbf{E}_m$.
- Describe the structure of the non-regular $\mathcal{D}$-classes of $\mathbf{E}_2$.

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