MAXIMAL AND MINIMAL CLOSED CLASSES
IN MULTIPLE-VALUED LOGIC

TAMÁS WALDHAUSER

Abstract. We consider classes of operations in multiple-valued logic that are 
closed under composition as well as under permutation of variables, identifi-
cation of variables (diagonalization) and introduction of inessential variables 
(cylindrification). Such closed classes on a given finite set form a complete 
lattice that includes the lattice of clones as the principal filter above the triv-
ial clone. We determine all maximal closed classes; it turns out that there 
is only one family of closed classes besides Rosenberg’s six families of maxi-
mal clones. For minimal closed classes we prove an analogon of Rosenberg’s 
five-type classification of minimal clones and we describe explicitly the unary 
minimal closed classes.

1. Introduction

Let $A$ be a nonempty finite set, and let $C$ be a class of finitary operations on $A$. If $C$ is closed under composition of operations and contains the projections, then $C$ is called a clone on $A$. There are countably infinitely many clones on a two-element base set, and all such clones were determined by Post [15]. For $|A| \geq 3$ there exists a continuum of clones on $A$ (see [11]), and it is widely accepted that an explicit description of clones is an extremely difficult task even for $|A| = 3$. The set of all clones on $A$ is a complete lattice, and many authors have investigated different parts of these lattices of clones. Here we focus on two special classes of clones: maximal and minimal clones. Maximal clones (i.e., coatoms in the lattice of clones) have been determined by Rosenberg [16] on arbitrary finite sets. The description of minimal clones (i.e., atoms in the lattice of clones) seems to be a considerably harder problem; a full description is available only for $|A| \leq 4$ (see [7, 19, 20, 18]). However, Rosenberg classified minimal clones into five types, and for two of the types he found necessary and sufficient conditions for minimality over arbitrary finite sets [17].

In this paper we generalize these two theorems of Rosenberg about maximal and minimal clones to more general classes of operations. We consider classes that are closed under composition but do not necessarily contain projections. However, we assume that our classes are closed under permutation of variables, identification of variables and introduction of inessential variables; in the case of clones, these properties are guaranteed by the presence of projections. The set of these closed classes on a given base set $A$ forms a complete lattice under inclusion, in which the lattice of clones appears as the principal filter generated by the trivial clone (i.e., the clone that consists of projections only). We determine the coatoms in the lattice of closed classes, thus extending Rosenberg’s theorem about maximal clones to these more general classes of operations. We will see that the coatoms are exactly the maximal clones and one more family of closed classes without projections. Since the bottom element of the lattice of closed classes is the empty class, the atoms are quite trivial to determine (one of them is the trivial clone). The true analogues of minimal clones turn out to be the closed classes on the “second floor” of the lattice, i.e., the covers of atoms, hence we will refer to these classes as minimal.

Key words and phrases. class of operations, composition of operations, subfunction, minor, equational class, clone, maximal clone, minimal clone.
closed classes. These include minimal clones, hence describing them is at least as difficult as describing minimal clones. We provide a classification of minimal closed classes in the spirit of Rosenberg’s classification of minimal clones, and we determine minimal closed classes for one of the five types, namely for the unary type.

In the next section we recall the definitions and results about classes of operations, composition, clones and relations that will be used in the sequel. For more background on these topics we refer the reader to the monographs [12] and [14]. Then in Section 3 we state and prove the above mentioned generalization of Rosenberg’s theorem on maximal clones, and in Section 4 we establish a generalization of Rosenberg’s classification of minimal clones.

2. Preliminaries

Throughout this paper, $A$ is a nonempty finite set, and $O_A = \bigcup_{n \geq 1} A^{A^n}$ denotes the set of all finitary operations on $A$. For a class $\mathcal{K} \subseteq O_A$ of operations, let $\mathcal{K}^{(n)}$ stand for the $n$-ary part of $\mathcal{K}$, i.e., $\mathcal{K}^{(n)} := \mathcal{K} \cap A^{A^n}$. For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$.

2.1. Composition of operations and classes of operations. For $f \in O_A^{(n)}$ and $g_1, \ldots, g_n \in O_A^{(k)}$, the composition of $f$ by $g_1, \ldots, g_n$ is the operation $f(g_1, \ldots, g_n) \in O_A^{(k)}$ defined by

$$(f(g_1, \ldots, g_n))(\mathbf{x}) = f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) \quad \text{for all } \mathbf{x} \in A^k.$$  

We can extend this definition to composition of classes of operations: for $\mathcal{K}, \mathcal{L} \subseteq O_A$ let $\mathcal{K} \circ \mathcal{L}$ denote the set

$$(1) \quad \{ f(g_1, \ldots, g_n) : k, n \in \mathbb{N}, f \in \mathcal{K}^{(n)}, g_1, \ldots, g_n \in \mathcal{L}^{(k)} \}.$$  

This composition is a binary operation on the power set of $O_A$. In general, it is not associative, but it becomes associative when restricted to so-called equational classes (see the next subsection).

2.2. Subfunctions and equational classes. The $i$-th $n$-ary projection for $1 \leq i \leq n \in \mathbb{N}$ is the operation $e_i^{(n)} \in O_A^{(n)}$ such that $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ for all $(x_1, \ldots, x_n) \in A^n$. For $f, g \in O_A$, we say that $g$ is a subfunction (or identification minor) of $f$ (notation: $g \preceq f$) if $g$ belongs to the class composition

$$\{ f \} \circ \{ e_i^{(n)} : n \in \mathbb{N}, i \in [n] \},$$  

i.e., if $g$ can be obtained from $f$ by permutation of variables, identification of variables (diagonalization) and introduction of inessential variables (cylindrification). The subfunction relation is a quasiorder on $O_A$, and the corresponding equivalence is defined by $f \equiv g \iff f \preceq g$ and $g \preceq f$. It is easy to see that two operations are equivalent if and only if they can be obtained from each other by permutation of variables and introduction or deletion of inessential variables. In the following we will not make a sharp distinction between equivalent functions. In particular, denoting the identity function on $A$ by id, we have $f \equiv \text{id}$ if and only if $f$ is a projection; therefore, we will simply write $\{ \text{id} \}$ for the set of projections.

A class $\mathcal{K} \subseteq O_A$ of operations on $A$ is called an equational class if it is an order ideal in the subfunction quasiorder. This terminology is motivated by the fact that definability by certain types of functional equations is equivalent to being closed under forming subfunctions [8] [13]. Although composition of classes of operations is not associative in general, equational classes form a semigroup under composition [4] [5]. Every clone is an idempotent element in this semigroup, and every idempotent is a composition-closed equational class (see the formal definition in the
next subsection, and see also Section 5). In [1] the study of the semigroup of equa-
tional classes was initiated with the intention of obtaining a better understanding 
of composition of operations and composition-closed classes such as clones.

2.3. Clones and closed classes. A clone on $A$ is a class $K \subseteq O_A$ that is closed 
under composition and contains all projections:

$$K \circ K \subseteq K \quad \text{and} \quad \{ \text{id} \} \subseteq K.$$  

The least clone containing a given class $K \subseteq O_A$ is denoted by $[K]$; it consists of 
those operations that can be built from members of $K$ and from projections by 
means of composition. The set of all clones on a fixed base set $A$ constitutes a lattice 
under inclusion, with the lattice operations being $C_1 \wedge C_2 = C_1 \cap C_2$ and 
$C_1 \vee C_2 = [C_1 \cup C_2]$. The least element of the lattice of clones over $A$ is $\{ \text{id} \}$, the 
clone containing only projections, which is called the trivial clone, and the greatest 
element is $O_A$, the clone of all operations on $A$. Atoms and coatoms of the lattice of 
clones are called minimal clones and maximal clones, respectively. As mentioned in 
the introduction, maximal clones on finite sets are completely known [10], while for 
minimal clones only a classification is available, with two of the five types completely 
described [17].

We consider in this paper equational classes that are closed under composition, 
that is, classes $K \subseteq O_A$ such that $K \circ K \subseteq K$ and $f \in K$, $g \leq f \implies g \in K$ for all 
g \in O_A, or, more compactly,

$$K \circ K \subseteq K \quad \text{and} \quad K \circ \{ \text{id} \} \subseteq K. $$

For brevity, in the following we will simply write closed class instead of composition-
closed equational class. For $K \subseteq O_A$, we denote by $[K]$ the least closed class 
containing $K$. The set of all closed classes on $A$ forms a lattice under inclusion, 
with the lattice operations being $K_1 \wedge K_2 = K_1 \cap K_2$ and $K_1 \vee K_2 = [K_1 \cup K_2]$. 
Note that the least element of this lattice is the empty class. For all $K \subseteq O_A$, we 
have $[K] = [K \cup \{ \text{id} \}]$, and $C \subseteq O_A$ is a close if and only if $C$ is a closed class 
and $\text{id} \in C$. Therefore, the lattice of clones appears in the lattice of closed classes 
as the principal filter generated by the trivial clone. The lattice of closed classes 
over a two-element base set has already continuum cardinality; this lattice has been 
described in [21].

Remark 1. Iterative algebras provide another generalization of clones. These 
classes are usually defined by the five Mal’tsev operations $\zeta$, $\tau$, $\Delta$, $\nabla$, $*$ (see [12] [14]), but they can also be defined by means of class composition as follows. A class 
$K \subseteq O_A$ is an iterative algebra iff

$$K \circ (K \cup \{ \text{id} \}) \subseteq K.$$  

It is clear that (2) $\implies$ (1) $\implies$ (3), hence every clone is an iterative algebra, and 
every iterative algebra is a closed class.

2.4. Relations and constraints. For an $m$-ary relation $R \subseteq A^m$ and a matrix 
$N \in A^{m \times n}$, we say that $N$ is an $R$-matrix if each column of $N$ belongs to $R$. If 
f \in O_A, then $fN$ stands for the $m$-tuple that is obtained by applying $f$ row-wise 
to $N$, and let $fR = \{ fN : N \in A^{m \times n} \text{ is an } R\text{-matrix} \}$. If $fR \subseteq R$, then we say 
that $f$ preserves the relation $R$. For a set $Q$ of relations, let $Pol(Q)$ denote the set 
of all operations that preserve every member of $Q$:

$$Pol(Q) = \{ f \in O_A : \forall R \in Q \quad fR \subseteq R \}.$$  

Preservation of relations induces a Galois connection between operations and re-
lations on $A$. The corresponding Galois closed sets are exactly the clones and the 
so-called relational clones (see [3] [9]). Thus, $C \subseteq O_A$ is a clone if and only if there 
exists a set of relations $Q$ such that $C = Pol(Q)$. Relational clones can also be 
characterized as sets of relations closed under certain constructions; since we will
not need this description, we do not give the details here. We will only use the fact that the relations in the smallest relational clone (generated by the unary total relation $A$) are exactly relations of the form
\begin{equation}
S = \{ a \in A^m : a_i = a_j \text{ whenever } (i,j) \in \varepsilon \},
\end{equation}
where $\varepsilon$ is an equivalence relation on $[m]$. In other words, a relation $S \subseteq A^m$ is preserved by all operations on $A$ if and only if $S$ is of form (5).

Composition-closed equational classes can be described by relational constraints, i.e., pairs $(R,S)$ of relations, where $R$ and $S$ have the same arity. For a set $Q$ of relational constraints, we define \( \text{Pol}^*(Q) \subseteq O_A \) as follows:
\[ \text{Pol}^*(Q) = \{ f \in O_A : \forall (R,S) \in Q \ fR \subseteq S \text{ and } fS \subseteq S \}. \]
It was shown in [21] that a class $K \subseteq O_A$ is a closed class if and only if there exists a set of relational constraints $Q$ such that $C = \text{Pol}^*(Q)$.

Remark 2. Iterative algebras (cf. Remark 1) also admit a characterization in terms of relational constraints, as shown by Harnau in [10]. Here one uses pairs of relations $(R,S)$ with $S \subseteq R$, and an operation $f$ is said to preserve such a pair if $fR \subseteq S$.

3. Maximal closed classes

First let us recall Rosenberg’s description of maximal clones. We do not define the types of relations appearing in the theorem, as we will not need them.

**Theorem 1** (Rosenberg’s theorem [16]). A clone $C \subseteq O_A$ is a maximal clone if and only if $C = \text{Pol} R$ for some relation $R$ satisfying one of the following six conditions:

(i) $R$ is a bounded partial order;
(ii) $R$ is the graph of a permutation of prime order;
(iii) $R$ is a nontrivial equivalence relation;
(iv) $R$ is a prime-affine relation;
(v) $R$ is a central relation;
(vi) $R$ is an $h$-regular relation.

**Proposition 1.** Every composition-closed equational class of $O_A$ is contained in a maximal composition-closed equational class.

**Proof.** It is well known that $O_A$ is a finitely generated clone. In fact, $O_A$ can be generated by a single operation; such a generator is called a Sheffer operation. For instance, the operation defined by
\[ g(x,y) = \begin{cases} x \oplus 1, & \text{if } x = y; \\ 0, & \text{if } x \neq y. \end{cases} \]
is a Sheffer operation on $A = \{0, 1, \ldots, n-1\}$, where $\oplus$ stands for addition modulo $n$ (see [22]). Thus, we have $[g] = O_A$, hence $[g, \text{id}] = O_A$. This shows that $O_A$ is a finitely generated closed class, and then by Zorn’s lemma we have that each closed class is contained in a maximal one.

We will prove that the maximal closed subclasses of $O_A$ are exactly the maximal clones together with the classes $M_{ab} := \{ f \in O_A : f(a, \ldots, a) = f(b, \ldots, b) \}$ for $a, b \in A, a \neq b$.

It is easy to verify directly that $M_{ab}$ is a closed class; alternatively, one can observe that $M_{ab} = \text{Pol}^* \left( \{ (a,b) \} , = \right)$.

**Lemma 1.** For any $a, b \in A, a \neq b$, the clone generated by $M_{ab}$ is $O_A$. 

Proof. It would be sufficient to verify that $M_{ab}$ is not contained in any of the maximal clones given in Theorem 1. Alternatively, since $M_{ab}$ contains all constants, one could use one of the criteria for functional completeness (e.g., the Werner-Wille theorem). However, it seems easier to give a direct proof as follows. Let $f \in O_A$ be an arbitrary operation, and let $n$ denote the arity of $f$. Choose any operation $g \in O_A$ of arity $n + 2$ such that
\begin{equation}
   g(x_1, \ldots, x_n, a, b) = f(x_1, \ldots, x_n)
\end{equation}
for all $(x_1, \ldots, x_n) \in A^n$, and
\begin{equation}
   g(x, \ldots, x) = a
\end{equation}
for all $x \in A$. (Clearly, there are many such operations $g$; the values of $g$ on tuples not listed above are irrelevant.) Condition (7) guarantees that $g \in M_{ab}$, and then by (6) we conclude $f \in [M_{ab}]$, as the constants $a$ and $b$ also belong to $M_{ab}$. □

**Theorem 2.** The maximal composition-closed equational classes on $A$ are the maximal clones listed in Theorem 1 and the classes $M_{ab}$ with $a, b \in A, a \neq b$.

Proof. Let $K$ be a maximal closed class that is not a clone. Since $K$ is closed, there exists a set $Q$ of relational constraints such that $K = \text{Pol}^*(Q)$. For each $(R, S) \in Q$ we have $\text{Pol}^*(R, S) \supseteq K$, hence $\text{Pol}^*(R, S) = K$ or $\text{Pol}^*(R, S) = O_A$ by the maximality of $K$. From $K = \bigcap_{(R, S) \in Q} \text{Pol}^*(R, S)$ it follows that there exists $(R, S) \in Q$ such that $\text{Pol}^*(R, S) = K$. Then we have $K = \text{Pol}^*(R, S) \subseteq \text{Pol}(S)$, hence either $\text{Pol}(S) = K$ or $\text{Pol}(S) = O_A$, again by the maximality of $K$. However, the first case is impossible, as $K$ is not a clone. Thus $\text{Pol}(S) = O_A$, which means that $S$ belongs to the smallest relational clone (generated by the unary total relation $A$), thus $S$ is of the form $[5]$ for some $m \in N$ and an equivalence relation $\varepsilon$ on $[m]$.

Next we show that $R \not\subseteq S$. Suppose for contradiction that $R \subseteq S$, and let $f \in O_A$ be an arbitrary operation. Then $fR \subseteq fS \subseteq S$, since $S$ is preserved by all operations. This implies that $\text{Pol}^*(R, S) = O_A$, which contradicts $\text{Pol}^*(R, S) = K$.

Thus $R \not\subseteq S$, and then there exists a tuple $r \in R \setminus S$. Taking into account that $S$ is given by (5), $r \notin S$ implies that there exist $i, j \in [m]$ such that $(i, j) \in \varepsilon$ but $r_i \neq r_j$. We claim that $K = M_{ab}$, where $a = r_i, b = r_j$.

Let $f \in K$ be an arbitrary operation, and let $N$ be the $m \times n$ matrix such that each column vector of $N$ is $r$ and $n$ is the arity of $f$. Clearly, $N$ is an $R$-matrix. Since $f \in K = \text{Pol}^*(R, S)$, we have $fN \in S$. The $i$-th and $j$-th entries of the tuple $fN$ are $(a, \ldots, a)$ and $(b, \ldots, b)$, respectively. Therefore, $fN \in S$ and $(i, j) \in \varepsilon$ imply that $f(a, \ldots, a) = f(b, \ldots, b)$ according to (6). This shows that $f \in M_{ab}$ for every $f \in K$, i.e., $K \subseteq M_{ab}$. By the maximality of $K$ we can conclude that $K = M_{ab}$.

We have proved that every maximal closed class is either a maximal clone or one of the classes $M_{ab}$ with $a \neq b$. It remains to prove that each of these classes is indeed maximal. By Proposition 1 every closed class is contained in a maximal one, hence it suffices to prove that the aforementioned classes are pairwise incomparable. It is clear that maximal clones are pairwise incomparable as well as the classes $M_{ab} (a, b \in A, a \neq b)$. Now let $C$ be a maximal clone and let $a, b \in A, a \neq b$. There are no projections in $M_{ab}$, hence $C \not\subseteq M_{ab}$, and $M_{ab} \not\subseteq C$ follows immediately from Lemma 1. □

**Corollary 1.** There are finitely many maximal composition-closed equational classes on $A$.

Proof. It follows from Theorem 1 that there are finitely many maximal clones on $A$, and it is clear that there are finitely many classes $M_{ab}$. □

**Example 1.** All closed classes on $\{0, 1\}$ have been described in [21]; in particular, the maximal closed classes turned out to be the five maximal clones of Boolean
functions (0-preserving functions, 1-preserving functions, monotone functions, linear functions, selfdual functions) together with the class
\[ \Omega_n = \{ f \in \mathcal{O}_{\{0,1\}} : f(0,\ldots,0) = f(1,\ldots,1) \}. \]

Theorem 2 indicates that the situation is similar over arbitrary finite sets, as the classes \( \mathcal{M}_{ab} \) are immediate generalizations of \( \Omega_n \).

4. Minimal closed classes

First we determine the atoms of the lattice of closed classes on \( A \). We say that a unary operation \( u \in O_A^{(1)} \) is idempotent, if \( u^2 = u \), i.e., \( u(u(x)) = u(x) \) holds for all \( x \in A \). Observe that \( u \) is idempotent if and only if \( u(x) = x \) for all \( x \in \text{ran } u := \{ u(x) : x \in A \} \). (Note that it is also customary to say that \( f \in O_A \) is idempotent if \( f(x,\ldots,x) = x \) for all \( x \in A \). We will not use this notion of idempotence in this paper.)

**Proposition 2.** A class \( K \subseteq O_A \) is an atom in the lattice of composition-closed equational classes on \( A \) if and only if \( K = \{ u \} \) for some idempotent unary operation \( u \in O_A^{(1)} \).

**Proof.** Let \( K \) be an atom in the lattice of closed classes, and let \( f \in K \) be an arbitrary operation. Since \( K \) is an equational class, the unary operation \( g(x) := f(x,\ldots,x) \) belongs to \( K \). Finiteness of \( A \) implies that some power of \( g \) is idempotent, i.e., there exists \( k \in \mathbb{N} \) such that \( u := g^k \) satisfies \( u^2 = u \). Clearly, \( u \in K \), as \( K \) is closed under composition. Therefore, \( \emptyset \not\subseteq \{ u \} \subseteq K \), and then \( \{ u \} = K \), since \( K \) has no proper nonempty closed subclasses. \( \square \)

**Remark 3.** If \( u \) is an idempotent unary operation, then \( \{ u \} = \{ u \} \); in other words, \( \{ u \} \) contains only the operations that are equivalent to \( u \), i.e., essentially unary operations \( f \) of the form \( f(x_1,\ldots,x_n) = u(x_i) \). For \( u = \text{id} \) we obtain the trivial clone \( \{ \text{id} \} = \{ \text{id} \} = \{ \text{id} \} \). In the following we will refer to the atoms described in Proposition 2 as trivial closed classes. The proof of Proposition 2 shows that every nontrivial nonempty closed class contains a trivial closed class.

To be in accordance with the terminology of clone theory, we shall say that a nonempty nontrivial closed class \( K \) is a minimal closed class, if the only nonempty nontrivial closed subclass of \( K \) is \( K \) itself. If \( K \) is a minimal closed class and \( \{ u \} \subseteq K \) is a trivial closed class, then \( K \) covers \( \{ u \} \) in the lattice of closed classes, and we will briefly express this fact by saying that \( K \) is a minimal closed class above \( \{ u \} \). Note that it may happen that \( K \) covers two trivial closed classes; see Remark 4.

Now we recall Rosenberg’s theorem on minimal clones, and then we present the corresponding result for minimal closed classes.

**Theorem 3** (Rosenberg’s theorem \[17\]). Let \( C \subseteq O_A \) be a minimal clone, and let \( f \) be an operation in \( C \setminus \{ \text{id} \} \) of minimum arity. Then \( K = \{ f \} \) and one of the following five conditions hold for \( f \):

(I) \( f \) is a unary operation;

(II) \( f \) is a binary operation such that for all \( x \in A \),
\[ f(x,x) = x; \]

(III) \( f \) is a ternary operation such that for all \( x,y \in A \),
\[ f(x,x,y) = f(x,y,x) = f(y,x,x) = x; \]

(IV) \( f \) is a ternary operation such that for all \( x,y \in A \),
\[ f(x,x,y) = f(x,y,x) = f(y,x,x) = y; \]

(V) \( f \) is of arity \( n \) with \( 3 \leq n \leq |A| \), and there exists an \( i \in [n] \) such that \( f(x_1,\ldots,x_n) = x_i \) whenever \([x_1,\ldots,x_n] \leq n\).

**Theorem 4.** Let \( K \subseteq O_A \) be a minimal composition-closed equational class above \( \{ u \} \), where \( u \in O_A^{(1)} \) is an idempotent unary operation. Let \( f \) be an operation in
\( \mathcal{K} \setminus \{u\} \) of minimum arity. Then \( \mathcal{K} = [f, u] \), and one of the following five conditions hold for \( f \):

(I) \( f \) is a unary operation;

(II) \( f \) is a binary operation such that for all \( x \in A \),
    \( f(x, x) = u(x) \);

(III) \( f \) is a ternary operation such that for all \( x, y \in A \),
    \( f(x, x, y) = f(y, x, x) = u(x) \);

(IV) \( f \) is a ternary operation such that for all \( x, y \in A \),
    \( f(x, x, y) = f(y, x, x) = u(y) \);

(V) \( f \) is of arity \( n \) with \( 3 \leq n \leq |A| \), and there exists an index \( i \in [n] \) such that
    \( f(x_1, \ldots, x_n) = u(x_i) \) whenever \( |\{x_1, \ldots, x_n\}| < n \).

Proof. Since \( \mathcal{K} \) is minimal, it is clear that \( \mathcal{K} = [f, u] \) for any \( f \in \mathcal{K} \setminus \{u\} \). Let \( f \in \mathcal{K} \setminus \{u\} \) be of minimum arity, and let us denote this minimal arity by \( n \). If \( n = 1 \), then \( f \) is of type (I). From now on we shall assume that \( n \geq 2 \). If \( g \) is any operation that is obtained from \( f \) by identifying some of its variables, then \( g \in \{u\} \) by the minimality of the arity of \( f \), hence \( g \equiv u \). If \( n = 2 \), then this immediately implies that (II) holds.

If \( n \geq 4 \), then by a generalization of Świerczkowski’s lemma (see Theorem 7 in [6]) there exists an index \( i \in [n] \) such that \( f(x_1, \ldots, x_n) = u(x_i) \) whenever \( x_1, \ldots, x_n \in A \) are not pairwise different. If \( n > |A| \), then this implies \( f(x_1, \ldots, x_n) = u(x_i) \) for all \( x_1, \ldots, x_n \in A \), i.e., \( f \equiv u \). Moreover, it is also straightforward to verify (by term induction) that every ternary operation in \([g, u] \setminus \{u\}\) satisfies (10) as well. The minimality of \( \mathcal{K} \) implies that \( \mathcal{K} = [g, u] \) (note that it follows from (10) that \( g \) is essentially ternary). Therefore, we have \( f \in [g, u] \setminus \{u\} \), hence \( f \) satisfies (10), too. In particular, we have \( f(a, a, b) = a \), which contradicts (9). Thus, the case \( (r, s, t) = (y, x, y) \) is impossible whenever \( u \) is not constant, and this completes the proof.

Corollary 2. There are finitely many minimal composition-closed equational classes in \( \mathcal{O}_A \), and every nonempty nontrivial composition-closed equational class contains a minimal one.

Proof. Let us denote by \( T \) the set of all closed classes of \( \mathcal{O}_A \) that are of the form \([f, u]\), where \( u \) is an idempotent unary operation and \( f \notin \{u\} \) is of arity at most \( \max(|A|, 3) \). (Note that we do not require here that \([f, u]\) is a minimal closed class.) We claim that every nontrivial closed class contains a closed subclass that
belongs to \( T \). To this extent, let \( \mathcal{K} \) be a nontrivial closed subclass of \( \mathcal{O}_A \), and let \( u \in \mathcal{K}^{(1)} \) with \( u^2 = u \). If \( f \in \mathcal{K} \setminus \{ u \} \) is of minimum arity, then following the first two paragraphs of the proof of Theorem 4, we see that the arity of \( f \) is at most \( \max(|A|, 3) \). Thus \( [f, u] \in T \) is the desired closed subclass of \( \mathcal{K} \), and this completes the proof of our claim.

By Theorem 4 every minimal closed class is a minimal element of the partially ordered set \((T, \subseteq)\). Conversely, let \( \mathcal{K} \) be a minimal element of \( T \). If \( \mathcal{K}_1 \) is a nontrivial closed subclass of \( \mathcal{K} \), then, by the claim in the previous paragraph, there exists \( \mathcal{K}_2 \subseteq T \) such that \( \mathcal{K}_2 \subseteq \mathcal{K}_1 \subseteq \mathcal{K} \). Since \( \mathcal{K}_2, \mathcal{K} \in T \) and \( \mathcal{K} \) is minimal in \( T \), we must have \( \mathcal{K}_2 = \mathcal{K}_1 = \mathcal{K} \). This shows that \( \mathcal{K} \) is a minimal closed class. Thus we have proved that the minimal closed classes coincide with the minimal elements of \( T \).

Now the first statement of the theorem follows immediately, as \( T \) is finite. For the second statement, we just need to recall that every nontrivial closed class \( \mathcal{K} \) contains a subclass \( \mathcal{K}_1 \in T \), and \( \mathcal{K}_1 \) contains a minimal element \( \mathcal{K}_2 \) of \( T \), again by the finiteness of \( T \). Then \( \mathcal{K}_2 \) is a minimal closed subclass contained in \( \mathcal{K} \).

Minimal clones of type (I) and type (IV) have been explicitly described by Rosenberg in [17]. A unary operation \( f \) generates a minimal clone if and only if either \( f \) is idempotent \((f^2 = f)\) or \( f \) is a permutation of prime order \((f^p = id\) for some prime \( p \)). A ternary operation \( f \) of type (IV) generates a minimal clone if and only if there exists a binary operation \(+\) on \( A \) such that \((A; +)\) is an Abelian group of exponent 2 and \( f(x, y, z) = x + y + z \). In the following theorem we describe minimal closed classes \( \mathcal{K} \) of type (I). In this case all operations in \( \mathcal{K} \) are essentially unary (and equivalent to some member of \( \mathcal{K}^{(1)} \)), hence it suffices to describe the unary part \( \mathcal{K}^{(1)} \).

**Theorem 5.** Let \( \mathcal{K} \subseteq \mathcal{O}_A \) be a minimal composition-closed equational class of type (I) above \([u]\), where \( u \in \mathcal{O}_A^{(1)} \) is an idempotent unary operation. Then there exists \( f \in \mathcal{K}^{(1)} \setminus \{u\} \) such that \( \mathcal{K} = [f, u] \) and one of the following three conditions holds:

(1) there exists a prime \( p \) such that \( f^p = u, fu = uf = f \); in this case we have \( \mathcal{K}^{(1)} = \{ f, f^2, \ldots, f^p \} \);

(2) \( f^2 = f, fu = uf = f \); in this case we have \( \mathcal{K}^{(1)} = \{ f, u \} \);

(3) \( f^2 = fu = uf = u \); in this case we have \( \mathcal{K}^{(1)} = \{ f, f^2 \} \).

**Proof.** Since every member of \( \mathcal{K} \) is essentially unary, we may work with its unary part, which constitutes a subsemigroup of the transformation semigroup \( \mathcal{O}_A^{(1)} \). The minimality of \( \mathcal{K} \) means that \( \mathcal{K}^{(1)} \) has exactly two subsemigroups containing \( u \), namely \( \{ u \} \) and \( \mathcal{K}^{(1)} \). Let \( f \in \mathcal{K}^{(1)} \setminus \{ u \} \) be an arbitrary operation, then \( \mathcal{K} = [f, u] \) and \( \mathcal{K}^{(1)} \) (as a semigroup) is generated by \( f \) and \( u \). Since \( \mathcal{K}^{(1)} \) is a finite semigroup, each of its elements has an idempotent power. In particular, there exists \( k \in \mathbb{N} \) such that \( f^k \) is idempotent. We separate two cases on whether \( f^k = u \) or not.

**Case 1:** \( f^k = u \). In this case \( \mathcal{K}^{(1)} \) is generated by \( f \), hence it is a cyclic semigroup. If the index of this cyclic semigroup is at least 2, then \( \{ f^2, f^3, \ldots \} \) is a proper subsemigroup of \( \mathcal{K}^{(1)} \). By minimality, this implies \( \{ f^2, f^3, \ldots \} = \{ u \} \), hence \( f^2 = u \) and \( f^3 = fu = uf = u \), and thus the conditions of (1) are fulfilled (in this case \( \mathcal{K}^{(1)} \) is a two-element zero semigroup). If the index of \( \mathcal{K}^{(1)} \) is 1, then \( \mathcal{K}^{(1)} \) is a group with identity element \( u \). Then it is clear that \( \mathcal{K} \) is minimal if and only if \( \mathcal{K}^{(1)} \) is a cyclic group of prime order, hence (1) is satisfied.

**Case 2:** \( f^k \neq u \). In this case \( f^k \) and \( u \) generate a subsemigroup of \( \mathcal{K}^{(1)} \) that properly contains \( \{ u \} \). By minimality, this subsemigroup must be all of \( \mathcal{K}^{(1)} \). Therefore, \( \mathcal{K}^{(1)} \) is generated by two idempotents, hence we may assume without loss of generality that \( f \) is idempotent (we replace the generating set \( \{ f, u \} \) with \( \{ f^k, u \} \)).

It is clear that \( \mathcal{K}^{(1)} \circ \{ u \} := \{ gu: g \in \mathcal{K}^{(1)} \} \) is a subsemigroup of \( \mathcal{K}^{(1)} \) that contains \( \{ u \} \). Therefore, we have either \( \mathcal{K}^{(1)} \circ \{ u \} = \mathcal{K}^{(1)} \) or \( \mathcal{K}^{(1)} \circ \{ u \} = \{ u \} \). In the former
case \( f = gu \) for some \( g \in \mathcal{K}^{(1)} \), hence \( fu = gu^2 = gu = f \), while in the latter case \( fu = u \). A similar argument, using the subsemigroup \( \{u\} \circ \mathcal{K}^{(1)} \), shows that \( uf = f \) or \( uf = u \), hence the conditions of (I\(b\)) are satisfied. (Note that we have four possibilities for the pair \((fu, uf)\): two of them yield a two-element semilattice, and the other two possibilities correspond to \( \mathcal{K}^{(1)} \) being a two-element left or right zero semigroup.) 

**Remark 4.** Note that if \( f \) belongs to types (II)–(V), then \( f(x, \ldots, x) = u(x) \), hence \([f, u] = [f]\). Also, if condition (I\(b\)) or (I\(c\)) of Theorem 3 holds, then \([f, u] \neq [f]\). However, if \( f \) corresponds to type (I\(b\)), then \([f, u] = [f]\), and in this case \([f, u]\) cannot be generated by a single operation. Observe also that in all types except (I\(b\)), there is only one idempotent unary operation in a minimal closed class, hence the operation \( u \) in Theorems 3 and 4 is unique, and then our minimal class is join-irreducible in the lattice of closed classes. A minimal class of type (I\(b\)) contains exactly two idempotent unary operations, hence it has two lower covers in the lattice of closed classes, and therefore it is not join-irreducible.

**Example 2.** There are three atoms in the lattice of closed classes of Boolean functions: \([\text{id}]\), \([0]\), \([1]\). The minimal closed classes above \([\text{id}]\) are the seven minimal clones: \([0]\) (type (I\(b\))), \([1]\) (type (I\(b\))), \([-x]\) (type (I\(a\))), \([x \land y]\) (type (II)), \([x \lor y]\) (type (II)), \([xy \lor xz \lor yz]\) (type (III)) and \([x + y + z]\) (type (IV)). The results of [21] imply that the minimal closed classes covering \([0]\) are \([0, 1]\) (type (I\(b\))), \([x + y]\) (type (II)), \([xy + y]\) (type (II)). The minimal closed classes covering \([1]\) are the duals of the latter classes, namely \([0, 1]\) (type (I\(b\))), \([x + y + 1]\) (type (II)), \([-\to]\) (type (II)). As observed in Remark 4, minimal closed classes of type (I\(b\)) cover two atoms: \([0]\) covers \([0]\) and \([\text{id}]\), \([1]\) covers \([1]\) and \([\text{id}]\) and \([0, 1]\) covers \([0]\) and \([1]\).

5. Concluding remarks

We described explicitly the maximal composition-closed equational classes of operations on finite sets, and gave a five-type classification of minimal closed classes in analogy with Rosenberg’s theorem on minimal clones. Explicit description of minimal closed classes remains an open problem, which involves describing minimal clones, a notoriously difficult problem. Nevertheless, it may be possible to determine minimal closed classes “modulo minimal clones” in some sense, e.g., to show that every minimal closed class can be constructed from a minimal clone by some canonical construction.

Finally, let us point out another related problem. As mentioned in Section 2 the set of all equational classes on \( A \) constitute a semigroup under the operation of composition of classes of operations as defined in [1]; we shall denote this semigroup by \( E_m \) if \( A = \{0, 1, \ldots, m - 1\} \). A first step in exploring the structure of this semigroup would be to determine its idempotent elements. For \( m = 2 \) (i.e., for Boolean functions) this was done in [21], and in [2] the regular elements of \( E_2 \) were also described. Clearly, if \( \mathcal{K} \in E_m \) is idempotent (\( \mathcal{K} \circ \mathcal{K} = \mathcal{K} \)), then \( \mathcal{K} \) is a closed class (\( \mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K} \)). The converse is also true if \( m = 2 \) (see [21]), but for \( m \geq 3 \) there exist classes \( \mathcal{K} \in E_m \) with \( \mathcal{K} \circ \mathcal{K} \subset \mathcal{K} \) (see Examples 3 and 4 below). It is also clear that every clone is an idempotent equational class, since if \( \mathcal{K} \) is a clone then \( \mathcal{K} \circ \mathcal{K} \supseteq \mathcal{K} \circ \{\text{id}\} = \mathcal{K} \).

The maximal closed classes of Theorem 2 are all idempotent. We have seen above that clones are idempotent, so it suffices to consider the classes \( \mathcal{M}_{ab} \). Let \( * \) be an arbitrary binary operation on \( A \) such that

\[
b * b = a \text{ and } a * x = x \text{ for all } x \in A.
\]

Then \( * \) belongs to \( \mathcal{M}_{ab} \), and for every \( f \in \mathcal{M}_{ab}^{(n)} \) we have \( f(x_1, \ldots, x_n) = a * f(x_1, \ldots, x_n) \in \mathcal{M}_{ab} \circ \mathcal{M}_{ab} \), since the constant function \( a \) is also a member.
Figure 1. The lattice of closed subclasses of $K_2$ (see Example 4)

of $M_{ab}$. This shows that $M_{ab} \supseteq M_{ab} \circ M_{ab}$, hence $M_{ab}$ is idempotent. It follows that the maximal idempotents of $E_m$ are the same as the maximal closed classes on $A = \{0, 1, \ldots, m - 1\}$.

Now let $K = [f, u]$ be a minimal closed class, as in Theorem 4. Since every element of $K$ can be obtained from $f$ and $u$ by composition, $K \circ K \supseteq K \setminus \{f, u\}$. Furthermore, $u^2 = u$ implies that $u \in K \circ K$. Therefore, $K$ is idempotent if and only if $f \in K \circ K$. Unary minimal closed classes of types (I$_a$) and (I$_b$) are all idempotent. On the other hand, if $K$ is a minimal closed class of type (I$_c$), then $K$ is not idempotent, since $K \circ K = \{u\}$ in this case. As the next example shows, there also exist binary minimal closed classes that are not idempotent.

Example 3. Let $f(x, y) = x \ast y$ be the binary operation on $A = \{0, 1, 2\}$ defined by $1 \ast 2 = 2 \ast 1 = 3$ and $x \ast y = 0$ for all $(x, y) \in A^2 \setminus \{(1, 2), (2, 1)\}$. Then $(A; \ast)$ is a semigroup, and $x \ast y \ast z = 0$ for every $x, y, z \in A$. Thus, $K := [f] = \{f, 0\}$ is a minimal closed class of type (I) covering the trivial class $\{0\}$. However, $K$ is not idempotent: we have $K \circ K = \{0\}$.

Our last example illustrates that, in some sense, idempotents of $E_m$ behave “less nicely” than closed classes.

Example 4. We consider some unary closed classes on $A = \{0, 1, 2\}$. Let $f_{abc}$ denote the unary operation on $A$ that is defined by $f_{abc}(0) = a$, $f_{abc}(1) = b$, $f_{abc}(2) = c$. In particular, $f_{012} = \text{id}$ and $f_{000}$ is the constant zero function. Let $K = \{f_{000}, f_{001}\}$, then $K \circ K = \{f_{000}\}$, therefore $K$ is closed but not idempotent. The classes $K_1 := \{f_{000}, f_{001}, f_{012}\}$ and $K_2 := \{f_{000}, f_{001}, f_{112}, f_{111}\}$ are both idempotents containing $K$, and we have $K = K_1 \cap K_2$. Thus, the intersection of idempotents is not always idempotent, and there may be no least idempotent containing a given closed class $K$. The lattice of closed subclasses of $K_2$ is shown in Figure 1, where idempotent classes are marked by filled circles, whereas non-idempotent classes are marked by empty circles. We can see from Figure 1 that although $K_2$ is minimal among idempotent classes that contain the minimal closed class $K$, the class $K_2$ is not a minimal idempotent subclass of $O_A$.

Acknowledgements

This research is supported by the Hungarian National Foundation for Scientific Research under grants no. K83219 and K104251, by the János Bolyai Research Scholarship, by the European Union and co-funded by the European Social Fund
under the project “Telemedicine-focused research activities on the field of Mathematics, Informatics and Medical sciences” of project number “TAMOP-4.2.2.A-11/1/KONV-2012-0073”.

REFERENCES


(T. Waldhauser) Bolyai Institute, University of Szeged, Aradi věrtanúk teré 1, H-6720 Szeged, Hungary

E-mail address: twaldha@math.u-szeged.hu