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ON THE INTERVAL OF STRONG PARTIAL CLONES OF BOOLEAN FUNCTIONS CONTAINING $\mathrm{Pol}(\{(0,0),(0,1),(1,0)\})$

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Abstract. D. Lau raised the problem of determining the cardinality of the set of all partial clones of Boolean functions whose total part is a given Boolean clone. The key step in the solution of this problem, which was obtained recently by the authors, was to show that the sublattice of strong partial clones on $\{0,1\}$ that contain all total functions preserving the relation $\rho_{0,2} = \{(0,0),(0,1),(1,0)\}$ is of continuum cardinality. In this paper we represent relations derived from $\rho_{0,2}$ in terms of graphs, and we define a suitable closure operator on graphs such that the lattice of closed sets of graphs is isomorphic to the dual of this uncountable sublattice of strong partial clones. With the help of this duality, we provide a rough description of the structure of this lattice, and we also obtain a new proof for its uncountability.

1. Introduction

Let A be a finite non-singleton set. Without loss of generality we assume that $A = \mathbf{k} := \{0,\dots,k-1\}$. For a positive integer n, an *n-ary partial function* on $\mathbf{k}$ is a map $f\colon \mathrm{dom}(f) \to \mathbf{k}$ where $\mathrm{dom}(f)$ is a subset of $\mathbf{k}^n$ called the *domain* of f. If $\mathrm{dom}(f) = \mathbf{k}^n$, then f is a *total function* (or *operation*) on $\mathbf{k}$. Let $\mathrm{Par}^{(n)}(\mathbf{k})$ denote the set of all n-ary partial functions on $\mathbf{k}$ and let $\mathrm{Par}(\mathbf{k}) := \bigcup_{n\geq 1} \mathrm{Par}^{(n)}(\mathbf{k})$. The set of all total operations on $\mathbf{k}$ is denoted by $\mathrm{Op}(\mathbf{k})$.

For $n,m \geq 1$, $f \in \mathrm{Par}^{(n)}(\mathbf{k})$ and $g_1,\dots,g_n \in \mathrm{Par}^{(m)}(\mathbf{k})$, the *composition* of f and $g_1,\dots,g_n$, denoted by $f[g_1,\dots,g_n] \in \mathrm{Par}^{(m)}(\mathbf{k})$, is defined by

$$\mathrm{dom}(f[g_1,\dots,g_n]) := \left\{\mathbf{a} \in \mathbf{k}^m : \mathbf{a} \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i) \text{ and } (g_1(\mathbf{a}),\dots,g_n(\mathbf{a})) \in \mathrm{dom}(f)\right\}$$

and

$$f[g_1,\dots,g_n](\mathbf{a}) := f(g_1(\mathbf{a}),\dots,g_n(\mathbf{a}))$$

for all $\mathbf{a} \in \mathrm{dom}(f[g_1,\dots,g_n])$.

For every positive integer n and each $1 \leq i \leq n$, let e_i^n denote the *n-ary i-th projection function* defined by $e_i^n(a_1,\dots,a_n) = a_i$ for all $(a_1,\dots,a_n) \in \mathbf{k}^n$. Furthermore, let

$$J_{\mathbf{k}} := \{e_i^n : 1 \leq i \leq n\}$$

be the set of all (total) projections on $\mathbf{k}$.

Definition 1.1. A *partial clone* on $\mathbf{k}$ is a composition closed subset of $\mathrm{Par}(\mathbf{k})$ containing $J_{\mathbf{k}}$.

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The partial clones on $\mathbf{k}$, ordered by inclusion, form a complete lattice $\mathcal{L}_{P_{\mathbf{k}}}$ in which the infimum is the set-theoretical intersection. That means that the intersection of an arbitrary family of partial clones on $\mathbf{k}$ is also a partial clone on $\mathbf{k}$.

Examples.

(1) $\Omega_k := \bigcup_{n\geq 1} \{f \in \mathrm{Par}^{(n)}(\mathbf{k}) : \mathrm{dom}(f) \neq \emptyset \Longrightarrow \mathrm{dom}(f) = \mathbf{k}^n\}$ is a partial clone on $\mathbf{k}$.

(2) For $a = 0, 1$ let T_a be the set of all total functions satisfying $f(a, \ldots, a) = a$, let M be the set of all monotone total functions and S be the set of all self-dual total functions on $\mathbf{2} = \{0, 1\}$. Then T_0, T_1, M and S are (total) clones on $\mathbf{2}$.

(3) Let

$$\begin{aligned} T_{0,2} := \{f \in \mathrm{Op}(\mathbf{2}) : [(a_1, b_1) \neq (1,1), \ldots, (a_n, b_n) \neq (1,1)] \\ \Longrightarrow (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \neq (1,1)\}. \end{aligned}$$

Then $T_{0,2}$ is a (total) clone on $\mathbf{2}$.

(4) Let

$$\begin{aligned} \widetilde{S} := \{f \in \mathrm{Par}(\mathbf{2}) : \{(a_1, \ldots, a_n), (\neg a_1, \ldots, \neg a_n)\} \subseteq \mathrm{dom}(f) \\ \Longrightarrow f(\neg a_1, \ldots, \neg a_n) = \neg f(a_1, \ldots, a_n)\}, \end{aligned}$$

where $\neg$ is the negation on $\mathbf{2}$. Then $\widetilde{S}$ is a partial clone on $\mathbf{2}$.

Definition 1.2. For $h \geq 1$, let ρ be an h-ary relation on $\mathbf{k}$ and f be an n-ary partial function on $\mathbf{k}$. We say that f *preserves* ρ if for every $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in \rho$, $(j = 1, \ldots, n)$ and whose rows $M_{i*} \in \mathrm{dom}(f)$ $(i = 1, \ldots, h)$, the h-tuple $(f(M_{1*}), \ldots, f(M_{h*})) \in \rho$. Define

$$\mathrm{pPol}(\rho) := \{f \in \mathrm{Par}(\mathbf{k}) : f \text{ preserves } \rho\}.$$

It is well known that $\mathrm{pPol}\,\rho$ is a partial clone called the *partial clone determined by the relation* ρ. Note that if there is no $h \times n$ matrix $M = [M_{ij}]$ whose columns $M_{*j} \in \rho$ and whose rows $M_{i*} \in \mathrm{dom}(f)$, then $f \in \mathrm{pPol}(\rho)$. We can naturally extend the pPol operator to sets of relations: if $\mathcal{R}$ is a set of relations, then let $\mathrm{pPol}(\mathcal{R}) = \bigcap_{\rho\in\mathcal{R}} \mathrm{pPol}(\rho)$. We denote the total part of $\mathrm{pPol}(\mathcal{R})$ by $\mathrm{Pol}(\mathcal{R})$, i.e., $\mathrm{Pol}(\mathcal{R}) = \mathrm{pPol}(\mathcal{R}) \cap \mathrm{Op}(\mathbf{k})$.

We say that $g \in \mathrm{Par}(\mathbf{k})$ is a *subfunction* of $f \in \mathrm{Par}(\mathbf{k})$ if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and g is the restriction of f to $\mathrm{dom}\, g$.

Definition 1.3. A *strong partial clone* is a partial clone that is closed under taking subfunctions.

A partial clone is strong if and only if it contains all partial projections (subfunctions of projections). For a set $P \subseteq \mathrm{Par}(\mathbf{k})$ we denote the least strong partial clone containing P by $\mathrm{Str}(P)$. Observe that if $C \subseteq \mathrm{Op}(\mathbf{k})$ is a total clone, then $\mathrm{Str}(C)$ is just the set of all subfunctions of members of C. It is easy to see that if a partial function f preserves a relation ρ, then all subfunctions of f also preserve ρ. Thus every partial clone of the form $\mathrm{pPol}(\rho)$ is strong.

In the examples above $T_a = \mathrm{Pol}(\{a\})$, $M = \mathrm{Pol}(\leq)$, $S = \mathrm{Pol}(\neq)$, $T_{0,2} = \mathrm{Pol}(\rho_{0,2})$ and $\widetilde{S} = \mathrm{pPol}(\neq)$, whereas Ω_k is not a strong partial clone. Here, for simplicity, we write $\leq$ for $\{(0,0), (0,1), (1,1)\}$, $\rho_{0,2}$ for $\{(0,0), (0,1), (1,0)\}$ and $\neq$ for $\{(0,1), (1,0)\}$.

The study of partial clones on $\mathbf{2} := \{0, 1\}$ was initiated by R. V. Freivald [8]. Among other things, he showed that the set of all monotone partial functions and the set of all self-dual partial functions are both maximal partial clones on $\mathbf{2}$. In fact, Freivald showed that there are exactly eight maximal partial clones on $\mathbf{2}$. To state Freivald's result, we introduce the following two relations: let

$$R_1 = \{(x,x,y,y) : x,y \in \mathbf{2}\} \cup \{(x,y,y,x) : x,y \in \mathbf{2}\}$$
$$R_2 = R_1 \cup \{(x,y,x,y) : x,y \in \mathbf{2}\}.$$

Theorem 1.4 ([8]). *There are 8 maximal partial clones on* $\mathbf{2}$*:* $\mathrm{pPol}(\{0\})$, $\mathrm{pPol}(\{1\})$, $\mathrm{pPol}(\{(0,1)\})$, $\mathrm{pPol}(\leq)$, $\mathrm{pPol}(\neq)$, $\mathrm{pPol}(R_1)$, $\mathrm{pPol}(R_2)$ *and* Ω_2.

Note that the set of total functions preserving R_2 form the maximal clone of all (total) linear functions over $\mathbf{2}$.

Also interesting is to determine the intersections of maximal partial clones. It is shown in [1] that the set of all partial clones on $\mathbf{2}$ that contain the maximal clone consisting of all *total* linear functions on $\mathbf{2}$ is of continuum cardinality (for details see [1, 11] and Theorem 20.7.13 of [17]). A consequence of this is that the interval of partial clones $[\mathrm{pPol}(R_2) \cap \Omega_2, \mathrm{Par}(\mathbf{2})]$ is of continuum cardinality.

A similar result, (but slightly easier to prove) is established in [10] where it is shown that the interval of partial clones $[\mathrm{pPol}(R_1) \cap \Omega_2, \mathrm{Par}(\mathbf{2})]$ is also of continuum cardinality. Notice that the three maximal partial clones $\mathrm{pPol}\, R_1$, $\mathrm{pPol}\, R_2$ and Ω_2 contain all unary functions (i.e., maps) on $\mathbf{2}$. Such partial clones are called *Słupecki type* partial clones in [11, 21]. These are the only three maximal partial clones of Słupecki type on $\mathbf{2}$.

For a complete study of the pairwise intersections of all maximal partial clones of Słupecki type on a finite non-singleton set $\mathbf{k}$, see [11]. The papers [12, 13, 18, 23, 24] focus on the case $k = 2$ where various interesting, and sometimes hard to obtain, results are established. For instance, the intervals

$$[\mathrm{pPol}(\{0\}) \cap \mathrm{pPol}(\{1\}) \cap \mathrm{pPol}(\{(0,1)\}) \cap \mathrm{pPol}(\leq), \mathrm{Par}(\mathbf{2})]$$

and

$$[\mathrm{pPol}(\{0\}) \cap \mathrm{pPol}(\{1\}) \cap \mathrm{pPol}(\{(0,1)\}) \cap \mathrm{pPol}(\neq), \mathrm{Par}(\mathbf{2})]$$

are shown to be finite and are completely described in [12]. Some of the results in [12] are included in [23, 24] where partial clones on $\mathbf{2}$ are handled via the one point extension approach (see section 20.2 in [17]).

In view of results from [1, 10, 12, 23, 24], it was thought that if $2 \leq i \leq 5$ and $M_1, \ldots, M_i$ are non-Słupecki maximal partial clones on $\mathbf{2}$, then the interval $[M_1 \cap \cdots \cap M_i, \mathrm{Par}(\mathbf{2})]$ is either finite or countably infinite. It was shown in [13] that the interval of partial clones $[\mathrm{pPol}(\leq) \cap \mathrm{pPol}(\neq), \mathrm{Par}(\mathbf{2})]$ is infinite. However, it remained an open problem to determine whether $[\mathrm{pPol}(\leq) \cap \mathrm{pPol}(\neq), \mathrm{Par}(\mathbf{2})]$ is countably or uncountably infinite. This problem was settled in [3]:

Theorem 1.5 ([3]). *The interval of partial clones* $[\mathrm{pPol}(\leq) \cap \mathrm{pPol}(\neq), \mathrm{Par}(\mathbf{2})]$ *that contain the strong partial clone of monotone self-dual partial functions, is of continuum cardinality on* $\mathbf{2}$.

The main construction in proving this result was later adapted in [4] to solve an intrinsically related problem that was first considered by D. Lau [16], and tackled recently by several authors, namely: Given a total clone C on $\mathbf{2}$, describe the interval of all partial clones on $\mathbf{2}$ whose total component is C. Let us introduce a notation for this interval and several variants:

$$\mathcal{I}(C) := \{P \subseteq \mathrm{Par}(\mathbf{2}) \colon P \text{ is a partial clone and } C = P \cap \mathrm{Op}(\mathbf{2})\};$$
$$\mathcal{I}_{\mathrm{Str}}(C) := \{P \subseteq \mathrm{Par}(\mathbf{2}) \colon P \text{ is a strong partial clone and } C = P \cap \mathrm{Op}(\mathbf{2})\};$$
$$\mathcal{I}_{\mathrm{Str}}^{\subseteq}(C) := \{P \subseteq \mathrm{Par}(\mathbf{2}) \colon P \text{ is a strong partial clone and } C \subseteq P \cap \mathrm{Op}(\mathbf{2})\}.$$

In [4] we established a complete classification of all intervals of the form $\mathcal{I}(C)$, for a total clone C on $\mathbf{2}$, and showed that each such $\mathcal{I}(C)$ is either finite or of continuum cardinality. Given the previous results by several authors, the missing case was settled by the following theorem. (Note that $\mathcal{I}(T_{0,2}) \supseteq \mathcal{I}_{\mathrm{Str}}(T_{0,2})$, hence if $\mathcal{I}_{\mathrm{Str}}(T_{0,2})$ has continuum cardinality, then it follows that $\mathcal{I}(T_{0,2})$ is also uncountable.)

Theorem 1.6 ([4]). *The interval of strong partial clones $\mathcal{I}_{\mathrm{Str}}(T_{0,2})$ is of continuum cardinality.*

Lau's problem is equivalent to the problem of determining the cardinalities of intervals of weak relational clones generating a given relational clone (see Subsection 2.1). This problem is important in the study of complexity of constraint satisfaction problems (CSPs), and has been posed in [15].

In this paper we provide an alternative proof of Theorem 1.6 based on a representation of relations that are invariant under $T_{0,2}$ by graphs. By defining an appropriate closure operator on graphs, we will show that there are a continuum of such closed sets of graphs, which in turn are in a one-to-one correspondence with strong partial clones containing $T_{0,2}$. As we will see, this construction will contribute to a better understanding of the structure of this uncountable sublattice of partial clones.

This paper is organized as follows. In Section 2 we recall some basic notions and preliminary results on relations and graphs that will be needed throughout. In Section 3 we introduce a representation of relations by graphs, and we show that the lattice $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ is dually isomorphic to the lattice of classes of graphs that are closed under some natural constructions such as disjoint unions and quotients. Motivated by this duality, in Section 4 and Section 5 we focus on this lattice of closed sets of graphs, and obtain some results about its structure. These results (after dualizing) yield the following information about $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$:

(a) $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ has a two-element chain at the bottom and a three-element chain at the top (Theorem 4.4);
(b) between these chains there is an uncountable "jungle" (see Figure 1), in which there is a continuum of elements below and above every element (Theorems 5.15 and 5.21);
(c) for each $n \in \{0, 1, 2, \dots, \aleph_0\}$, there exist elements in $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ with exactly n lower covers (Theorem 5.13).

This paper is an extended version of the conference paper [5] presented at the 44th IEEE International Symposium on Multiple-Valued Logic, where (a) has been proved as well as a weaker form of (b).

2. Preliminaries

2.1. **Relations.** An n-ary relation $\rho \subseteq \mathbf{k}^n$ over $\mathbf{k}$ can be regarded as a map $\mathbf{k}^n \to \{0, 1\}$, such that $\rho(a_1, \dots, a_n)$ is 1 iff $(a_1, \dots, a_n) \in \rho$. This allows us to speak about inessential coordinates: the i-th coordinate of ρ is *inessential* if the corresponding map $\mathbf{k}^n \to \{0, 1\}$ does not depend on its i-th variable. Sometimes it will be convenient to think of a relation ρ as an $n \times |\rho|$ matrix, whose columns are the tuples belonging to ρ (the order of the columns is irrelevant).

For a set $\mathcal{R}$ of relations, let $\langle \mathcal{R} \rangle_{\nexists}$ denote the set of relations definable by quantifier-free primitive positive formulas over $\mathcal{R} \cup \{\omega_{\mathbf{k}}\}$, where $\omega_{\mathbf{k}} = \{(a, a) \colon a \in \mathbf{k}\}$ is the equality relation on $\mathbf{k}$. Formally, an n-ary relation σ belongs to $\langle \mathcal{R} \rangle_{\nexists}$ if and only if there exist relations $\rho_1, \dots, \rho_t \in \mathcal{R} \cup \{\omega_{\mathbf{k}}\}$ of arities $r_1, \dots, r_t$, respectively, and there are variables $z_i^{(j)} \in \{x_1, x_2, \ldots, x_n\}$ $(j = 1, \ldots, t;\ i = 1, \ldots, r_j)$ such that

$$\sigma(x_1, \ldots, x_n) = \bigwedge_{j=1}^{t} \rho_j\bigl(z_1^{(j)}, \ldots, z_{r_j}^{(j)}\bigr).$$

We say that $\mathcal{R}$ is a *weak relational clone* if $\mathcal{R}$ is closed under quantifier-free primitive positive definability, i.e., $\langle \mathcal{R} \rangle_{\nexists} = \mathcal{R}$. (The terms weak partial co-clone and weak co-clone are also used for this notion.)

Let $\operatorname{Inv}(P)$ denote the set of *invariant relations* of a set $P \subseteq \operatorname{Par}(\mathbf{k})$ of partial functions:

$$\operatorname{Inv}(P) := \{\rho \subseteq \mathbf{k}^n : \rho \text{ is preserved by each } f \in P\}.$$

The operators pPol and Inv give rise to a Galois connection between partial functions and relations, and the corresponding Galois closed classes are strong partial clones and weak relational clones.

Theorem 2.1 ([20])**.** *For any set $P \subseteq \operatorname{Par}(\mathbf{k})$ of partial functions and for any set $\mathcal{R}$ of relations on $\mathbf{k}$, we have*

$$\operatorname{Str}(P) = \operatorname{pPol}\operatorname{Inv}(P) \quad \textit{and} \quad \langle \mathcal{R} \rangle_{\nexists} = \operatorname{Inv}\operatorname{pPol}(\mathcal{R}).$$

Remark 2.2. Theorem 2.1 implies that the lattice of strong partial clones is dually isomorphic to the lattice of weak relational clones. In particular, for any total Boolean clone C, the interval $\mathcal{I}_{\mathrm{Str}}(C)$ is dually isomorphic to the interval $\{\mathcal{R} \colon \langle \mathcal{R} \rangle_{\nexists} = \mathcal{R} \text{ and } \operatorname{Pol}(\mathcal{R}) = C\}$ in the lattice of weak relational clones.

Now we introduce some simple constructions for relations that allow us to give an alternative description of the closure $\langle \mathcal{R} \rangle_{\nexists}$.

- For $\rho \subseteq \mathbf{k}^n$ and $\sigma \subseteq \mathbf{k}^m$, the *direct product* of ρ and σ is the relation $\rho \times \sigma \subseteq \mathbf{k}^{n+m}$ defined by

$$\rho \times \sigma = \bigl\{(a_1, \ldots, a_{n+m}) \in \mathbf{k}^{n+m} \colon (a_1, \ldots, a_n) \in \rho,\ (a_{n+1}, \ldots, a_{n+m}) \in \sigma\bigr\}.$$

- Let $\rho \subseteq \mathbf{k}^n$ and let ε be an equivalence relation on $\{1, 2, \ldots, n\}$. Define $\Delta_\varepsilon(\rho) \subseteq \mathbf{k}^n$ by

$$\Delta_\varepsilon(\rho) = \{(a_1, \ldots, a_n) \in \rho \colon a_i = a_j \text{ whenever } (i, j) \in \varepsilon\}.$$

  We say that $\Delta_\varepsilon(\rho)$ is obtained from ρ by *diagonalization*.
- If two relations ρ and σ, considered as matrices, can be obtained from each other by permuting rows, by adding or deleting repeated rows, and by adding or deleting inessential coordinates, then a partial function f preserves ρ if and only if f preserves σ. In this case we say that ρ and σ are *essentially the same*, and we write $\rho \approx \sigma$. Observe that the relations $\mathbf{k}$ (unary total relation) and $\omega_{\mathbf{k}}$ (binary equality relation) are essentially the same.

The following characterization of weak relational clones is straightforward to verify.

Fact 2.3. *For an arbitrary set $\mathcal{R}$ of relations on $\mathbf{k}$, we have $\langle \mathcal{R} \rangle_{\nexists} = \mathcal{R}$ if and only if the following conditions are satisfied:*

(i) *if $\rho, \sigma \in \mathcal{R}$, then $\rho \times \sigma \in \mathcal{R}$;*
(ii) *if $\rho \in \mathcal{R}$, then $\Delta_\varepsilon(\rho) \in \mathcal{R}$ (for all appropriate equivalence relations ε);*
(iii) *$\mathbf{k} \in \mathcal{R}$ (here $\mathbf{k}$ is the unary total relation);*
(iv) *if $\rho \in \mathcal{R}$ and $\sigma \approx \rho$, then $\sigma \in \mathcal{R}$.*

2.2. **Graphs.** We consider finite undirected graphs without multiple edges. For any graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. An edge $uv \in E(G)$ is called a *loop* if $u = v$. A map $\varphi\colon V(G) \to V(H)$ is a *homomorphism* from G to H if for all $uv \in E(G)$ we have $\varphi(u)\varphi(v) \in E(H)$. We use the notation $G \to H$ to denote the fact that there is a homomorphism from G to H. The *homomorphic image* of G under φ is the subgraph $\varphi(G)$ of H given by $V(\varphi(G)) = \{\varphi(v)\colon v \in V(G)\}$ and $E(\varphi(G)) = \{\varphi(u)\varphi(v)\colon uv \in E(G)\}$. If $\varphi(G)$ is an induced subgraph of H, then we say that φ is a *faithful homomorphism*; this means that every edge of H between two vertices in $\varphi(V(G))$ is the image of an edge of G under φ. If $\varphi\colon G \to H$ is a surjective faithful homomorphism, then φ is said to be a *complete homomorphism.* In this case H is the homomorphic image of G under φ (i.e., $H = \varphi(G)$), and we shall denote this by $G \twoheadrightarrow H$.

If ε is an equivalence relation on the set of vertices $V(G)$ of a graph G, then we can form the *quotient graph* G/ε as follows: the vertices of G/ε are the equivalence classes of ε, and two such equivalence classes C, D are connected by an edge in G/ε if and only if there exist $u \in C, v \in D$ such that $uv \in E(G)$. Note that a vertex of G/ε has no loop if and only if the corresponding equivalence class is an independent set in G (i.e., there are no edges inside this equivalence class in G). There is a canonical correspondence between quotients and homomorphic images: the quotient G/ε is a homomorphic image of G (under the natural homomorphism sending every vertex to the ε-class to which it belongs), and if $\varphi\colon G \twoheadrightarrow H$ is a complete homomorphism, then H is isomorphic to the quotient of G corresponding to the kernel of φ.

For $n \in \mathbb{N}$, the *complete graph* K_n is the graph on n vertices that has no loops but has an edge between any two distinct vertices, i.e.,

$$E(K_n) = \{uv\colon u, v \in V(K_n) \text{ and } u \neq v\}.$$

Note that this defines K_n only up to isomorphism (as the vertex set is not specified). In fact, in the following we will not distinguish between isomorphic graphs. For $n = 1$ we get the graph K_1 consisting of a single isolated vertex. We will denote the one-vertex graph with a loop by L.

The disjoint union of graphs G and H will be denoted by $G \mathbin{\dot\cup} H$. Observe that there exist natural homomorphisms $G \to G \mathbin{\dot\cup} H$ and $H \to G \mathbin{\dot\cup} H$. By $k \cdot G := G \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} G$ we denote the disjoint union of k copies of G.

A homomorphism $G \to K_n$ is a *proper coloring* of G by n colors (regard the vertices of K_n as n different colors; properness means that adjacent vertices of G must receive different colors). The *chromatic number* $\chi(G)$ of a loopless graph is the least number of colors required in a proper coloring of G. Observe that if $G \to H$, then $\chi(G) \leq \chi(H)$, since $G \to H \to K_n$ implies $G \to K_n$ for all natural numbers n. A graph is *bipartite* if and only if $\chi(G) \leq 2$, i.e., G is 2-colorable.

The *girth* of a graph is the length of its shortest cycle (if there is a cycle at all), and the *odd girth* of a graph G is the length of the shortest cycle of odd length in G (if there is an odd cycle at all, i.e., if G is not bipartite). The odd girth can be described in terms of homomorphisms as follows. Let C_n denote the cycle of length n without loops (just like K_n, this graph is defined only up to isomorphism). Then the odd girth of a non-bipartite graph G is the least odd number n such that $C_n \to G$. It follows that if $G \to H$, then the odd girth of H is at most as large as the odd girth of G. P. Erdős has proved that for any pair of natural numbers (k, g) with $k, g \geq 3$ there exists a graph with chromatic number k and girth g [7].

Since the relation $\to$ is reflexive and transitive, it is a quasiorder on the set of all (isomorphism types of) finite graphs. The corresponding equivalence relation is called *homomorphic equivalence*, and factoring out by this equivalence, we obtain the *homomorphism order* of graphs. The above mentioned theorem of Erdős implies that this homomorphism order has infinite width: if G_k is a graph with chromatic number and odd girth equal to $2k+1$ for each $k \in \mathbb{N}$, then $\{G_1, G_2, \dots\}$ is an infinite antichain. The homomorphism order is dense almost everywhere: E. Welzl showed that if G is strictly less than H (that is $G \to H$ and $H \nrightarrow G$), then there exists a graph lying between G and H, except in the case when G and H are homomorphically equivalent to K_1 and K_2, respectively [25].

Let $\mathcal{G}$ denote the set of (isomorphism types of) finite undirected graphs without multiple edges and without isolated vertices. We make one exception to the ban on isolated vertices: we include the one-point graph K_1 in $\mathcal{G}$. We allow loops, and a vertex having a loop is not considered as isolated; in particular, $L \in \mathcal{G}$ (recall that L denotes the graph with a single vertex having a loop). In Section 5 we will work only with loopless non-bipartite graphs, so let us introduce the notation $\mathcal{G}_1$ for the set of loopless non-bipartite members of $\mathcal{G}$. Observe that no graph from $\mathcal{G}_1$ is homomorphically equivalent to K_1 or K_2, hence Welzl's theorem implies that $(\mathcal{G}_1; \to)$ is a dense quasiordered set. We shall need the following strengthening of this density result.

Theorem 2.4 ([19])**.** *If $G, H \in \mathcal{G}_1$ such that $G \to H$ and $H \nrightarrow G$, then there exists an infinite antichain $\{T_1, T_2, \dots\} \subseteq \mathcal{G}_1$ between G and H, i.e., $G \to T_i \to H$ and $T_i \nrightarrow T_j$ for all $i, j \in \mathbb{N}, i \neq j$.*

3. Representing relations in $\langle \rho_{0,2} \rangle_{\nexists}$ by graphs

Recall that $\rho_{0,2}$ is the binary relation $\rho_{0,2} = \{0,1\}^2 \setminus \{(1,1)\}$ on $\mathbf{2}$, and $T_{0,2} = \mathrm{Pol}(\rho_{0,2})$ is the corresponding total clone. The interval $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ is dually isomorphic to the interval $\{\mathcal{R} \colon \langle \mathcal{R} \rangle_{\nexists} = \mathcal{R} \text{ and } T_{0,2} \subseteq \mathrm{Pol}(\mathcal{R})\}$ in the lattice of weak relational clones (cf. Remark 2.2). According to the next proposition, this latter interval is in turn isomorphic the lattice of weak relational subclones of $\langle \rho_{0,2} \rangle_{\nexists}$.

Proposition 3.1. *For any weak relational clone $\mathcal{R}$ on $\mathbf{2}$, we have $T_{0,2} \subseteq \mathrm{Pol}(\mathcal{R})$ if and only if $\mathcal{R} \subseteq \langle \rho_{0,2} \rangle_{\nexists}$.*

Proof. The condition $T_{0,2} \subseteq \mathrm{Pol}(\mathcal{R})$ is equivalent to $\mathrm{Str}(T_{0,2}) \subseteq \mathrm{pPol}(\mathcal{R})$, whereas $\mathcal{R} \subseteq \langle \rho_{0,2} \rangle_{\nexists}$ is equivalent to $\mathrm{pPol}(\rho_{0,2}) \subseteq \mathrm{pPol}(\mathcal{R})$. Therefore, it suffices to prove that $\mathrm{pPol}(\rho_{0,2}) = \mathrm{Str}(T_{0,2})$, i.e., that if a partial function f preserves $\rho_{0,2}$, then it extends to a total function $\widehat{f}$ still preserving $\rho_{0,2}$. It is easy to see that setting $\widehat{f}(\mathbf{a}) = 0$ for all $\mathbf{a} \notin \mathrm{dom}(f)$ gives the required extension of f. $\square$

Let us write $\mathrm{Sub}\big(\langle \rho_{0,2} \rangle_{\nexists}\big)$ for the lattice of weak relational clones contained in $\langle \rho_{0,2} \rangle_{\nexists}$. By Proposition 3.1, $\mathrm{Sub}(\langle \rho_{0,2} \rangle_{\nexists})$ is dually isomorphic to $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$. Since the only Boolean clones properly containing $T_{0,2}$ are T_0 and $\mathrm{Op}(\mathbf{2})$, we have $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2}) = \mathcal{I}_{\mathrm{Str}}(T_{0,2}) \cup \mathcal{I}_{\mathrm{Str}}(T_0) \cup \mathcal{I}_{\mathrm{Str}}(\mathrm{Op}(\mathbf{2}))$. The intervals $\mathcal{I}_{\mathrm{Str}}(T_0)$ and $\mathcal{I}_{\mathrm{Str}}(\mathrm{Op}(\mathbf{2}))$ are singletons (see [1], but we will also reprove these facts in Remark 4.5), hence the main task is to describe the structure of $\mathcal{I}_{\mathrm{Str}}(T_{0,2})$.

We will represent relations in $\langle \rho_{0,2} \rangle_{\nexists}$ by graphs, and we will introduce an appropriate closure operator on graphs such that the closed sets of graphs are in a one-to-one correspondence with the $\langle \cdot \rangle_{\nexists}$-closed subsets of $\langle \rho_{0,2} \rangle_{\nexists}$. This will allow us to give a simple proof for the uncountability of $\mathcal{I}_{\mathrm{Str}}(T_{0,2})$ and to obtain some new results about the structure of this lattice.

If $G \in \mathcal{G}$ is a graph with $V(G) = \{v_1, \dots, v_n\}$, then we can define a relation $\mathrm{rel}(G) \subseteq \mathbf{2}^n$ by

$$\mathrm{rel}(G)(x_1, \dots, x_n) = \bigwedge_{v_i v_j \in E(G)} \rho_{0,2}(x_i, x_j).$$

Note that if we enumerate the vertices of G in a different way, then we may obtain a different relation; however, these two relations differ only in the order of their rows, hence they are essentially the same. Clearly, $\mathrm{rel}(G) \in \langle\rho_{0,2}\rangle_{\nexists}$ for every $G \in \mathcal{G}$; moreover, for any $\sigma \in \langle\rho_{0,2}\rangle_{\nexists}$ there exists $G \in \mathcal{G}$ such that σ and $\mathrm{rel}(G)$ are essentially the same. Indeed, $\sigma \in \langle\rho_{0,2}\rangle_{\nexists}$ implies that σ is of the form

$$\sigma(x_1, \ldots, x_n) = \bigwedge_{j=1}^{t} \rho_{0,2}(x_{u_j}, x_{v_j}) \wedge \bigwedge_{j=t+1}^{s} (x_{u_j} = x_{v_j}),$$

where $u_j, v_j \in \{1, 2, \ldots, n\}$ $(j = 1, \ldots, s)$. Now if we define a graph G by $V(G) = \{1, 2, \ldots, n\}$ and

$$E(G) = \{u_1v_1, \ldots, u_tv_t\},$$

then we have $\sigma \approx \mathrm{rel}(G/\varepsilon)$, where ε is the least equivalence relation on $V(G)$ that contains the pairs $(u_{t+1}, v_{t+1}), \ldots, (u_s, v_s)$. Removing isolated vertices (if there are any) from G/ε, we obtain a graph $G' \in \mathcal{G}$ such that $\sigma \approx \mathrm{rel}(G')$. (Recall that isolated vertices are not allowed in $\mathcal{G}$ with the sole exception of K_1. This does not result in a loss of generality, since isolated vertices in a graph H correspond to inessential coordinates in the relation $\mathrm{rel}(H)$.)

It may happen that nonisomorphic graphs induce essentially the same relation. This is captured by the following equivalence relation. Let us say that the graphs $G, H \in \mathcal{G}$ are *loopvivalent* (notation: $G \circlearrowright H$) if the following two conditions are satisfied:

- G has a loop if and only if H has a loop;
- the subgraphs spanned by the edges connecting loopless vertices in G and H are isomorphic.

Remark 3.2. Observe that for loopless graphs loopvivalence is equivalent to isomorphy. If G has a loop, then we can obtain a canonical representative of the loopvivalence class of G as follows. Delete all looped vertices from G, and if any of the remaining vertices become isolated, then delete these isolated vertices, too. Denoting the resulting (loopless) graph by G^*, we have $G \circlearrowright G^* \dot\cup L$; furthermore, $G^* \dot\cup L$ is the "simplest" graph that is loopvivalent to G. As an example, consider a graph G on two vertices, which are connected by an edge, and at least one of them has a loop. Then G^* is empty (cf. [14]), hence G is loopvivalent to L.

Lemma 3.3. *For any $G, H \in \mathcal{G}$, we have* $\mathrm{rel}(G) \approx \mathrm{rel}(H) \iff G \circlearrowright H$.

Proof. Let $G \in \mathcal{G}$ be an arbitrary graph with $V(G) = \{v_1, \ldots, v_n\}$. Since $\rho_{0,2} = \mathbf{2}^2 \setminus \{(1,1)\}$, a tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{2}^n$ belongs to $\mathrm{rel}(G)$ if and only if $\mathbf{a}^{-1}(1) := \{v_i : a_i = 1\} \subseteq V(G)$ is an independent set. Thus the tuples in $\mathrm{rel}(G)$ are in a one-to-one correspondence with the independent sets of G. Therefore, for any $G, H \in \mathcal{G}$ with $V(G) = V(H) = \{v_1, \ldots, v_n\}$, we have $\mathrm{rel}(G) = \mathrm{rel}(H)$ if and only if G and H have the same independent sets. This holds if and only if G and H have the same loops and they have the same edges between loopless vertices. Indeed, a vertex v_i has a loop if and only if the set $\{v_i\}$ is not independent, and there is an edge between loopless vertices v_i and v_j if and only if the set $\{v_i, v_j\}$ is not independent. Moreover, edges between a looped vertex and any other vertex are irrelevant in determining independent sets, since a set containing a looped vertex can never be independent.

Now let us determine the possible repeated rows of the matrix of $\mathrm{rel}(G)$. If two vertices v_i and v_j both have a loop, then the i-th and the j-th rows of the matrix of $\mathrm{rel}(G)$ are identical (in fact, they are constant 0, as a looped vertex cannot belong to any independent set). On the other hand, if, say, v_i does not have a loop, then $\{v_i\}$ is an independent set, and the corresponding tuple $\mathbf{a} \in \mathrm{rel}(G)$ satisfies $1 = a_i \neq a_j = 0$, hence the i-th and the j-th rows of the matrix of $\mathrm{rel}(G)$ are different. Thus the matrix of $\mathrm{rel}(G)$ has repeated rows if and only if G has more than one loop, and in this case the repeated rows are the constant 0 rows corresponding to the looped vertices.

From the above considerations it follows that for any $G, H \in \mathcal{G}$ we have $\mathrm{rel}(G) \approx \mathrm{rel}(H)$ if and only if $G \circlearrowright H$. $\square$

Now let us translate the four conditions of Fact 2.3 to an appropriate closure operator on $\mathcal{G}$. Let us say that a set $\mathcal{K} \subseteq \mathcal{G}$ of graphs is $\circlearrowright$-closed if it is closed under disjoint unions, homomorphic images and loopvivalence, and contains K_1:

(i) if $G, H \in \mathcal{K}$, then $G \dot\cup H \in \mathcal{K}$;
(ii) if $G \in \mathcal{K}$ and $G \twoheadrightarrow H$, then $H \in \mathcal{K}$;
(iii) $K_1 \in \mathcal{K}$;
(iv) if $G \in \mathcal{K}$ and $G \circlearrowright H$, then $H \in \mathcal{K}$.

The $\circlearrowright$-closure of $\mathcal{K} \subseteq \mathcal{G}$ is the smallest $\circlearrowright$-closed set $\langle \mathcal{K} \rangle_{\circlearrowright}$ that contains $\mathcal{K}$. Let us denote the lattice of $\circlearrowright$-closed subsets of $\mathcal{G}$ by $\mathrm{Sub}(\mathcal{G})$. Later we shall also need another closure operator on loopless graphs, which we call $\not\circlearrowright$-closure. We say that a set $\mathcal{K} \subseteq \mathcal{G}_1$ is $\not\circlearrowright$-closed if it is closed under disjoint unions and loopless homomorphic images:

(i) if $G, H \in \mathcal{K}$, then $G \dot\cup H \in \mathcal{K}$;
(ii) if $G \in \mathcal{K}$ and $G \twoheadrightarrow H$, then $H \in \mathcal{K}$, provided that H has no loops.

The least $\not\circlearrowright$-closed subset of $\mathcal{G}_1$ containing $\mathcal{K}$ will be denoted by $\langle \mathcal{K} \rangle_{\not\circlearrowright}$.

The next lemma gives a visual interpretation of $\not\circlearrowright$-closure that we will often use in the sequel: a graph G belongs to $\langle \mathcal{K} \rangle_{\not\circlearrowright}$ if and only if G can be built by "gluing together" loopless homomorphic images of members of $\mathcal{K}$.

Lemma 3.4. *For arbitrary $\mathcal{K} \subseteq \mathcal{G}_1$ and $G \in \mathcal{G}_1$ the following three conditions are equivalent:*

(i) *$G \in \langle \mathcal{K} \rangle_{\not\circlearrowright}$;*
(ii) *$H_1 \dot\cup \cdots \dot\cup H_k \twoheadrightarrow G$ for some $k \in \mathbb{N}$ and $H_1, \ldots, H_k \in \mathcal{K}$;*
(iii) *every edge of G is contained in a subgraph that is a homomorphic image of a member of $\mathcal{K}$.*

Proof. It is easy to see that a disjoint union of quotients of some graphs is also a quotient of the disjoint union of these graphs, thus (i) $\Longrightarrow$ (ii). To prove (ii) $\Longrightarrow$ (iii), suppose that $H_1, \ldots, H_k \in \mathcal{K}$ and $\varphi\colon H_1 \dot\cup \cdots \dot\cup H_k \twoheadrightarrow G$ is a complete homomorphism, and let e be an arbitrary edge of G. By completeness of φ, the edge e is contained in $\varphi(H_i)$ for some i, and then $\varphi(H_i)$ will be the required subgraph of G.

Finally, for (iii) $\Longrightarrow$ (i), assume that for every edge $e \in E(G)$ there is a (not necessarily induced) subgraph S_e of G that is the homomorphic image of some member of $\mathcal{K}$ and $e \in E(S_e)$. Clearly, this implies $S_e \in \langle \mathcal{K} \rangle_{\not\circlearrowright}$, so it suffices to prove that $G \in \langle \{S_e \colon e \in E(G)\} \rangle_{\not\circlearrowright}$. Let $\iota_e\colon S_e \to G$ be the inclusion map for every $e \in E(G)$, and let us combine these maps into a homomorphism $\varphi\colon \dot\bigcup_{e \in E(G)} S_e \to G$. Since e is included in the image of S_e, the homomorphism φ is complete, and this shows that G indeed belongs to the $\not\circlearrowright$-closure of $\{S_e \colon e \in E(G)\}$. $\square$

Remark 3.5. Note that the (proof of) implication (iii) $\Longrightarrow$ (i) of Lemma 3.4 applies also to $\circlearrowright$-closure. As an illustration, observe that any graph without isolated vertices can be built from edges and looped vertices, hence $\mathcal{G} = \langle K_2, L \rangle_{\circlearrowright} = \langle K_2 \rangle_{\circlearrowright}$ (we can omit L as it is a homomorphic image of K_2).

As the main result of this section, we prove that $\circlearrowright$-closure is indeed the appropriate closure operator on $\mathcal{G}$ that reflects the structure of the lattices $\mathrm{Sub}\big(\langle \rho_{0,2} \rangle_{\nexists}\big)$ and $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$.

Proposition 3.6. *The lattice $\mathrm{Sub}\big(\langle \rho_{0,2} \rangle_{\nexists}\big)$ of weak relational subclones of $\langle \rho_{0,2} \rangle_{\nexists}$ is isomorphic to the lattice $\mathrm{Sub}(\mathcal{G})$ of $\circlearrowright$-closed subsets of $\mathcal{G}$.*

Proof. For $\mathcal{K} \subseteq \mathcal{G}$ and $\mathcal{R} \subseteq \langle \rho_{0,2} \rangle_{\nexists}$, let

$$\begin{aligned}\Phi(\mathcal{K}) &= \{\sigma \in \langle \rho_{0,2} \rangle_{\nexists} \colon \exists G \in \mathcal{K} \text{ such that } \sigma \approx \mathrm{rel}(G)\};\\ \Psi(\mathcal{R}) &= \{G \in \mathcal{G} \colon \mathrm{rel}(G) \in \mathcal{R}\}.\end{aligned}$$

Observe that $\mathrm{rel}(G \dot\cup H) \approx \mathrm{rel}(G) \times \mathrm{rel}(H)$ and $\mathrm{rel}(G/\varepsilon) \approx \Delta_\varepsilon(\mathrm{rel}(G))$ for all $G, H \in \mathcal{G}$ and for every equivalence relation ε on $V(G)$, and we have $\mathrm{rel}(K_1) \approx \mathbf{2}$. Using these observations it is straightforward to verify that $\langle \mathcal{K} \rangle_{\circlearrowright} = \mathcal{K} \implies \langle \Phi(\mathcal{K}) \rangle_{\nexists} = \Phi(\mathcal{K})$ and $\langle \mathcal{R} \rangle_{\nexists} = \mathcal{R} \implies \langle \Psi(\mathcal{R}) \rangle_{\circlearrowright} = \Psi(\mathcal{R})$. Thus we obtain maps $\Phi \colon \mathrm{Sub}(\mathcal{G}) \to \mathrm{Sub}(\langle \rho_{0,2} \rangle_{\nexists})$ and $\Psi \colon \mathrm{Sub}(\langle \rho_{0,2} \rangle_{\nexists}) \to \mathrm{Sub}(\mathcal{G})$, and it is clear that both maps are order-preserving. Therefore, it only remains to show that Φ and Ψ are inverses of each other: for every $\mathcal{K} \in \mathrm{Sub}(\mathcal{G})$ and $\mathcal{R} \in \mathrm{Sub}(\langle \rho_{0,2} \rangle_{\nexists})$ we have

$$\begin{aligned}\Psi\Phi(\mathcal{K}) &= \{G \in \mathcal{G} \colon \mathrm{rel}(G) \in \Phi(\mathcal{K})\}\\ &= \{G \in \mathcal{G} \colon \exists H \in \mathcal{K} \text{ such that } \mathrm{rel}(G) \approx \mathrm{rel}(H)\}\\ &= \{G \in \mathcal{G} \colon \exists H \in \mathcal{K} \text{ such that } G \circlearrowright H\}\\ &= \mathcal{K};\\ \Phi\Psi(\mathcal{R}) &= \{\sigma \in \langle \rho_{0,2} \rangle_{\nexists} \colon \exists G \in \Psi(\mathcal{R}) \text{ such that } \sigma \approx \mathrm{rel}(G)\}\\ &= \{\sigma \in \langle \rho_{0,2} \rangle_{\nexists} \colon \exists G \in \mathcal{G} \text{ such that } \mathrm{rel}(G) \in \mathcal{R} \text{ and } \sigma \approx \mathrm{rel}(G)\}\\ &= \mathcal{R}.\end{aligned}$$

□

Corollary 3.7. *The lattice $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ of strong partial clones containing $T_{0,2}$ is dually isomorphic to the lattice $\mathrm{Sub}(\mathcal{G})$ of $\circlearrowright$-closed subsets of $\mathcal{G}$ (see Figure 1).*

4. The bottom and the top of $\mathrm{Sub}(\mathcal{G})$

Building upon Corollary 3.7, in the rest of the paper we study the lattice of $\circlearrowright$-closed subsets of $\mathcal{G}$. In this section we take a closer look at the bottom and the top of the lattice: we prove that there is a 3-element chain at the bottom and a 2-element chain at the top of $\mathrm{Sub}(\mathcal{G})$; see Figure 1. Between these chains there is a "jungle" that embeds the power set of a countably infinite set, hence it has continuum cardinality. We shall explore this jungle in Section 5.

The smallest $\circlearrowright$-closed subset of $\mathcal{G}$ is $\langle \emptyset \rangle_{\circlearrowright} = \langle K_1 \rangle_{\circlearrowright} = \{K_1\}$. Any graph containing an edge has L (the graph having only one vertex with a loop on it) as a homomorphic image, hence the second smallest $\circlearrowright$-closed set is $\langle L \rangle_{\circlearrowright}$, which consists of K_1 and all graphs having a loop and no edges between loopless vertices. In the next lemma we prove that the third smallest $\circlearrowright$-closed subset of $\mathcal{G}$ is $\langle K_2 \dot\cup L \rangle_{\circlearrowright}$. It is easy to see with the help of Remark 3.5 that $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \setminus \{K_1\}$ is the set of all graphs containing at least one loop.

Lemma 4.1. *At the bottom of the lattice $\mathrm{Sub}(\mathcal{G})$ we have the three-element chain $\langle K_1 \rangle_{\circlearrowright} \prec \langle L \rangle_{\circlearrowright} \prec \langle K_2 \dot\cup L \rangle_{\circlearrowright}$. All other $\circlearrowright$-closed subsets of $\mathcal{G}$ contain $\langle K_2 \dot\cup L \rangle_{\circlearrowright}$.*

Proof. Let $\mathcal{K} \subseteq \mathcal{G}$ be a $\circlearrowright$-closed set such that $\langle L \rangle_{\circlearrowright} \subset \mathcal{K}$. Then $\mathcal{K}$ contains a graph G with an edge uv where u and v are distinct loopless vertices. Let us form the disjoint union $G \dot\cup L$, and let us identify all vertices of this graph except for u and v. Then we obtain a graph $G' \in \mathcal{K}$ with $V(G') = \{u, v, w\}$ and $\{uv, ww\} \subseteq E(G') \subseteq \{uv, ww, uw, vw\}$. Deleting the edges uw and vw (if they are present) we arrive at a graph G'' with $V(G'') = \{u, v, w\}$ and $E(G'') = \{uv, ww\}$. Since $G'' \circlearrowright G'$, we have $G'' \in \mathcal{K}$; moreover, G'' is isomorphic to $K_2 \dot\cup L$, hence $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \subseteq \mathcal{K}$. This proves that $\langle K_2 \dot\cup L \rangle_{\circlearrowright}$ is the third smallest $\circlearrowright$-closed subset of $\mathcal{G}$. □

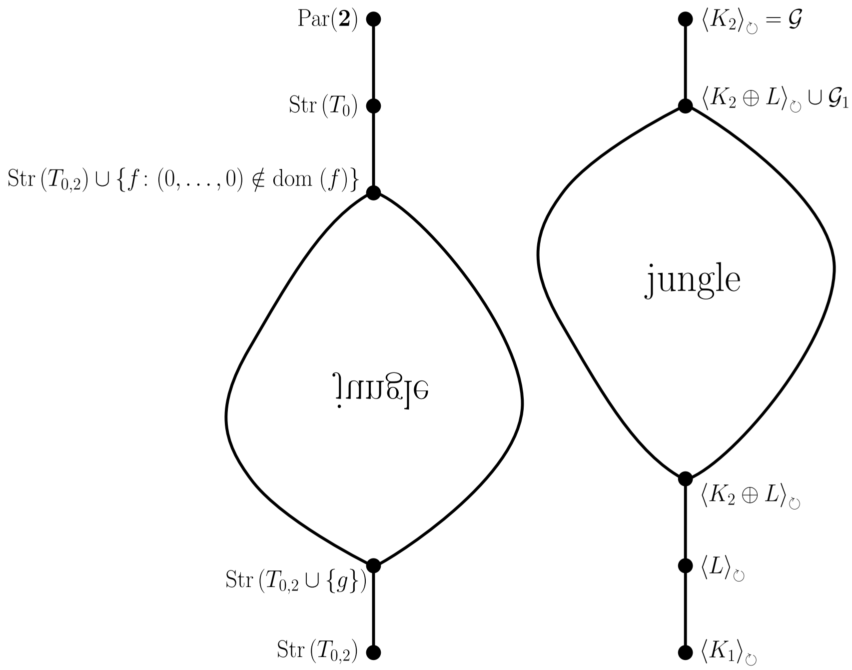

FIGURE 1. The lattices $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ and $\mathrm{Sub}(\mathcal{G})$

As we will see later, we have to stop our climbing up in the lattice here, as there is no fourth smallest $\circlearrowright$-closed set, so let us now focus on the top of the lattice $\mathrm{Sub}(\mathcal{G})$. The largest $\circlearrowright$-closed set is clearly $\mathcal{G}$, which, as we observed in Remark 3.5, can be generated by K_2. The following lemma describes the second largest $\circlearrowright$-closed set (recall that $\mathcal{G}_1$ denotes the set of all loopless non-bipartite members of $\mathcal{G}$).

Lemma 4.2. *At the top of the lattice* $\mathrm{Sub}(\mathcal{G})$ *we have the two-element chain* $\mathcal{G} = \langle K_2 \rangle_{\circlearrowright} \succ \langle K_2 \dot\cup L \rangle_{\circlearrowright} \cup \mathcal{G}_1$. *All other $\circlearrowright$-closed subsets of $\mathcal{G}$ are contained in* $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \cup \mathcal{G}_1$.

Proof. Let us consider a $\circlearrowright$-closed set $\mathcal{K}$ such that $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \subseteq \mathcal{K}$. If $\mathcal{K}$ contains a graph G that is bipartite and has at least one edge (which cannot be a loop, because of bipartiteness), then we have $G \twoheadrightarrow K_2 \in \mathcal{K}$. Then we can conclude $\mathcal{K} \supseteq \langle K_2 \rangle_{\circlearrowright} = \mathcal{G}$ (cf. Remark 3.5). Thus every proper $\circlearrowright$-closed subset of $\mathcal{G}$ must be contained in $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \cup \mathcal{G}_1$. It remains to show that the set $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \cup \mathcal{G}_1$ is $\circlearrowright$-closed. To verify this, one just needs to observe that if at least one of G and H is not bipartite, then $G \dot\cup H$ is not bipartite either; furthermore, if G is not bipartite and $G \twoheadrightarrow H$, then H is not bipartite either (otherwise we would have $G \twoheadrightarrow H \to K_2$, hence $G \to K_2$, contradicting the non-bipartiteness of G). Therefore, the second largest $\circlearrowright$-closed subset of $\mathcal{G}$ is indeed $\langle K_2 \dot\cup L \rangle_{\circlearrowright} \cup \mathcal{G}_1$. $\square$

We will see in the next section that there is no third largest $\circlearrowright$-closed subset of $\mathcal{G}$, therefore we finish our climbing down here and summarize our findings in the following theorem.

Theorem 4.3. *A set $\mathcal{K} \subseteq \mathcal{G}$ is $\circlearrowright$-closed if and only if either*

(i) $\mathcal{K} = \langle K_1 \rangle_{\circlearrowright} = \{K_1\}$, *or*
(ii) $\mathcal{K} = \langle L \rangle_{\circlearrowright}$, *or*
(iii) $\mathcal{K} = \langle K_2 \rangle_{\circlearrowright} = \mathcal{G}$, *or*
(iv) $\mathcal{K} = \langle K_2 \dot\cup L \rangle_{\circlearrowright} \cup \mathcal{H}$, *where $\mathcal{H} \subseteq \mathcal{G}_1$ is $\not\circlearrowright$-closed.*

Proof. By Lemmas 4.1 and 4.2, the sets listed in the first three items are $\circlearrowright$-closed (as well as the fourth item with $\mathcal{H} = \emptyset$ and $\mathcal{H} = \mathcal{G}_1$); moreover, any other $\circlearrowright$-closed set $\mathcal{K}$ satisfies $\langle K_2 \dot\cup L\rangle_{\circlearrowright} \subseteq \mathcal{K} \subseteq \langle K_2 \dot\cup L\rangle_{\circlearrowright} \cup \mathcal{G}_1$. Let $\mathcal{K}$ be such a set, and let $\mathcal{H} \subseteq \mathcal{G}_1$ be the set of all loopless non-bipartite members of $\mathcal{K}$; then we have $\mathcal{K} = \langle K_2 \dot\cup L\rangle_{\circlearrowright} \cup \mathcal{H}$. To finish the proof, one just has to verify that $\mathcal{K}$ is $\circlearrowright$-closed if and only if $\mathcal{H}$ is closed under disjoint unions and loopless homomorphic images. $\square$

We conclude this section with the description of the bottom and the top of $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$. It is immediate from Theorem 4.3 and Corollary 3.7 that there is a three-element chain at the top, and a two-element chain at the bottom of $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$. In the next theorem we describe explicitly the five strong partial clones in these chains.

Theorem 4.4. *At the top of the lattice $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ we have a three-element chain $\mathrm{Par}(\mathbf{2}) \succ \mathrm{Str}(T_0) \succ \mathrm{Str}(T_{0,2}) \cup \{f \in \mathrm{Par}(\mathbf{2})\colon (0,\dots,0) \notin \mathrm{dom}(f)\}$, while at the bottom we have the two-element chain $\mathrm{Str}(T_{0,2}) \prec \mathrm{Str}(T_{0,2} \cup \{g\})$, where g is the binary partial function defined by $\mathrm{dom}(g) = \{(0,1),(1,0)\}$ and $g(0,1) = g(1,0) = 1$. All other strong partial clones in $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ lie between these two chains (see Figure 1).*

Proof. We just need to translate the results of Lemma 4.1 and Lemma 4.2 to the lattice $\mathrm{Sub}\big(\langle\rho_{0,2}\rangle_{\nexists}\big)$ with the help of Proposition 3.6, and then pass to the lattice $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ by the operator pPol (note that this last step turns the lattice upside down).

It is obvious that $\Phi(\langle K_1\rangle_{\circlearrowright}) = \langle \mathbf{2}\rangle_{\nexists}$ is the trivial relational clone, and the corresponding strong partial clone is $\mathrm{pPol}(\mathbf{2}) = \mathrm{Par}(\mathbf{2})$. Similarly, since $\mathrm{rel}(L)$ is the unary relation $\{0\}$, we have $\Phi(\langle L\rangle_{\circlearrowright}) = \langle\{0\}\rangle_{\nexists}$, and $\mathrm{pPol}(\{0\}) = \mathrm{Str}(T_0)$. The relation corresponding to $K_2 \dot\cup L$ is

$$\mathrm{rel}(K_2 \dot\cup L) = \{(0,0,0),(0,1,0),(1,0,0)\} = \rho_{0,2} \times \{0\}.$$

All partial functions with $(0,\dots,0) \notin \mathrm{dom}(f)$ automatically preserve this relation, and it is straightforward to verify that if $(0,\dots,0) \in \mathrm{dom}(f)$, then $f \in \mathrm{pPol}(\rho_{0,2} \times \{0\})$ holds if and only if $f \in \mathrm{pPol}(\rho_{0,2}) = \mathrm{Str}(T_{0,2})$.

For the chain at the bottom, observe that $\mathrm{rel}(K_2) = \rho_{0,2}$, thus we have $\Phi(\langle K_2\rangle_{\circlearrowright}) = \langle\rho_{0,2}\rangle_{\nexists}$, and the corresponding strong partial clone is clearly $\mathrm{pPol}(\rho_{0,2}) = \mathrm{Str}(T_{0,2})$. Finally, let us consider the strong partial clone $\mathcal{C} := \mathrm{pPol}(\Phi(\langle K_2 \dot\cup L\rangle_{\circlearrowright} \cup \mathcal{G}_1))$. The function g defined in the statement of the theorem does not preserve $\rho_{0,2}$, therefore $\mathrm{Str}(T_{0,2}) \subset \mathrm{Str}(T_{0,2} \cup \{g\})$. It follows from Theorem 4.3 that $\mathcal{C}$ is the unique upper cover of $\mathrm{Str}(T_{0,2})$, hence it suffices to verify that $\mathrm{Str}(T_{0,2} \cup \{g\}) \subseteq \mathcal{C}$, i.e., that g preserves $\mathrm{rel}(K_2 \dot\cup L)$ and $\mathrm{rel}(G)$ for all $G \in \mathcal{G}_1$. The former is trivial, as $(0,0) \notin \mathrm{dom}(g)$. For the latter, let us consider an arbitrary non-bipartite graph G with $V(G) = \{v_1,\dots,v_n\}$, and let $\mathbf{a},\mathbf{b} \in \{0,1\}^n$ such that $\mathbf{a},\mathbf{b} \in \mathrm{rel}(G)$ and $(a_i,b_i) \in \mathrm{dom}(g)$ for every i. Since $\mathrm{dom}(g) = \{(0,1),(1,0)\}$, the sets $\mathbf{a}^{-1}(1)$ and $\mathbf{b}^{-1}(1)$ form a partition of $V(G)$, and both sets are independent by the definition of $\mathrm{rel}(G)$ (cf. the proof of Lemma 3.3). However, this means that G is 2-colorable, contradicting the non-bipartiteness of G. Thus Definition 1.2 is satisfied emptily: there is no matrix M such that its columns belong to $\mathrm{rel}(G)$ and its rows belong to $\mathrm{dom}(g)$. $\square$

Remark 4.5. The total parts of $\mathrm{Par}(\mathbf{2})$ and $\mathrm{Str}(T_0)$ are $\mathrm{Op}(\mathbf{2})$ and T_0, while the total part of $\mathrm{Str}(T_{0,2}) \cup \{f \in \mathrm{Par}(\mathbf{2})\colon (0,\dots,0) \notin \mathrm{dom}(f)\}$ is $T_{0,2}$. Therefore, we have $\mathcal{I}_{\mathrm{Str}}(\mathrm{Op}(\mathbf{2})) = \{\mathrm{Par}(\mathbf{2})\}$ and $\mathcal{I}_{\mathrm{Str}}(T_0) = \{\mathrm{Str}(T_0)\}$, while $\mathcal{I}_{\mathrm{Str}}(T_{0,2})$ can be obtained from $\mathcal{I}^{\subseteq}_{\mathrm{Str}}(T_{0,2})$ by removing these two elements from the top of the lattice.

5. The jungle

After Theorem 4.3, it remains to describe the structure of the interval

$$\big[\langle K_2 \dot\cup L\rangle_{\circlearrowright}, \langle K_2 \dot\cup L\rangle_{\circlearrowright} \cup \mathcal{G}_1\big]$$

of $\mathrm{Sub}(\mathcal{G})$. By Theorem 4.3, the map $\langle K_2 \dot\cup L\rangle_{\circlearrowleft} \cup \mathcal{H} \mapsto \mathcal{H}$ is an isomorphism from this interval to the lattice of $\not\circlearrowleft$-closed subsets of $\mathcal{G}_1$, which we shall denote by $\mathrm{Sub}(\mathcal{G}_1)$. Therefore, in this section we focus on the lattice $\mathrm{Sub}(\mathcal{G}_1)$. Thus, in the sequel we will assume that all homomorphisms map to loopless graphs; in particular, we never identify vertices connected by an edge. We will prove several properties of $\mathrm{Sub}(\mathcal{G}_1)$ indicating that this lattice is quite complicated, hence it deserves to be called a jungle.

5.1. **Decomposing the jungle.** Let us consider the partition $\mathcal{G}_1 = \mathcal{A} \dot\cup \mathcal{B}$, where

$$\mathcal{A} = \{G \in \mathcal{G}_1 \colon \textbf{a}\text{ll components of } G \text{ are non-bipartite}\},$$
$$\mathcal{B} = \{G \in \mathcal{G}_1 \colon \text{at least one component of } G \text{ is } \textbf{b}\text{ipartite}\}.$$

Observe that $\langle \mathcal{A}\rangle_{\not\circlearrowleft} = \mathcal{A}$, but $\mathcal{B}$ is not $\not\circlearrowleft$-closed. In this subsection we show that for any $\mathcal{H} \subseteq \mathcal{G}_1$, one can determine $\langle \mathcal{H}\rangle_{\not\circlearrowleft}$ by computing the $\not\circlearrowleft$-closure of $\mathcal{H} \cap \mathcal{A}$ and $\mathcal{H} \cap \mathcal{B}$ separately; moreover, $\langle \mathcal{H} \cap \mathcal{B}\rangle_{\not\circlearrowleft}$ is particularly easy to describe, since it is just an upset in the homomorphism order of graphs (see Theorem 5.4). As a corollary, we obtain that $\mathrm{Sub}(\mathcal{G}_1)$ can be embedded into the direct product of the lattice of $\not\circlearrowleft$-closed subsets of $\mathcal{A}$ and the lattice of upsets of the quasiordered set $(\mathcal{A}; \to)$ (see Corollary 5.8). First we introduce some notation, and then we prove some preparatory results about the connection between $\not\circlearrowleft$-closure and upsets.

For any graph $H \in \mathcal{G}_1$, let $H_A \in \mathcal{A}$ be the union of the non-bipartite components of H. If $H \in \mathcal{A}$ then $H_A = H$, whereas for $H \in \mathcal{B}$ we have $H = H_A \dot\cup B$ with some bipartite graph B. Note that H_A is never empty, as every graph in $\mathcal{G}_1$ is non-bipartite. For a set $\mathcal{H} \subseteq \mathcal{G}_1$, let $\mathcal{H}^{\uparrow}$ denote the upset generated by $\mathcal{H}$ in the quasiordered set $(\mathcal{G}_1; \to)$, i.e., let

$$\mathcal{H}^{\uparrow} = \{G \in \mathcal{G}_1 \colon H \to G \text{ for some } H \in \mathcal{H}\}.$$

Lemma 5.1. *For every $\mathcal{H} \subseteq \mathcal{G}_1$, we have $\langle \mathcal{H}\rangle_{\not\circlearrowleft} \subseteq \mathcal{H}^{\uparrow}$; consequently, if $\mathcal{H} \subseteq \mathcal{G}_1$ is an upset in $(\mathcal{G}_1; \to)$, then $\langle \mathcal{H}\rangle_{\not\circlearrowleft} = \mathcal{H}$.*

Proof. If $G \in \langle \mathcal{H}\rangle_{\not\circlearrowleft}$, then, by Lemma 3.4, there is a complete homomorphism $\varphi \colon H_1 \dot\cup \cdots \dot\cup H_k \twoheadrightarrow G$ for some $k \in \mathbb{N}$ and $H_1, \dots, H_k \in \mathcal{H}$. Restricting φ to H_1, we get a homomorphism (not necessarily complete) $H_1 \to G$, which shows that $G \in \mathcal{H}^{\uparrow}$. If $\mathcal{H}$ is an upset, then $\mathcal{H} \subseteq \langle \mathcal{H}\rangle_{\not\circlearrowleft} \subseteq \mathcal{H}^{\uparrow} = \mathcal{H}$, therefore $\langle \mathcal{H}\rangle_{\not\circlearrowleft} = \mathcal{H}$. $\square$

Remark 5.2. It follows from Lemma 5.1 that if $\{H_1, H_2, \dots\} \subseteq \mathcal{G}_1$ is an infinite antichain in the homomorphism order, then the map $I \mapsto \{H_i \colon i \in I\}$ embeds the power set of $\mathbb{N}$ into $\mathrm{Sub}(\mathcal{G}_1)$. As mentioned in Subsection 2.2, such antichains do exist, hence $\mathrm{Sub}(\mathcal{G}_1)$ has continuum cardinality.

Lemma 5.3. *For every $H \in \mathcal{B}$, the graphs H and $H_A \dot\cup K_2$ are homomorphically equivalent, and $\langle H\rangle_{\not\circlearrowleft} = \langle H_A \dot\cup K_2\rangle_{\not\circlearrowleft} = H^{\uparrow}$.*

Proof. Let $H \in \mathcal{B}$, and let us consider the decomposition $H = H_A \dot\cup B$, where B is the union of the bipartite components of H. Since H has no isolated vertices, B has at least one edge, hence $K_2 \to B$, and also $B \twoheadrightarrow K_2$, as B is bipartite. This implies that the graphs $H = H_A \dot\cup B$ and $H_A \dot\cup K_2$ are homomorphically equivalent.

For the other statements of the lemma, let us verify the following chain of containments:

$$(H_A \dot\cup K_2)^{\uparrow} \subseteq \langle H_A \dot\cup K_2\rangle_{\not\circlearrowleft} \subseteq \langle H\rangle_{\not\circlearrowleft} \subseteq H^{\uparrow}. \tag{5.1}$$

To prove the first containment, let $G \in \mathcal{G}_1$ such that $H_A \dot\cup K_2 \to G$; then there is also a homomorphism $\varphi \colon H_A \to G$. For every edge $e = uv \in E(G)$, let S_e denote the subgraph of G that is obtained by adding the edge e to $\varphi(H_A)$: let $V(S_e) = V(\varphi(H_A)) \cup \{u, v\}$ and $E(S_e) = E(\varphi(H_A)) \cup \{e\}$. We can extend φ to a homomorphism

$\varphi_e \colon H_A \mathbin{\dot\cup} K_2 \to S_e$ that maps the edge of K_2 onto e. This shows that condition (iii) of Lemma 3.4 is satisfied with $\mathcal{H} = \{H_A \mathbin{\dot\cup} K_2\}$, therefore $G \in \langle H_A \mathbin{\dot\cup} K_2\rangle_{\not\emptyset}$.

The second containment of (5.1) follows from the fact that $H_A \mathbin{\dot\cup} K_2$ is a homomorphic image of $H = H_A \mathbin{\dot\cup} B$, since $B \twoheadrightarrow K_2$. The third containment is immediate from Lemma 5.1.

To finish the proof, recall that H and $H_A \mathbin{\dot\cup} K_2$ are homomorphically equivalent, hence $(H_A \mathbin{\dot\cup} K_2)^{\uparrow} = H^{\uparrow}$, and then all containments of (5.1) are actually equalities. $\square$

Theorem 5.4. *For every set $\mathcal{H} \subseteq \mathcal{G}_1$, we have*

$$\langle \mathcal{H}\rangle_{\not\emptyset} = \langle \mathcal{H} \cap \mathcal{A}\rangle_{\not\emptyset} \cup (\mathcal{H} \cap \mathcal{B})^{\uparrow}.$$

Proof. If $G \in \langle \mathcal{H}\rangle_{\not\emptyset}$, then $H_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} H_k \twoheadrightarrow G$ for some $k \in \mathbb{N}$ and $H_1, \dots, H_k \in \mathcal{H}$, by Lemma 3.4. If $H_i \in \mathcal{A}$ for every i, then $G \in \langle \mathcal{H} \cap \mathcal{A}\rangle_{\not\emptyset}$. Otherwise there is an i such that $H_i \in \mathcal{B}$, and then $H_i \to G$. This proves that $\langle \mathcal{H}\rangle_{\not\emptyset} \subseteq \langle \mathcal{H} \cap \mathcal{A}\rangle_{\not\emptyset} \cup (\mathcal{H} \cap \mathcal{B})^{\uparrow}$.

For the reverse containment, let us suppose that $G \in \langle \mathcal{H} \cap \mathcal{A}\rangle_{\not\emptyset} \cup (\mathcal{H} \cap \mathcal{B})^{\uparrow}$. If $G \in \langle \mathcal{H} \cap \mathcal{A}\rangle_{\not\emptyset}$, then we have obviously $G \in \langle \mathcal{H}\rangle_{\not\emptyset}$, as $\langle \mathcal{H} \cap \mathcal{A}\rangle_{\not\emptyset} \subseteq \langle \mathcal{H}\rangle_{\not\emptyset}$. Otherwise there exists $H \in \mathcal{H} \cap \mathcal{B}$ such that $H \to G$. It follows from Lemma 5.3 that $G \in \langle H\rangle_{\not\emptyset}$, and then $G \in \langle \mathcal{H}\rangle_{\not\emptyset}$. $\square$

Remark 5.5. In view of Lemma 5.3, we may identify the graphs H and $H_A \mathbin{\dot\cup} K_2$ for every $H \in \mathcal{B}$, when investigating homomorphisms and $\not\emptyset$-closed sets in $\mathcal{G}_1$, i.e., we can assume without loss of generality that the bipartite components (if any) of our graphs are always K_2. Therefore, we will write subsets of $\mathcal{B}$ in the form $\mathcal{H}^{\dot\cup K_2} := \{H \mathbin{\dot\cup} K_2 \colon H \in \mathcal{H}\}$ with $\mathcal{H} \subseteq \mathcal{A}$. In particular, we have $\mathcal{B} = \mathcal{A}^{\dot\cup K_2}$. (If one does not wish to make the aforementioned identification, then $\mathcal{H}^{\dot\cup K_2}$ should be interpreted as the set of all graphs of the form $H \mathbin{\dot\cup} B$, where $H \in \mathcal{H}$ and B is a bipartite graph without isolated vertices.)

Theorem 5.6. *A set $\mathcal{H} \subseteq \mathcal{G}_1$ is $\not\emptyset$-closed if and only if there exist $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{A}$ such that*

(i) *$\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2^{\dot\cup K_2}$;*
(ii) *$\mathcal{H}_1$ is $\not\emptyset$-closed;*
(iii) *$\mathcal{H}_2$ is an upset (order filter) in $(\mathcal{A}; \to)$, i.e., $\mathcal{H}_2^{\uparrow} \cap \mathcal{A} = \mathcal{H}_2$;*
(iv) *$\mathcal{H}_2 \subseteq \mathcal{H}_1$.*

Proof. Let us put $\mathcal{H}_1 = \mathcal{H} \cap \mathcal{A}$, and let $\mathcal{H}_2$ denote the collection of the non-bipartite parts of the members of $\mathcal{H} \cap \mathcal{B}$, i.e., $\mathcal{H}_2 = \{H_A \colon H \in \mathcal{H} \cap \mathcal{B}\}$. Then, performing the identification of Remark 5.5, we have $\mathcal{H} \cap \mathcal{B} = \mathcal{H}_2^{\dot\cup K_2}$, hence $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2^{\dot\cup K_2}$. For every graph G with at least one edge, G and $G \mathbin{\dot\cup} K_2$ are homomorphically equivalent; therefore, $(\mathcal{H} \cap \mathcal{B})^{\uparrow} = \left(\mathcal{H}_2^{\dot\cup K_2}\right)^{\uparrow} = \mathcal{H}_2^{\uparrow}$. By the same token, we have $\mathcal{H}_2^{\uparrow} \cap \mathcal{B} = \left(\mathcal{H}_2^{\uparrow} \cap \mathcal{A}\right)^{\dot\cup K_2}$.

By Theorem 5.4 and by the above observations, we have

$$\langle \mathcal{H}\rangle_{\not\emptyset} = \langle \mathcal{H}_1\rangle_{\not\emptyset} \cup \mathcal{H}_2^{\uparrow} = \langle \mathcal{H}_1\rangle_{\not\emptyset} \cup \left(\mathcal{H}_2^{\uparrow} \cap \mathcal{A}\right) \cup \left(\mathcal{H}_2^{\uparrow} \cap \mathcal{A}\right)^{\dot\cup K_2}. \tag{5.2}$$

Clearly, $\langle \mathcal{H}\rangle_{\not\emptyset} = \mathcal{H}$ holds if and only if $\langle \mathcal{H}\rangle_{\not\emptyset} \cap \mathcal{A} \subseteq \mathcal{H} \cap \mathcal{A}$ and $\langle \mathcal{H}\rangle_{\not\emptyset} \cap \mathcal{B} \subseteq \mathcal{H} \cap \mathcal{B}$. From (5.2) we see that $\langle \mathcal{H}\rangle_{\not\emptyset} \cap \mathcal{B} = \left(\mathcal{H}_2^{\uparrow} \cap \mathcal{A}\right)^{\dot\cup K_2}$, thus

$$\langle \mathcal{H}\rangle_{\not\emptyset} \cap \mathcal{B} \subseteq \mathcal{H} \cap \mathcal{B} \iff \left(\mathcal{H}_2^{\uparrow} \cap \mathcal{A}\right)^{\dot\cup K_2} \subseteq \mathcal{H}_2^{\dot\cup K_2} \iff \mathcal{H}_2^{\uparrow} \cap \mathcal{A} \subseteq \mathcal{H}_2,$$

which is equivalent to (iii). Again from (5.2) we have $\langle \mathcal{H}\rangle_{\not\emptyset} \cap \mathcal{A} = \langle \mathcal{H}_1\rangle_{\not\emptyset} \cup \left(\mathcal{H}_2^{\uparrow} \cap \mathcal{A}\right)$, hence

$$\langle \mathcal{H}\rangle_{\not\emptyset} \cap \mathcal{A} \subseteq \mathcal{H} \cap \mathcal{A} \iff \langle \mathcal{H}_1\rangle_{\not\emptyset} \subseteq \mathcal{H}_1 \text{ and } \mathcal{H}_2^{\uparrow} \cap \mathcal{A} \subseteq \mathcal{H}_1,$$

which, taking (iii) also into account, is equivalent to (ii) and (iv). $\square$

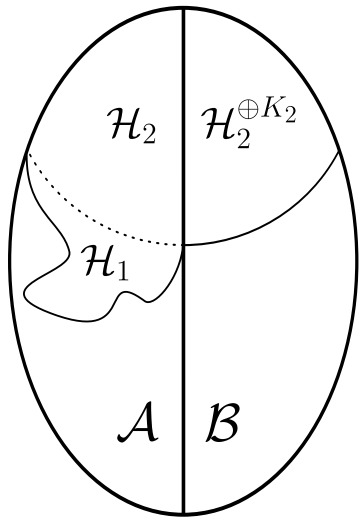

FIGURE 2. The structure of a $\not\emptyset$-closed subset of $\mathcal{G}_1$

Remark 5.7. The structure of $\not\emptyset$-closed subsets of $\mathcal{G}_1$ as described by Theorem 5.6 can be visualized as follows (see Figure 2): we take an upset $\mathcal{H}_2$ in $(\mathcal{A};\to)$; together with its "copy" $\mathcal{H}_2^{\dot\cup K_2}$ in $\mathcal{B}$, and then we extend $\mathcal{H}_2$ to a (possibly) larger $\not\emptyset$-closed subset $\mathcal{H}_1 \subseteq \mathcal{A}$.

Corollary 5.8. *The lattice* $\mathrm{Sub}(\mathcal{G}_1)$ *is isomorphic to the sublattice*

$$\{(\mathcal{H}_1,\mathcal{H}_2)\colon \mathcal{H}_2 \subseteq \mathcal{H}_1\} \subseteq \mathrm{Sub}(\mathcal{A}) \times \mathrm{Upsets}(\mathcal{A})$$

of the direct product of the lattice of $\not\emptyset$-closed subsets of $\mathcal{A}$ and the lattice of upsets of the quasiordered set $(\mathcal{A};\to)$.

5.2. **The upper part of the jungle.** The results of the previous subsection show that in order to understand the structure of $\mathrm{Sub}(\mathcal{G}_1)$, it suffices to describe the intervals $[\emptyset,\mathcal{A}]$ and $[\mathcal{A},\mathcal{G}_1]$. Let us now explore the part of the jungle that lies above $\mathcal{A}$. By choosing $\mathcal{H}_1 = \mathcal{A}$ in Theorem 5.6, we see that the $\not\emptyset$-closed sets $\mathcal{H}$ containing $\mathcal{A}$ are of the form $\mathcal{A} \,\dot\cup\, \mathcal{H}_2^{\dot\cup K_2}$, where $\mathcal{H}_2$ is an upset in $(\mathcal{A};\to)$. Thus, we have the following description of the upper part of the jungle.

Theorem 5.9. *The interval* $[\mathcal{A},\mathcal{G}_1]$ *in* $\mathrm{Sub}(\mathcal{G}_1)$ *is isomorphic to* $\mathrm{Upsets}(\mathcal{A})$.

Proof. Using the notation of Theorem 5.6, the map $\mathcal{H} \mapsto \mathcal{H}_2$ establishes the required isomorphism. □

Observe that the union of two upsets is an upset, hence the lattice $[\mathcal{A},\mathcal{G}_1]$ is distributive. Building upon the isomorphism given in Theorem 5.9, we show that each subinterval of $[\mathcal{A},\mathcal{G}_1]$ is either finite or has continuum cardinality.

Theorem 5.10. *If $\mathcal{H}$ and $\mathcal{K}$ are $\not\emptyset$-closed subsets of $\mathcal{G}_1$ such that $\mathcal{A} \subseteq \mathcal{H} \subset \mathcal{K}$, then the interval $[\mathcal{H},\mathcal{K}]$ is either a finite Boolean lattice or it embeds the power set of* $\mathbb{N}$.

Proof. According to Theorem 5.9, we can work in the lattice $\mathrm{Upsets}(\mathcal{A})$; let $\mathcal{H}_2$ and $\mathcal{K}_2$ be the upsets corresponding to $\mathcal{H}$ and $\mathcal{K}$. Assume first that the difference $\mathcal{K}_2 \setminus \mathcal{H}_2$ contains two comparable graphs: there exist $G, H \in \mathcal{K}_2 \setminus \mathcal{H}_2$ such that $G \to H$ and $H \nrightarrow G$. By Theorem 2.4, there is an infinite antichain between G and H, i.e., there are graphs $T_1, T_2, \ldots$ such that $G \to T_i \to H$ and T_i and T_j are incomparable for all $i, j \in \mathbb{N}, i \neq j$. For every set $S \subseteq \mathbb{N}$ we can construct an upset $\mathcal{U}_S = \mathcal{H}_2 \cup \{T_i\colon i \in S\}^{\uparrow}$, and it is straightofrward to verify that the map $S \mapsto \mathcal{U}_S$ embeds the power set of $\mathbb{N}$ into the interval $[\mathcal{H}_2, \mathcal{K}_2]$ of the lattice of upsets of $(\mathcal{A}; \to)$.

Now let us assume that $\mathcal{K}_2 \setminus \mathcal{H}_2$ contains no comparable elements, i.e., it is an antichain. Then the interval $[\mathcal{H}_2, \mathcal{K}_2]$ is isomorphic to the power set of $\mathcal{K}_2 \setminus \mathcal{H}_2$. Depending on whether $\mathcal{K}_2 \setminus \mathcal{H}_2$ is finite or infinite, we obtain either a finite Boolean lattice or the power set of $\mathbb{N}$. □

Remark 5.11. Both cases of Theorem 5.10 do appear: if $G, H \in \mathcal{A}$ are such that $G \to H$ and $H \nrightarrow G$, then the interval $[H^{\uparrow}, G^{\uparrow}]$ in $\mathrm{Upsets}(\mathcal{A})$ embeds the power set of $\mathbb{N}$, while if $T_1, \ldots, T_n$ is an antichain in $\mathcal{A}$ then the interval between $\mathcal{H} = T_1^{\uparrow} \cup \cdots \cup T_n^{\uparrow} \setminus \{T_1, \ldots, T_n\}$ and $\mathcal{K} = T_1^{\uparrow} \cup \cdots \cup T_n^{\uparrow}$ is isomorphic to the power set of $\{1, \ldots, n\}$.

Corollary 5.12. *Every interval above $\mathcal{A}$ in $\mathrm{Sub}(\mathcal{G}_1)$ is either finite or has continuum cardinality.*

Theorem 5.13. *For each $n \in \{0, 1, 2, \ldots, \aleph_0\}$, there exist elements in $\mathrm{Sub}(\mathcal{G}_1)$ with exactly n upper covers.*

Proof. By Theorem 5.9, if $\mathcal{H} \subseteq \mathcal{A}$ has n upper covers in $\mathrm{Upsets}(\mathcal{A})$, then $\mathcal{A} \mathbin{\dot\cup} \mathcal{H}^{\dot\cup K_2}$ has n upper covers in the lattice of $\not\circlearrowleft$-closed subsets of $\mathcal{G}_1$. For $n = 0$ let us take an infinite ascending chain $G_1 \to G_2 \to \ldots$ in $\mathcal{A}$ (for example, let $G_i = K_{i+2}$), and let $\mathcal{U} = \{H \in \mathcal{A} \colon H \nrightarrow G_i \text{ for every } i \in \mathbb{N}\}$; this is clearly an upset. If $\mathcal{V}$ is an upset such that $\mathcal{U} \subset \mathcal{V}$ and $H \in \mathcal{V} \setminus \mathcal{U}$, then $H \to G_i$ for some $i \in \mathbb{N}$. This implies that $G_i \in \mathcal{V}$, thus $\mathcal{U} \subset \mathcal{U} \cup G_{i+1}^{\uparrow} \subset \mathcal{U} \cup G_i^{\uparrow} \subseteq \mathcal{V}$. Therefore, $\mathcal{V}$ is not an upper cover of $\mathcal{U}$, hence $\mathcal{U}$ has no upper covers.

For $n \in \{1, 2, \ldots, \aleph_0\}$, let $\{G_i \colon i \in I\}$ be an antichain in $\mathcal{A}$ of size n. Let us define $\mathcal{U}$ in the same way as above: $\mathcal{U} = \{H \in \mathcal{A} \colon H \nrightarrow G_i \text{ for every } i \in I\}$. Then $\mathcal{U}$ is an upset and $\mathcal{U} \cup \{G_i\}$ covers $\mathcal{U}$ for every $i \in I$. Moreover, if $\mathcal{V}$ is an upset with $\mathcal{U} \subset \mathcal{V}$, then $\mathcal{U} \cup \{G_i\} \subseteq \mathcal{V}$ for some $i \in I$. Indeed, for any element $H \in \mathcal{V} \setminus \mathcal{U}$, we have $H \to G_i$ for some $i \in I$, hence $G_i \in \mathcal{V}$, as $\mathcal{V}$ is an upset. This shows that the only covers of $\mathcal{U}$ are $\mathcal{U} \cup \{G_i\}$ $(i \in I)$. □

Remark 5.14. Choosing the ascending chain $K_3 \to K_4 \to \ldots$ in the first half of the proof of Theorem 5.13, we obtain $\mathcal{U} = \emptyset$, since every finite graph has a finite chromatic number. This shows that the empty set has no upper cover in $\mathrm{Upsets}(\mathcal{A})$, consequently $\mathcal{A} \mathbin{\dot\cup} \emptyset^{\dot\cup K_2} = \mathcal{A}$ has no upper cover in $\mathrm{Sub}(\mathcal{G}_1)$.

To conclude this subsection, we prove, as promised in Section 4, that $\langle K_2 \mathbin{\dot\cup} L \rangle_{\circlearrowleft} \cup \mathcal{G}_1$ has no lower covers in $\mathrm{Sub}(\mathcal{G})$, or, equivalently, that $\mathcal{G}_1$ has no lower covers in $\mathrm{Sub}(\mathcal{G}_1)$. Actually, we shall prove more: no matter how small a step we take downwards from $\mathcal{G}_1$, we already have passed an uncountable part of the jungle.

Theorem 5.15. *For every $\not\circlearrowleft$-closed set $\mathcal{H} \subset \mathcal{G}_1$, the interval $\left[\mathcal{H}, \mathcal{G}_1\right]$ has continuum cardinality.*

Proof. Let us consider the decomposition $\mathcal{H} = \mathcal{H}_1 \mathbin{\dot\cup} \mathcal{H}_2^{\dot\cup K_2}$ as in Theorem 5.6. If $\mathcal{H}_2 = \mathcal{A}$, then also $\mathcal{H}_1 = \mathcal{A}$, since $\mathcal{H}_1 \supseteq \mathcal{H}_2$, and then $\mathcal{H} = \mathcal{A} \mathbin{\dot\cup} \mathcal{A}^{\dot\cup K_2} = \mathcal{A} \mathbin{\dot\cup} \mathcal{B} = \mathcal{G}_1$ (cf. Remark 5.5), contrary to our assumption.

Thus $\mathcal{H}_2 \subset \mathcal{A}$, hence $\mathcal{A} \mathbin{\dot\cup} \mathcal{H}_2^{\dot\cup K_2} \subset \mathcal{G}_1$; moreover, $\mathcal{A} \mathbin{\dot\cup} \mathcal{H}_2^{\dot\cup K_2}$ is $\not\circlearrowleft$-closed by Theorem 5.6. Let $G \in \mathcal{A} \setminus \mathcal{H}_2$, and let $H \in \mathcal{A}$ be a graph below G, i.e., $H \to G$ and $G \nrightarrow H$ (for example, let $H = C_{g+2}$, where g is the odd girth of G). Since $G \notin \mathcal{H}_2$ and $\mathcal{H}_2$ is an upset, it follows that $H \notin \mathcal{H}_2$. Therefore, $\mathcal{A} \setminus \mathcal{H}_2$ contains two comparable graphs (namely G and H), and then (the proof of) Theorem 5.10 shows that there is a continuum of $\not\circlearrowleft$-closed sets in the interval $\left[\mathcal{A} \mathbin{\dot\cup} \mathcal{H}_2^{\dot\cup K_2}, \mathcal{G}_1\right]$. Clearly, we have $\mathcal{H} = \mathcal{H}_1 \mathbin{\dot\cup} \mathcal{H}_2^{\dot\cup K_2} \subseteq \mathcal{A} \mathbin{\dot\cup} \mathcal{H}_2^{\dot\cup K_2}$, hence these $\not\circlearrowleft$-closed sets are all above $\mathcal{H}$. □

5.3. **The lower part of the jungle.** The lower part of the jungle, i.e., the interval $[\emptyset, \mathcal{A}] = \mathrm{Sub}(\mathcal{A})$, seems to be more complicated than the upper part. We only prove here the analogue of Theorem 5.15: every nonempty $\not\emptyset$-closed subset of $\mathcal{G}_1$ has a continuum of $\not\emptyset$-closed subsets. This implies immediately the promised result that $\emptyset$ has no upper covers in $\mathrm{Sub}(\mathcal{G}_1)$, or, equivalently, $\langle K_2 \dot\cup L\rangle_{\circlearrowleft}$ has no upper covers in $\mathrm{Sub}(\mathcal{G})$. The proof of this result relies on the following construction of "blowing up" a graph by replacing its vertices by complete graphs. For an arbitrary graph G and natural number ℓ, let G^ℓ denote the graph defined by

$$V(G^\ell) = V(G) \times \{1, \dots, \ell\} = \{(v, i)\colon v \in V(G), i \in \{1, \dots, \ell\}\};$$
$$E(G^\ell) = \{((v, i), (v', i'))\colon vv' \in E(V) \text{ or } v = v' \text{ and } i \neq i'\}.$$

(This is a special case of the so-called strong product of graphs, namely $G^\ell = G \boxtimes K_\ell$.)

Lemma 5.16. *For every $G \in \mathcal{G}_1$ and $\ell \geq 2$, we have $G^\ell \in \langle K_{2\ell}\rangle_{\not\emptyset}$.*

Proof. If uv is an edge in G, then $\{u, v\}\times\{1, \dots, \ell\}$ is a clique of size 2ℓ in G^ℓ, and these cliques cover every edge of G^ℓ. Therefore, by Lemma 3.4, we have $G^\ell \in \langle K_{2\ell}\rangle_{\not\emptyset}$. □

Lemma 5.17. *If $n > m \geq 5$ are odd numbers, then $C_m^\ell \nrightarrow C_n^\ell$.*

Proof. As mentioned in the proof of Lemma 5.16, every edge of an arbitrary graph G gives rise to a clique of size 2ℓ in G^ℓ. Moreover, if G contains no triangles (cliques of size 3), then these are the only cliques of size 2ℓ in G. Therefore, a homomorphism $\varphi\colon G^\ell \to H^\ell$ for triangle-free graphs G and H induces a mapping $\psi\colon E(G) \to E(H)$ such that if two edges $e_1, e_2 \in E(G)$ have a common endpoint, then $\psi(e_1)$ and $\psi(e_2)$ also have a common endpoint.

Now let us assume that $\varphi\colon C_m^\ell \to C_n^\ell$ is a homomorphism for some odd numbers $n > m > 3$. Since C_n and C_m are triangle-free, we can consider the corresponding map $\psi\colon E(C_m) \to E(C_n)$. Let $e_1, \dots, e_m$ be the edges of C_m in the cyclical order. Then $\psi(e_1), \dots, \psi(e_m)$ determine a connected subgraph with at most m edges in C_n. Since $n > m$, there is a vertex $v \in V(C_n)$ that does not belong to this subgraph. Then $\varphi(C_m^\ell)$ is disjoint from $\{v\} \times \{1, \dots, \ell\} \subseteq V(C_n^\ell)$, hence φ maps C_m^ℓ into P_n^ℓ, where P_n is the path of length n obtained from C_n by removing the vertex v. Clearly, $P_n \to K_2$, consequently $P_n^\ell \to K_{2\ell}$. Thus we have $C_m^\ell \to P_n^\ell \to K_{2\ell}$, which implies that $\chi(C_m^\ell) \leq 2\ell$. However, it is easy to see that C_m^ℓ is not 2ℓ-colorable. (Actually, by a result of Stahl [22], $\chi(C_{2k+1}^\ell) = 2\ell + 1 + \left[\frac{\ell-1}{k}\right]$.) This contradiction shows that $C_m^\ell \nrightarrow C_n^\ell$. □

Lemma 5.18. *If $n \geq 4$ and $m \geq 3$, then $K_m^\ell \nrightarrow C_n^\ell$.*

Proof. If $n \geq 4$ then C_n contains no triangles, and then the largest cliques in C_n^ℓ are the cliques of size 2ℓ (cf. the proofs of Lemma 5.16 and Lemma 5.17). If $m \geq 3$ then the size of $K_m^\ell = K_{m\ell}$ is $m\ell > 2\ell$, hence $K_m^\ell \nrightarrow C_n^\ell$. □

Lemma 5.19. *Let $N = \{5, 7, 9, \dots\}$ and let $T_n = K_n^\ell \dot\cup C_n^\ell$. Then for every $n \in N$, we have $T_n \notin \langle\{T_m\colon m \in N, m \neq n\}\rangle_{\not\emptyset}$.*

Proof. Assume that $T_n \in \langle\{T_m\colon m \in N, m \neq n\}\rangle_{\not\emptyset}$ for some odd integer $n \geq 5$. By Lemma 3.4, there exist $m_1, \dots, m_k \in N \setminus \{n\}$ such that there is a complete homomorphism $\varphi\colon T_{m_1} \dot\cup \cdots \dot\cup T_{m_k} \twoheadrightarrow T_n$. If $m_i > n$ then $T_{m_i} \nrightarrow T_n$, as T_{m_i} has a clique of size $m_i\ell$, whereas the largest clique in T_n is of size $n\ell$. Therefore, we have $m_i < n$ for $i = 1, \dots, k$. By Lemma 5.17 and Lemma 5.18, $C_{m_i}^\ell \nrightarrow C_n^\ell$ and $K_{m_i}^\ell \nrightarrow C_n^\ell$, hence φ maps T_{m_i} into K_n^ℓ for each i. However, this contradicts the surjectivity of φ. □

Lemma 5.20. *For every $G \in \mathcal{G}_1$ and $n \geq 3$ we have $K_n \in \langle G \rangle_{\not\emptyset}$ if and only if $\chi(G) \leq n$.*

Proof. If $K_n \in \langle G \rangle_{\not\emptyset}$, then, by Lemma 3.4, there exists a complete homomorphism $\varphi\colon k \cdot G \twoheadrightarrow K_n$ for some $k \geq 1$. Restricting φ to any one of the k copies of G, we get a homomorphism (not necessarily complete) $G \to K_n$, and this shows that $\chi(G) \leq n$.

Now assume that $\chi(G) \leq n$, and let us use the numbers $1, 2, \ldots, n$ for the n colors in proper n-colorings of G. Let us fix an edge $uv \in E(G)$, and for each pair of colors $i \neq j$ let us choose a proper n-coloring of G such that u and v receive the colors i and j, respectively. Joining all these $\binom{n}{2}$ colorings we obtain a homomorphism $\binom{n}{2} \cdot G \to K_n$, which is complete, as each edge $ij \in E(K_n)$ is the image of one of the $\binom{n}{2}$ copies of the edge uv. This proves that $K_n \in \langle G \rangle_{\not\emptyset}$. □

Theorem 5.21. *For every $\not\emptyset$-closed set $\mathcal{H} \supset \emptyset$, the interval $\big[\emptyset, \mathcal{H}\big]$ has continuum cardinality.*

Proof. Let H be an arbitrary element of $\mathcal{H}$, and let $\ell = \chi(H)$. According to Lemma 5.20, we have $K_\ell \in \langle H \rangle_{\not\emptyset} \subseteq \mathcal{H}$. By Lemma 5.19, the map $S \mapsto \langle \{T_m : m \in S\} \rangle_{\not\emptyset}$ embeds the power set of N into $\mathrm{Sub}(\mathcal{G}_1)$. Moreover, $\langle \{T_m : m \in S\} \rangle_{\not\emptyset} \subseteq \mathcal{H}$ for every $S \subseteq N$, since, by Lemma 5.16, $T_n \in \langle K_{2\ell} \rangle_{\not\emptyset} \subseteq \langle K_\ell \rangle_{\not\emptyset} \subseteq \mathcal{H}$ for all $n \in N$. □

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