LOCALLY MONOTONE BOOLEAN AND PSEUDO-BOOLEAN FUNCTIONS

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ABSTRACT. We propose local versions of monotonicity for Boolean and pseudo-Boolean functions: say that a pseudo-Boolean (Boolean) function is p-locally monotone if none of its partial derivatives changes in sign on tuples which differ in less than p positions. As it turns out, this parameterized notion provides a hierarchy of monotonicities for pseudo-Boolean (Boolean) functions.

Local monotonicities are shown to be tightly related to lattice counterparts of classical partial derivatives via the notion of permutable derivatives. More precisely, p-locally monotone functions are shown to have p-permutable lattice derivatives and, in the case of symmetric functions, these two notions coincide. We provide further results relating these two notions, and present a classification of p-locally monotone functions, as well as of functions having p-permutable derivatives, in terms of certain forbidden "sections", i.e., functions which can be obtained by substituting constants for variables. This description is made explicit in the special case when $p = 2$.

1. INTRODUCTION

Throughout this paper, let $[n] = \{1, \ldots, n\}$ and $\mathbb{B} = \{0, 1\}$. We are interested in the so-called Boolean functions $f \colon \mathbb{B}^n \to \mathbb{B}$ and pseudo-Boolean functions $f \colon \mathbb{B}^n \to \mathbb{R}$, where n denotes the arity of f. The pointwise ordering of functions is denoted by $\leq$, i.e., $f \leq g$ means that $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{B}^n$. The negation of $x \in \mathbb{B}$ is defined by $\overline{x} = x \oplus 1$, where $\oplus$ stands for addition modulo 2. For $x, y \in \mathbb{B}$, we set $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$.

For $k \in [n]$, $\mathbf{x} \in \mathbb{B}^n$, and $a \in \mathbb{B}$, let $\mathbf{x}_k^a$ be the tuple in $\mathbb{B}^n$ whose i-th component is a, if $i = k$, and x_i, otherwise. We use the shorthand notation $\mathbf{x}_{jk}^{ab}$ for $(\mathbf{x}_j^a)_k^b = (\mathbf{x}_k^b)_j^a$. More generally, for $S \subseteq [n]$, $\mathbf{a} \in \mathbb{B}^n$, and $\mathbf{x} \in \mathbb{B}^S$, let $\mathbf{a}_S^{\mathbf{x}}$ be the tuple in $\mathbb{B}^n$ whose i-th component is x_i, if $i \in S$, and a_i, otherwise.

Let $i \in [n]$ and $f \colon \mathbb{B}^n \to \mathbb{R}$. A variable x_i is said to be *essential* in f, or that f *depends* on x_i, if there exists $\mathbf{a} \in \mathbb{B}^n$ such that $f(\mathbf{a}_i^0) \neq f(\mathbf{a}_i^1)$. Otherwise, x_i is said to be *inessential* in f. Let $S \subseteq [n]$ and $f \colon \mathbb{B}^n \to \mathbb{R}$. We say that $g \colon \mathbb{B}^S \to \mathbb{R}$ is an *S-section* of f if there exists $\mathbf{a} \in \mathbb{B}^n$ such that $g(\mathbf{x}) = f(\mathbf{a}_S^{\mathbf{x}})$ for all $\mathbf{x} \in \mathbb{B}^S$. By a *section* of f we mean an S-section of f for some $S \subseteq [n]$, i.e., any function which can be obtained from f by replacing some of its variables by constants.

The *(discrete) partial derivative* of $f \colon \mathbb{B}^n \to \mathbb{R}$ with respect to its k-th variable is the function $\Delta_k f \colon \mathbb{B}^n \to \mathbb{R}$ defined by $\Delta_k f(\mathbf{x}) = f(\mathbf{x}_k^1) - f(\mathbf{x}_k^0)$; see [9, 12]. Note that $\Delta_k f$ does not depend on its k-th variable, hence it could be regarded as a function of arity $n - 1$, but for notational convenience we define it as an n-ary function.

A pseudo-Boolean function $f \colon \mathbb{B}^n \to \mathbb{R}$ can always be represented by a multilinear polynomial of degree at most n (see [13]), that is,

$$(1) \qquad f(\mathbf{x}) = \sum_{S \subseteq [n]} a_S \prod_{i \in S} x_i \,,$$

where $a_S \in \mathbb{R}$. For instance, the multilinear expression for a binary pseudo-Boolean function is given by

$$(2) \qquad a_0 + a_1 x_1 + a_2 x_2 + a_{12} x_1 x_2 \,.$$

This representation is very convenient for computing the partial derivatives of f. Indeed, $\Delta_k f$ can be obtained by applying the corresponding formal derivative to the multilinear representation of f. Thus, from (1), we immediately obtain

$$(3) \qquad \Delta_k f(\mathbf{x}) \;=\; \sum_{S \ni k} a_S \prod_{i \in S \setminus \{k\}} x_i \,.$$

We say that f is *isotone* (resp. *antitone*) *in its k-th variable* if $\Delta_k f(\mathbf{x}) \geq 0$ (resp. $\Delta_k f(\mathbf{x}) \leq 0$) for all $\mathbf{x} \in \mathbb{B}^n$. If f is either isotone or antitone in its k-th variable, then we say that f is *monotone in its k-th variable*. If f is isotone (resp. antitone, monotone) in all of its variables, then f is an *isotone* (resp. *antitone, monotone*) *function*.[1] It is clear that any section of an isotone (resp. antitone, monotone) function is also isotone (resp. antitone, monotone). Thus defined, a function $f \colon \mathbb{B}^n \to \mathbb{R}$ is monotone if and only if none of its partial derivatives changes in sign on $\mathbb{B}^n$.

Noteworthy examples of monotone functions include the so-called *pseudo-polynomial functions* [2, 3] which play an important role, for instance, in the qualitative approach to decision making; for general background see, e.g., [1, 6]. In the current setting, pseudo-polynomial functions can be thought of as compositions $p \circ (\varphi_1, \ldots, \varphi_n)$ of (lattice) polynomial functions $p \colon [a,b]^n \to [a,b]$, $a < b$, with unary functions $\varphi_i \colon \mathbb{B} \to [a,b]$, $i \in [n]$. Interestingly, pseudo-polynomial functions $f \colon \mathbb{B}^n \to \mathbb{R}$ coincide exactly with those pseudo-Boolean functions that are monotone.

Theorem 1. *A pseudo-Boolean function is monotone if and only if it is a pseudo-polynomial function.*

Proof. Clearly, every pseudo-polynomial function is monotone. For the converse, suppose that $f \colon \mathbb{B}^n \to \mathbb{R}$ is monotone and let $a \in \mathbb{R}$ be the minimum and $b \in \mathbb{R}$ the maximum of f. Constant functions are obviously pseudo-polynomial functions, therefore we assume $a < b$. Define $\varphi_i \colon \mathbb{B} \to \{a,b\}$ by $\varphi_i(0) = a$ and $\varphi_i(1) = b$ if f is isotone in its i-th variable and $\varphi_i(0) = b$ and $\varphi_i(1) = a$ otherwise. Let $p \colon \{a,b\}^n \to [a,b]$ be given by $p = f \circ (\varphi_1^{-1}, \ldots, \varphi_n^{-1})$. Thus defined, p is isotone (i.e., order-preserving) in each variable and hence, by Theorem D in [10, p. 237], there exists a polynomial function $p' \colon [a,b]^n \to [a,b]$ such that $p'|_{\{a,b\}^n} = p$. Therefore f is the pseudo-polynomial function $p' \circ (\varphi_1, \ldots, \varphi_n)$. $\qquad\square$

In the special case of Boolean functions, monotone functions are most frequent among functions of small (essential) arity. For instance, among binary functions $f \colon \mathbb{B}^2 \to \mathbb{B}$, there are exactly two non-monotone functions, namely the Boolean sum $x_1 \oplus x_2$ and its negation $x_1 \oplus x_2 \oplus 1$. Each of these functions is in fact highly non-monotone in the sense that any of its partial derivatives changes in sign when negating its unique essential variable; this is not the case, e.g., with $f(x_1, x_2, x_3) = x_1 - x_1 x_2 + x_2 x_3$ which is non-monotone but none of its partial derivatives changes in sign when negating any of its variables (see Example 6).

This fact motivates the study of these "skew" functions, i.e., these highly non-monotone functions. To formalize this problem we propose the following parameterized relaxations of monotonicity: a function $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-*locally monotone* if none of its partial derivatives changes in sign when negating up to p of its variables, or equivalently, on tuples which differ in less than p positions. With this terminology, our problem reduces to asking which Boolean functions are not 2-locally monotone. As we will see (Corollary 10), these are precisely those functions that have the Boolean sum or its negation as a binary section.

In this paper we extend this study to pseudo-Boolean functions and show that these parameterized relaxations of monotonicity are tightly related to the following lattice versions of partial derivatives. For $f \colon \mathbb{B}^n \to \mathbb{R}$ and $k \in [n]$, let $\wedge_k f \colon \mathbb{B}^n \to \mathbb{R}$ and

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[1]Note that the terms "positive" and "nondecreasing" (resp. "negative" and "nonincreasing") are often used instead of isotone (resp. antitone), and it is also customary to use the word "monotone" only for isotone functions.

$\vee_k f \colon \mathbb{B}^n \to \mathbb{R}$ be the *partial lattice derivatives* defined by

$$\wedge_k f(\mathbf{x}) \;=\; f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1) \qquad \text{and} \qquad \vee_k f(\mathbf{x}) \;=\; f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1).$$

The latter, known as the k-th *join derivative* of f, was proposed by Fadini [7] while the former, known as the k-th *meet derivative* of f, was introduced by Thayse [16]. In [17] these lattice derivatives were shown to be related to so-called prime implicants and implicates of Boolean functions which play an important role in the consensus method for Boolean and pseudo-Boolean functions. For further background and applications see, e.g., [4, 5, 8, 15, 18].

Observe that, just like in the case of the partial derivative $\Delta_k f$, the k-th variable of each of the lattice derivatives $\wedge_k f$ and $\vee_k f$ is inessential.

The following proposition assembles some basic properties of lattice derivatives.

Proposition 2. *For any pseudo-Boolean functions $f, g \colon \mathbb{B}^n \to \mathbb{R}$ and $j, k \in [n]$, $j \neq k$, the following hold:*

   (i) *$\wedge_k \wedge_k f = \wedge_k f$ and $\vee_k \vee_k f = \vee_k f$;*
   (ii) *if $f \leq g$, then $\wedge_k f \leq \wedge_k g$ and $\vee_k f \leq \vee_k g$;*
   (iii) *$\wedge_j \wedge_k f = \wedge_k \wedge_j f$ and $\vee_j \vee_k f = \vee_k \vee_j f$;*
   (iv) *$\vee_k \wedge_j f \leq \wedge_j \vee_k f$.*

From equations (1) and (3) it follows that every function is (up to an additive constant) uniquely determined by its partial derivatives. As it turns out, this does not hold when lattice derivatives are considered. However, as we shall see (Theorem 22), there are only two types of such exceptions.

Now, if an n-ary pseudo-Boolean function is 2-locally monotone, then for every $j, k \in [n]$, $j \neq k$, we have $\vee_k \wedge_j f = \wedge_j \vee_k f$ (see Lemma 11). This motivates the notion of permutable lattice derivatives. As it turns out, p-local monotonicity of f implies permutability of p of its lattice derivatives (see Theorem 21). However the converse does not hold (see Example 24).

The structure of this paper goes as follows. In Section 2 we formalize the notion of p-local monotonicity and show that it gives rise to a hierarchy of monotonicities whose largest member is the class of all n-ary pseudo-Boolean functions (this is the case when $p = 1$) and whose smallest member is the class of n-ary monotone functions (this is the case when $p = n$). We also provide a characterization of p-locally monotone functions in terms of "forbidden" sections; as mentioned, this characterization is made explicit in the special case when $p = 2$. In Section 3 we introduce the notion of permutable lattice derivatives. Similarly to local monotonicity, the notion of permutable lattice derivatives gives rise to nested classes, each of which is also described in terms of its sections. In the Boolean case and for $p = 2$, these two parameterized notions are shown to coincide; this does not hold for pseudo-Boolean functions even when $p = 2$ (see Example 13). (At the end of Section 3 we also provide some game-theoretic interpretations of p-local monotonicity and p-permutability of lattice derivatives.) However, in Section 4, we show that a symmetric function is p-locally monotone if and only if it has p-permutable lattice derivatives. In the last section we discuss directions for future research.

2. Local monotonicities

The following definition formulates a local version of monotonicity given in terms of Hamming distance between tuples. In what follows we assume that $p \in [n]$.

Definition 3. We say that $f \colon \mathbb{B}^n \to \mathbb{R}$ is *p-locally monotone* if, for every $k \in [n]$ and every $\mathbf{x}, \mathbf{y} \in \mathbb{B}^n$, we have

$$\sum_{i \in [n] \setminus \{k\}} |x_i - y_i| \;<\; p \quad \Rightarrow \quad \Delta_k f(\mathbf{x}) \, \Delta_k f(\mathbf{y}) \;\geq\; 0.$$

Any p-locally monotone pseudo-Boolean function is also p'-locally monotone for every $p' \leq p$. Every function $f \colon \mathbb{B}^n \to \mathbb{R}$ is 1-locally monotone, and f is n-locally

monotone if and only if it is monotone. Thus p-local monotonicity is a relaxation of monotonicity, and the nested classes of p-locally monotone functions for $p = 1, \ldots, n$ provide a hierarchy of monotonicities for n-ary pseudo-Boolean functions. The weakest nontrivial condition is 2-local monotonicity, therefore we will simply say that f is *locally monotone* whenever f is 2-locally monotone.[2] If f is p-locally monotone for some $p < n$ but not $(p+1)$-locally monotone, then we say that f is *exactly p-locally monotone*, or that the *degree of local monotonicity* of f is p.

If $f \colon \mathbb{B}^n \to \mathbb{B}$ is a Boolean function, then $\Delta_k f(\mathbf{x}) \in \{-1, 0, 1\}$ for all $\mathbf{x} \in \mathbb{B}^n$, hence the condition $\Delta_k f(\mathbf{x}) \, \Delta_k f(\mathbf{y}) \geq 0$ in the definition of p-local monotonicity is equivalent to

$$(4) \qquad |\Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y})| \; \leq \; 1.$$

From this it follows that a Boolean function $f \colon \mathbb{B}^n \to \mathbb{B}$ is locally monotone if and only if

$$(5) \qquad \big| \Delta_k f(\mathbf{x}) - \Delta_k f(\mathbf{y}) \big| \; \leq \; \sum_{i \in [n] \setminus \{k\}} |x_i - y_i|.$$

(see [14, Lemma 5.1] for a proof of (5) in a slightly more general framework). In a sense, the latter identity means that $\Delta_k f$ is "1-Lipschitz continuous".

The following proposition is just a reformulation of the definition of p-local monotonicity.

Proposition 4. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if and only if, for every $k \in [n]$, $S \subseteq [n] \setminus \{k\}$, with $|S| = p - 1$, and every $\mathbf{a} \in \mathbb{B}^n$, $\mathbf{x}, \mathbf{y} \in \mathbb{B}^S$, we have*

$$(6) \qquad \Delta_k f(\mathbf{a}_S^{\mathbf{x}}) \, \Delta_k f(\mathbf{a}_S^{\mathbf{y}}) \; \geq \; 0.$$

Equivalently, a pseudo-Boolean function is p-locally monotone if and only if none of its partial derivatives changes in sign when negating less than p of its variables.

As a special case, we have that $f \colon \mathbb{B}^n \to \mathbb{R}$ is locally monotone if and only if, for every $j, k \in [n]$, $j \neq k$, and every $\mathbf{x} \in \mathbb{B}^n$, we have

$$(7) \qquad \Delta_k f(\mathbf{x}_j^0) \, \Delta_k f(\mathbf{x}_j^1) \; \geq \; 0.$$

Equivalently, a pseudo-Boolean function is locally monotone if and only if none of its partial derivatives changes in sign when negating any of its variables.

By (4) we see that, for Boolean functions $f \colon \mathbb{B}^n \to \mathbb{B}$, inequality (7) can be replaced with $|\Delta_{jk} f(\mathbf{x})| \leq 1$, where $\Delta_{jk} f(\mathbf{x}) = \Delta_j \Delta_k f(\mathbf{x}) = \Delta_k \Delta_j f(\mathbf{x})$.

Example 5. As observed, the binary Boolean sum

$$f_1(x_1, x_2) \; = \; x_1 \oplus x_2 \; = \; x_1 + x_2 - 2x_1 x_2$$

and the binary Boolean equivalence

$$f_2(x_1, x_2) \; = \; \overline{f}_1(x_1, x_2) \; = \; x_1 \oplus x_2 \oplus 1 \; = \; 1 - x_1 - x_2 + 2x_1 x_2$$

are not locally monotone. Indeed, we have $|\Delta_{12} f_1(x_1, x_2)| = |\Delta_{12} f_2(x_1, x_2)| = 2$.

Example 6. Consider the ternary Boolean function $f \colon \mathbb{B}^3 \to \mathbb{B}$ given by

$$f(x_1, x_2, x_3) \; = \; x_1 - x_1 x_2 + x_2 x_3.$$

Since $\Delta_2 f$ may change in sign ($\Delta_2 f(\mathbf{x}) = x_3 - x_1$), the function f is not monotone. However, f is locally monotone since $|\Delta_{12} f(\mathbf{x})| = 1$, $|\Delta_{13} f(\mathbf{x})| = 0$, and $|\Delta_{23} f(\mathbf{x})| = 1$. Thus f is exactly 2-locally monotone. Example 26 in Section 4 provides, for each $p \geq 2$, examples of exactly p-locally monotone functions.

Fact 7. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if and only if so is $\alpha f + \beta$ for every $\alpha, \beta \in \mathbb{R}$, with $\alpha \neq 0$. The same holds for any function obtained from f by negating some of its variables.*

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[2]In [11], local monotonicity is used to refer to Boolean functions which are monotone (i.e., isotone or antitone in each variable).

The next theorem gives a characterization of p-locally monotone functions in terms of their sections.

Theorem 8. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone if and only if every p-ary section of f is monotone.*

Proof. We just need to observe that the inequality (6) is equivalent to $\Delta_k g(\mathbf{x})\Delta_k g(\mathbf{y}) \geq 0$, where g is the p-ary section of f defined by $g(\mathbf{x}) = f(\mathbf{a}_{S \cup \{k\}}^{\mathbf{x}})$, where S is a $(p-1)$-subset of $[n] \setminus \{k\}$. Thus f is p-locally monotone if and only if $\Delta_k g(\mathbf{x})\Delta_k g(\mathbf{y}) \geq 0$ holds for every $\mathbf{x}, \mathbf{y} \in \mathbb{B}^{S \cup \{k\}}$, and for every $S \cup \{k\}$-section g of f. $\square$

By combining (7) with Theorem 8, we can easily verify the following corollary.

Corollary 9. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ is locally monotone if and only if every binary section (2) of f satisfies $a_1(a_1 + a_{12}) \geq 0$ and $a_2(a_2 + a_{12}) \geq 0$.*

Since every binary Boolean function is monotone except for $x \oplus y$ and $x \oplus y \oplus 1$, we also obtain the following corollary.

Corollary 10. *A Boolean function $f \colon \mathbb{B}^n \to \mathbb{B}$ is locally monotone if and only if neither $x \oplus y$ nor $x \oplus y \oplus 1$ is a section of f.*

3. Permutable lattice derivatives

The aim of this section is to relate commutation of lattice derivatives to p-local monotonicity. The starting point is the characterization of locally monotone Boolean functions given in Theorem 12.

Lemma 11. *If $f \colon \mathbb{B}^n \to \mathbb{R}$ is locally monotone, then $\vee_k \wedge_j f = \wedge_j \vee_k f$ for all $j, k \in [n]$, $j \neq k$.*

Proof. Let $f \colon \mathbb{B}^n \to \mathbb{R}$ be a locally monotone function, and let $j, k \in [n]$, $j \neq k$. Setting $a = f(\mathbf{x}_{jk}^{00})$, $b = f(\mathbf{x}_{jk}^{01})$, $c = f(\mathbf{x}_{jk}^{10})$, and $d = f(\mathbf{x}_{jk}^{11})$, the desired equality $\vee_k \wedge_j f(\mathbf{x}) = \wedge_j \vee_k f(\mathbf{x})$ takes the form

$$(8) \qquad (a \wedge c) \vee (b \wedge d) = (a \vee b) \wedge (c \vee d).$$

Since f is 2-locally monotone, the binary section $g(u, v) = f(\mathbf{x}_{jk}^{uv})$ is monotone, according to Theorem 8. If g is isotone in u, then $a \leq c$ and $b \leq d$, while if g is antitone in u, then $a \geq c$ and $b \geq d$. Similarly, we have either $a \leq b$ and $c \leq d$ or $a \geq b$ and $c \geq d$, depending on whether g is isotone or antitone in v. Thus we need to consider four cases, and in each one of them it is straightforward to verify (8). $\square$

Theorem 12. *A Boolean function $f \colon \mathbb{B}^n \to \mathbb{B}$ is locally monotone if and only if $\vee_k \wedge_j f = \wedge_j \vee_k f$ holds for all $j, k \in [n]$, $j \neq k$.*

Proof. If f is locally monotone, then $\vee_k \wedge_j f = \wedge_j \vee_k f$ by Lemma 11. If f is not locally monotone, then Corollary 10 implies that there exists $\mathbf{a} \in \mathbb{B}^n$ and $j, k \in [n]$, $j \neq k$, such that the binary section $g(u, v) = f(\mathbf{a}_{jk}^{uv})$ is of the form $g(u, v) = u \oplus v$ or $g(u, v) = u \oplus v \oplus 1$. Then we have

$$\vee_k \wedge_j f(\mathbf{a}) = (g(0, 0) \wedge g(1, 0)) \vee (g(0, 1) \wedge g(1, 1)) = 0,$$
$$\wedge_j \vee_k f(\mathbf{a}) = (g(0, 0) \vee g(0, 1)) \wedge (g(1, 0) \vee g(1, 1)) = 1,$$

showing that $\vee_k \wedge_j f \neq \wedge_j \vee_k f$. $\square$

As the next example shows, Theorem 12 is not valid for pseudo-Boolean functions.

Example 13. Let f be the binary pseudo-Boolean function defined by $f(0, 0) = 1$, $f(0, 1) = 4$, $f(1, 0) = 2$ and $f(1, 1) = 3$. Then we have $\vee_2 \wedge_1 f = \wedge_1 \vee_2 f = 3$ and $\vee_1 \wedge_2 f = \wedge_2 \vee_1 f = 2$. However, f is not locally monotone since $\Delta_1 f(\mathbf{x}_2^0)\Delta_1 f(\mathbf{x}_2^1) = -1$.

Lemma 11 and Theorem 12 motivate the following notion of permutability of lattice derivatives, and its relation to local monotonicities.

Definition 14. We say that a pseudo-Boolean function $f \colon \mathbb{B}^n \to \mathbb{R}$ *has p-permutable lattice derivatives* if, for every p-subset $\{k_1, \ldots, k_p\} \subseteq [n]$, every choice of the operators $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ $(i = 1, \ldots, p)$, and every permutation $\pi \in S_p$, the following identity holds:

$$O_{k_1} \cdots O_{k_p} f \;=\; O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f.$$

If $f \colon \mathbb{B}^n \to \mathbb{R}$ has n-permutable lattice derivatives, then we simply say that f *has permutable lattice derivatives*.

Every function $f \colon \mathbb{B}^n \to \mathbb{R}$ has 1-permutable lattice derivatives. We will see in Theorem 23 that if a function $f \colon \mathbb{B}^n \to \mathbb{R}$ has p-permutable lattice derivatives, then it also has p'-permutable lattice derivatives for every $p' \leq p$.

Fact 15. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ has p-permutable lattice derivatives if and only if so has $\alpha f + \beta$ for every $\alpha, \beta \in \mathbb{R}$, with $\alpha \neq 0$. The same holds for any function obtained from f by negating some of its variables.*

Fact 16. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ has p-permutable lattice derivatives if and only if every p-ary section of f has permutable lattice derivatives.*

In the particular case when $p = 2$, we have the following description of functions having 2-permutable lattice derivatives. The proof is a straightforward verification of cases.

Proposition 17. *A function $f \colon \mathbb{B}^n \to \mathbb{R}$ has 2-permutable lattice derivatives if and only if every binary section (2) of f satisfies $a_1 a_{12} \geq 0$ or $a_2 a_{12} \geq 0$ or $|a_{12}| \leq |a_1| \vee |a_2|$.*

Lemma 11 shows that the class of 2-locally monotone pseudo-Boolean functions is a subclass of that of pseudo-Boolean functions which have 2-permutable lattice derivatives. Example 13 then shows that this inclusion is strict. Now, according to Theorem 12, a Boolean function is 2-locally monotone if and only if it has 2-permutable lattice derivatives. Example 24 shows that the analogous equivalence does not hold for $p > 2$. However, p-local monotonicity implies p-permutability of lattice derivatives of any pseudo-Boolean function (see Theorem 21). To this extent, let us first study how the degree of local monotonicity is affected by taking lattice derivatives.

Lemma 18. *If $f \colon \mathbb{B}^n \to \mathbb{R}$ is monotone, then $\wedge_j f$ and $\vee_j f$ are also monotone for all $j \in [n]$.*

Proof. Clearly, if f is monotone, then so are $f_j^0(\mathbf{x}) = f(\mathbf{x}_j^0)$ and $f_j^1(\mathbf{x}) = f(\mathbf{x}_j^1)$, for all $j \in [n]$. Moreover, if f is isotone (resp. antitone) in x_k, then both f_j^0 and f_j^1 are also isotone (resp. antitone) in x_k. Since $\wedge$ and $\vee$ are isotone functions, we have that for every $j \in [n]$, both $\wedge_j f(\mathbf{x}) = f_j^0(\mathbf{x}) \wedge f_j^1(\mathbf{x})$ and $\vee_j f(\mathbf{x}) = f_j^0(\mathbf{x}) \vee f_j^1(\mathbf{x})$ are monotone. $\square$

Theorem 19. *If $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone, then $\wedge_j f$ and $\vee_j f$ are $(p-1)$-locally monotone for all $j \in [n]$.*

Proof. Suppose that $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone. By Theorem 8, it suffices to show that all $(p-1)$-ary sections of $\wedge_j f$ and $\vee_j f$ are monotone. We consider only $\vee_j f$, the other case can be dealt with in a similar way.

Let h be a $(p-1)$-ary section of $\vee_j f$ defined by $h(\mathbf{x}) = \vee_j f(\mathbf{a}_S^{\mathbf{x}})$ for all $\mathbf{x} \in \mathbb{B}^S$, where $\mathbf{a} \in \mathbb{B}^n$ and $S \subseteq [n]$ is a $(p-1)$-subset. Let $T = S \cup \{j\}$, and let us define $g \colon \mathbb{B}^T \to \mathbb{R}$ by $g(\mathbf{y}) = f(\mathbf{a}_T^{\mathbf{y}})$ for all $\mathbf{y} \in \mathbb{B}^T$. Clearly, g is a section of f, and the arity of g is either $p-1$ or p, depending on whether j belongs to S or not. A simple calculation shows that $h(\mathbf{y}|_S) = \vee_j g(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{B}^T$, where $\mathbf{y}|_S$ stands for the restriction of $\mathbf{y}$ to S. This means that if $j \notin S$, then h can be obtained from $\vee_j g$ by deleting its inessential j-th variable, and $h = \vee_j g$ if $j \in S$. Since f is p-locally monotone, g is monotone by Theorem 8, thus we can conclude with the help of Lemma 18 that h is monotone as well. $\square$

Corollary 20. *Let $0 \leq \ell < p \leq n$. If $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone, then, for every ℓ-subset $\{k_1, \ldots, k_\ell\} \subseteq [n]$ and every choice of the operators $O_{k_i} \in \{\vee_{k_i}, \wedge_{k_i}\}$ $(i = 1, \ldots, \ell)$, the function $O_{k_1} \cdots O_{k_\ell} f$ is $(p - \ell)$-locally monotone. In particular, if $\ell \leq p - 2$, then $O_{k_1} \cdots O_{k_\ell} f$ is locally monotone.*

Remark 1. We will see in Example 26 of Section 4 that Theorem 19 cannot be sharpened, i.e., the lattice derivatives of a p-locally monotone function are not necessarily p-locally monotone, not even in the case of Boolean functions.

With the help of Corollary 20 we can now prove the promised implication between p-local monotonicity and p-permutability of lattice derivatives, thus generalizing Lemma 11.

Theorem 21. *If $f \colon \mathbb{B}^n \to \mathbb{R}$ is p-locally monotone, then it has p-permutable lattice derivatives.*

Proof. Let $f \colon \mathbb{B}^n \to \mathbb{R}$ be a p-locally monotone function, let $\{k_1, \ldots, k_p\}$ be a p-subset of $[n]$, and let $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ for $i = 1, \ldots, p$. We need to show that for any permutation $\pi \in S_p$ the following identity holds:

$$O_{k_1} \cdots O_{k_p} f \;=\; O_{k_{\pi(1)}} \cdots O_{k_{\pi(p)}} f.$$

Since S_p is generated by transpositions of the form $(i \; i + 1)$, it suffices to prove that

$$O_{k_1} \cdots O_{k_{i-1}} O_{k_i} O_{k_{i+1}} O_{k_{i+2}} \cdots O_{k_p} f \;=\; O_{k_1} \cdots O_{k_{i-1}} O_{k_{i+1}} O_{k_i} O_{k_{i+2}} \cdots O_{k_p} f,$$

and for this it is sufficient to verify that

$$(9) \qquad\qquad O_{k_i} O_{k_{i+1}} g \;=\; O_{k_{i+1}} O_{k_i} g,$$

where g stands for the function $O_{k_{i+2}} \cdots O_{k_p} f$. From Corollary 20 it follows that g is locally monotone, and then Lemma 11 proves (9) if one of $O_{k_i}, O_{k_{i+1}}$ is a meet and the other is a join derivative. (If both are meet or both are join, then (9) is trivial.) $\qquad\square$

A natural question regarding lattice derivatives is whether a function can be reconstructed from its derivatives. As the next theorem shows, the answer is positive for almost all functions.

Theorem 22. *Let $f, g \colon \mathbb{B}^n \to \mathbb{R}$ be pseudo-Boolean functions such that for all $k \in [n]$ we have $\vee_k f = \vee_k g$ and $\wedge_k f = \wedge_k g$. Then either $f = g$ or there exists a one-to-one function $\alpha \colon \mathbb{B} \to \mathbb{R}$ such that $f(\mathbf{x}) = \alpha(x_1 \oplus \cdots \oplus x_n)$ and $g(\mathbf{x}) = \alpha(x_1 \oplus \cdots \oplus x_n \oplus 1)$ for all $\mathbf{x} \in \mathbb{B}^n$.*

Proof. To make the proof more vivid, we present it through the analysis of the following game. Alice picks a secret function $f \colon \mathbb{B}^n \to \mathbb{R}$, and Bob tries to identify this function by asking the values of its lattice derivatives. If he can do this, then he wins, otherwise Alice is the winner. We show that Bob has a winning strategy unless f is a function of the special form in the statement of the theorem.

Let us regard $\mathbb{B}^n$ as the set of vertices of the n-dimensional cube, and let Alice write the values of f to the corresponding vertices. Now the possible winning strategy for Bob is based on the following four basic observations.

1. *Bob can determine the unordered pair of numbers written to the endpoints of any edge of the cube.* Indeed, the endpoints of an edge are of the form $\mathbf{x}_k^0, \mathbf{x}_k^1$, and it is clear that $\{\wedge_k f(\mathbf{x}), \vee_k f(\mathbf{x})\} = \{f(\mathbf{x}_k^0), f(\mathbf{x}_k^1)\}$.
2. *If Bob can find the value of f at one point, then he can win.* According to the previous observation, knowing the value at one vertex of the cube, Bob can figure out the values written to the neighboring vertices. Since the graph of the cube is connected, he can determine all values of f this way.
3. *If $f(\mathbf{x}_k^0) = f(\mathbf{x}_k^1)$ for some $\mathbf{x} \in \mathbb{B}^n$, $k \in [n]$, then Bob can win.* This follows immediately from the first two observations.
4. *If the range of f contains at least three elements, then Bob can win.* We can suppose that the previous observation does not apply, i.e., for every edge Bob detects a two-element set. If f takes on at least three different values, then, by

the connectedness of the cube, there exists a vertex $\mathbf{x}$ and two edges incident with this vertex such that the two-element sets E_1 and E_2 corresponding to these edges are different. Then $E_1 \cap E_2$ must be a one-element set[3] containing the value of $f(\mathbf{x})$, and then Bob can win as explained in the second observation.

From these observations we can conclude that Bob has a winning strategy unless the range of f contains exactly two numbers and $f(\mathbf{x}_k^0) \neq f(\mathbf{x}_k^1)$, for all $\mathbf{x} \in \mathbb{B}^n, k \in [n]$. This means that f is of the following form for some $u \neq v \in \mathbb{R}$:

$$f(\mathbf{x}) = \begin{cases} u, & \text{if } |\mathbf{x}| \text{ is even}; \\ v, & \text{if } |\mathbf{x}| \text{ is odd}, \end{cases}$$

where $|\mathbf{x}| = \sum_{i=1}^n x_i$. In other words, $f(\mathbf{x}) = \alpha(x_1 \oplus \cdots \oplus x_n)$, where $\alpha(0) = u, \alpha(1) = v$. In this case Bob can determine f only up to interchanging u and v, i.e., he cannot distinguish f from $g(\mathbf{x}) = \alpha(x_1 \oplus \cdots \oplus x_n \oplus 1)$, so he has only 50% chance to win. (Indeed, f and g have the same lattice derivatives, namely their meet derivatives are all constant $u \wedge v$, while their join derivatives are all constant $u \vee v$.) $\qquad\square$

The following theorem shows that, as in the case of local monotonicity, the classes of functions having permutable lattice derivatives form a chain under inclusion.

Theorem 23. *If $f \colon \mathbb{B}^n \to \mathbb{R}$ has $(p+1)$-permutable lattice derivatives, then f has p-permutable lattice derivatives.*

Proof. Let $f \colon \mathbb{B}^n \to \mathbb{R}$ be a function that has $(p+1)$-permutable lattice derivatives. Using the same notation as in the proof of Theorem 21, it suffices to prove that

$$O_{k_1} \cdots O_{k_{i-1}} O_{k_i} O_{k_{i+1}} O_{k_{i+2}} \cdots O_{k_p} f = O_{k_1} \cdots O_{k_{i-1}} O_{k_{i+1}} O_{k_i} O_{k_{i+2}} \cdots O_{k_p} f.$$

Let g_1 and g_2 be the $(n-p)$-ary functions obtained from the left-hand side and from the right-hand side of this equality by deleting their inessential variables $x_{k_1}, \ldots, x_{k_p}$. If $O_{k_i} = \wedge_{k_i}, O_{k_{i+1}} = \wedge_{k_{i+1}}$ or $O_{k_i} = \vee_{k_i}, O_{k_{i+1}} = \vee_{k_{i+1}}$, then $g_1 = g_2$ holds trivially. Let us now assume that $O_{k_i} = \vee_{k_i}, O_{k_{i+1}} = \wedge_{k_{i+1}}$; the remaining case $O_{k_i} = \wedge_{k_i}, O_{k_{i+1}} = \vee_{k_{i+1}}$ is similar.

By Proposition 2, we have $g_1 \leq g_2$. Since the two (types of) functions given in Theorem 22 are order-incomparable, if $g_1 \neq g_2$, then the lattice derivatives of g_1 and g_2 cannot all coincide. Thus there exists $j \in [n] \setminus \{k_1, \ldots, k_p\}$ and $O_j \in \{\wedge_j, \vee_j\}$ such that $O_j g_1 \neq O_j g_2$. Taking into account the definition of g_1 and g_2, we can rewrite this inequality as

$$O_j O_{k_1} \cdots O_{k_{i-1}} O_{k_i} O_{k_{i+1}} O_{k_{i+2}} \cdots O_{k_p} f \neq O_j O_{k_1} \cdots O_{k_{i-1}} O_{k_{i+1}} O_{k_i} O_{k_{i+2}} \cdots O_{k_p} f,$$

which contradicts the fact that f has $(p+1)$-permutable lattice derivatives. $\qquad\square$

If $f \colon \mathbb{B}^n \to \mathbb{B}$ is a Boolean function with p-permutable lattice derivatives for some $p \geq 2$, then f has 2-permutable lattice derivatives by Theorem 23, and then Theorem 12 implies that f is 2-locally monotone. Unfortunately, nothing more can be said about the degree of local monotonicity of a Boolean function with p-permutable lattice derivatives. Indeed, the next example shows that there exist n-ary Boolean functions with n-permutable lattice derivatives that are exactly 2-locally monotone.

Example 24. Let $f_n \colon \mathbb{B}^n \to \mathbb{B}$ be the function that takes the value 1 on all tuples of the form

$$\mathbf{x} = (\overbrace{1, \ldots, 1}^{m}, 0, \ldots, 0) \text{ with } 0 \leq m \leq n,$$

and takes the value 0 everywhere else. Using Corollary 10, it is not difficult to verify that f_n is 2-locally monotone. However, if $n \geq 3$, then f_n is not 3-locally monotone, since

$$\begin{aligned} \Delta_2 f(0,0,0,0,\ldots,0) &= -1, \\ \Delta_2 f(1,0,1,0,\ldots,0) &= 1. \end{aligned}$$

Thus f_n is exactly 2-locally monotone.

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[3]If $E_1 \cap E_2$ is empty, then Alice is cheating!

We will show by induction on n that f_n has n-permutable lattice derivatives. First we compute the meet derivatives

$$\wedge_k f_n(\mathbf{x}) = \begin{cases} 1\,, & \text{if } x_1 = \cdots = x_{k-1} = 1 \text{ and } x_{k+1} = \cdots = x_n = 0\,; \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Since $\wedge_k f$ takes the value 1 only at one tuple, it is monotone. The join derivative $\vee_k f_n$ is essentially the same as the function f_{n-1} (up to the inessential k-th variable of $\vee_k f_n$), that is,

$$(10) \qquad\qquad \vee_k f_n(\mathbf{x}) \;=\; f_{n-1}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n).$$

Now it follows that if $\{k_1, \ldots, k_n\} = [n]$ and $O_{k_i} \in \{\wedge_{k_i}, \vee_{k_i}\}$ $(i = 1, \ldots, n)$, then

$$(11) \qquad\qquad O_{k_1} \cdots O_{k_{n-1}} O_{k_n} f \;=\; O_{k_{\pi(1)}} \cdots O_{k_{\pi(n-1)}} O_{k_n} f$$

holds for every permutation $\pi \in S_{n-1}$. (If $O_{k_n} = \wedge_{k_n}$, then we use Theorem 21 and the fact that $\wedge_{k_n} f$ is monotone, and if $O_{k_n} = \vee_{k_n}$, then we use (10) and the induction hypothesis.) On the other hand, from the 2-local monotonicity of f we can conclude that

$$(12) \qquad\qquad O_{k_1} \cdots O_{k_{n-2}} O_{k_{n-1}} O_{k_n} f \;=\; O_{k_1} \cdots O_{k_{n-2}} O_{k_n} O_{k_{n-1}} f$$

with the help of Theorem 12. Since S_n is generated by S_{n-1} and the transposition $(n-1\ n)$, we see from (11) and (12) that f has n-permutable lattice derivatives.

We finish this section with game-theoretic interpretations of the parameterized notions of local monotonicity and permutability of lattice derivatives. Identifying $\mathbb{B}^n$ with the power set of $[n]$, we can regard a pseudo-Boolean function $f\colon \mathbb{B}^n \to \mathbb{R}$ as a cooperative game, where $[n]$ is the set of players and $f(C)$ is the worth of coalition $C \subseteq [n]$.

The partial derivative $\Delta_k f(C)$ gives the (marginal) contribution of the k-th player to coalition C. Note that the same player might have a positive contribution to some coalitions and a negative contribution to other coalitions. Such a setting can model situations where some players have conflicts, which prevents them from cooperating. The lattice derivative $\vee_k f(C)$ gives the outcome if the k-th player acts benevolently and joins (or leaves) the coalition C only if this increases the worth. Similarly, $\wedge_k f(C)$ represents the outcome if the k-th player acts malevolently.

Games corresponding to locally monotone functions have the property that if two coalitions are close to each other, then any given player relates in the same way to these coalitions. More precisely, f is p-locally monotone if and only if whenever two coalitions differ in less than p players, then the contribution of any player is either nonnegative to both coalitions or it is nonpositive to both.

Finally, let us interpret permutability of lattice derivatives. Let P be a p-subset of $[n]$, and let $C \subseteq [n] \setminus P$. Suppose that some players of P are benevolent and some of them are malevolent, and they are asked one by one to join coalition C if they want to. We obtain the least possible outcome if the malevolent players are asked first, and we get the greatest outcome if the benevolent players make their choices first. The function f has p-permutable lattice derivatives if and only if these extremal outcomes coincide, i.e., if the order in which the players make their choices is irrelevant for every p-subset of $[n]$.

4. Symmetric functions

In the previous sections we saw that both notions of local monotonicity and of permutable lattice derivatives lead to two hierarchies of pseudo-Boolean functions which are related by the fact that each p-local monotone class is contained in the corresponding class of functions having p-permutable lattice derivatives. Now, in general this containment is strict. However, under certain assumptions (see, e.g., Theorem 12), p-local monotonicity is equivalent to p-permutability of lattice derivatives. Hence it is natural to ask for conditions under which these two notions are equivalent.

In this section we provide a partial answer to this problem by focusing on symmetric pseudo-Boolean functions, i.e., functions $f\colon \mathbb{B}^n \to \mathbb{R}$ that are invariant under all permutations of their variables. Quite surprisingly, in this case the notions of p-local monotonicity and p-permutability of lattice derivatives become equivalent.

Symmetric functions of arity n are in a one-to-one correspondence with sequences of real numbers of length $n+1$, where the function corresponding to the sequence $\alpha = \alpha_0, \dots, \alpha_n$ is given by $f(\mathbf{x}) = \alpha_{|\mathbf{x}|}$ ($\mathbf{x} \in \mathbb{B}^n$). Clearly, f is isotone if and only if the corresponding sequence is nondecreasing, i.e., $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n$. Similarly, f is antitone if and only if $\alpha_0 \ge \alpha_1 \ge \cdots \ge \alpha_n$, and f is monotone if and only if f is either isotone or antitone.[4]

It is easy to see that if f is symmetric, then every section of f is also symmetric; moreover, if f corresponds to the sequence $\alpha = \alpha_0, \dots, \alpha_n$, then the p-ary sections of f are precisely the symmetric functions corresponding to the subsequences[5] $\alpha_i, \alpha_{i+1}, \dots, \alpha_{i+p}$ of α of length $p+1$. This observation and Theorem 8 lead to the following description of p-locally monotone symmetric pseudo-Boolean functions.

Proposition 25. *Let $f\colon \mathbb{B}^n \to \mathbb{R}$ be a symmetric function corresponding to the sequence $\alpha = \alpha_0, \dots, \alpha_n$. Then f is p-locally monotone if and only if each subsequence of length $p+1$ of α is either nondecreasing or nonincreasing.*

Unlike in the previous sections, here it will be more convenient to discard the inessential k-th variable of the lattice derivatives $\wedge_k f$ and $\vee_k f$, and regard the latter as $(n-1)$-ary functions. Clearly, if f is symmetric, then so are its lattice derivatives. Moreover, if f corresponds to the sequence $\alpha = \alpha_0, \dots, \alpha_n$, then $\wedge_k f$ and $\vee_k f$ correspond to the sequences

$$\alpha_0 \wedge \alpha_1, \alpha_1 \wedge \alpha_2, \dots, \alpha_{n-1} \wedge \alpha_n \quad \text{and}$$
$$\alpha_0 \vee \alpha_1, \alpha_1 \vee \alpha_2, \dots, \alpha_{n-1} \vee \alpha_n \,,$$

respectively, for all $k \in [n]$. Since these sequences do not depend on k, we will write $\wedge f$ and $\vee f$ instead of $\wedge_k f$ and $\vee_k f$, and we will abbreviate

$$\underbrace{\wedge \cdots \wedge}_{\ell} f \quad \text{and} \quad \underbrace{\vee \cdots \vee}_{\ell} f$$

by $\wedge^\ell f$ and $\vee^\ell f$, respectively.

The next example shows that Theorem 19 cannot be sharpened.

Example 26. Let $f\colon \mathbb{B}^n \to \mathbb{B}$ be the symmetric function corresponding to the sequence

$$\alpha \; = \; 0, 0, \overbrace{1, \dots, 1}^{p}, \overbrace{0, \dots, 0}^{p}, 1, 1$$

where $n = 2p + 4$ and $p \ge 2$. It follows from Proposition 25 that f is exactly p-locally monotone. To compute $\wedge f$, it is handy to construct a table whose first row contains the sequence α, and in the second row we write $\alpha_i \wedge \alpha_{i+1}$ between α_i and α_{i+1}:

$$\begin{array}{llllllllllllll} f: & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ \wedge f: & & 0 & 0 & 1 & 1 \cdots 1 & 1 & 0 & 0 & 0 \cdots 0 & 0 & 0 & 1 \end{array}$$

Thus $\wedge f$ corresponds to the sequence

$$0, 0, \overbrace{1, \dots, 1}^{p-1}, \overbrace{0, \dots, 0}^{p+1}, 1,$$

and a similar calculation yields that $\vee f$ corresponds to the sequence

$$0, \overbrace{1, \dots, 1}^{p+1}, \overbrace{0, \dots, 0}^{p-1}, 1, 1.$$

Now Proposition 25 shows that $\wedge f$ and $\vee f$ are exactly $(p-1)$-locally monotone.

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[4]Since if f is isotone (resp. antitone) in one variable, then it is isotone (resp. antitone) in all variables.

[5]Here by a subsequence we mean a sequence of consecutive entries of the original sequence.

Remark 2. Example 26 shows that the degree of local monotonicity can decrease, when taking lattice derivatives, and Theorem 19 states that it can decrease by at most one. Other examples can be found to illustrate the cases when this degree stays the same, or even increases. For instance, consider the function $f(\mathbf{x}) = x_1 \oplus \cdots \oplus x_n$, which is not even 2-locally monotone, but its lattice derivatives are constant.

We conclude this section by proving that for symmetric functions the notions of p-local monotonicity and p-permutability of lattice derivatives coincide.

Theorem 27. *If $f \colon \mathbb{B}^n \to \mathbb{R}$ is symmetric, then f is p-locally monotone if and only if f has p-permutable lattice derivatives.*

Proof. By Theorem 21, it is enough to show that if a symmetric function $f \colon \mathbb{B}^n \to \mathbb{R}$ is not p-locally monotone, then it does not have p-permutable lattice derivatives. So suppose that f is a symmetric function which is not p-locally monotone, and which corresponds to the sequence $\alpha = \alpha_0, \dots, \alpha_n$. Let $\alpha_i, \dots, \alpha_{i+\ell}$ be a shortest subsequence of α that is neither nondecreasing nor nonincreasing. Proposition 25 implies that there is indeed such a subsequence for $\ell + 1 \leq p + 1$. From the minimality of ℓ it follows that the subsequence $\alpha_i, \dots, \alpha_{i+\ell-1}$ is either nondecreasing or nonincreasing. We may assume without loss of generality that the first case holds; the second case is the dual of the first one. Then we must have $\alpha_{i+\ell-1} > \alpha_{i+\ell}$, since otherwise the whole subsequence $\alpha_i, \dots, \alpha_{i+\ell}$ would be nondecreasing. Thus we have the following inequalities:

$$(13) \qquad \alpha_i \ \leq \ \alpha_{i+1} \ \leq \ \cdots \ \leq \ \alpha_{i+\ell-1} \ > \ \alpha_{i+\ell}.$$

From the minimality of ℓ, we can also conclude that the subsequence $\alpha_{i+1}, \dots, \alpha_{i+\ell}$ is either nondecreasing or nonincreasing. As $\alpha_{i+\ell-1} > \alpha_{i+\ell}$, the first case is impossible, therefore $\alpha_{i+1}, \dots, \alpha_{i+\ell}$ is nonincreasing, and we must have $\alpha_i < \alpha_{i+1}$ since otherwise the whole subsequence $\alpha_i, \dots, \alpha_{i+\ell}$ would be nonincreasing:

$$(14) \qquad \alpha_i \ < \ \alpha_{i+1} \ \geq \ \cdots \ \geq \ \alpha_{i+\ell-1} \ \geq \ \alpha_{i+\ell}.$$

Comparing (13) and (14), we obtain

$$\alpha_i \ < \ \alpha_{i+1} \ = \ \cdots \ = \ \alpha_{i+\ell-1} \ > \ \alpha_{i+\ell}.$$

To simplify notation, we set $\beta := \alpha_i, \gamma := \alpha_{i+1}, \delta := \alpha_{i+\ell}$. With this notation we have that α contains the subsequence $\beta, \gamma, \dots, \gamma, \delta$ of length $\ell + 1$ with $\beta, \delta < \gamma$. In the following we will use this observation to prove that f does not have ℓ-permutable lattice derivatives.

Let us compute the sequence corresponding to $\vee \wedge^{\ell-1} f$. We can construct a table as in Example 26, but this time the table has $\ell + 1$ rows (in the last row μ stands for $\beta \vee \delta$):

$f:$	α_0	$\cdots$	β	γ	γ	γ	$\cdots$	γ	γ	γ	δ	$\cdots$	α_n
$\wedge f:$			$\cdots$	β	γ	γ		$\cdots$		γ	γ	δ	$\cdots$
$\wedge^2 f:$			$\cdots$	β	γ			$\cdots$			γ	δ	$\cdots$
			$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$\wedge^{\ell-2} f:$				$\cdots$	β	γ	δ	$\cdots$					
$\wedge^{\ell-1} f:$				$\cdots$	β	δ	$\cdots$						
$\vee \wedge^{\ell-1} f:$				$\cdots$	μ	$\cdots$							

A similar table can be constructed for $\wedge^{\ell-1} \vee f$:

$f:$	α_0	$\cdots$	β	γ	γ	γ	$\cdots$	γ	γ	γ	δ	$\cdots$	α_n
$\vee f:$			$\cdots$	γ	γ	γ		$\cdots$		γ	γ	γ	$\cdots$
$\wedge \vee f:$			$\cdots$	γ	γ			$\cdots$			γ	γ	$\cdots$
			$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$				
$\wedge^{\ell-3} \vee f:$				$\cdots$	γ	γ	γ	$\cdots$					
$\wedge^{\ell-2} \vee f:$				$\cdots$	γ	γ	$\cdots$						
$\wedge^{\ell-1} \vee f:$				$\cdots$	γ	$\cdots$							

Since $\beta, \delta < \gamma$, we have $\mu < \gamma$, and this means that the sequences corresponding to $\vee \wedge^{\ell-1} f$ and $\wedge^{\ell-1} \vee f$ differ in at least one position, therefore f does not have ℓ-permutable lattice derivatives. As $\ell \leq p$, this implies that f does not have p-permutable lattice derivatives either, according to Theorem 23. $\square$

Remark 3. As a consequence of Theorem 27, we can observe that any exactly p-locally monotone symmetric function (for instance, the functions considered in Example 26) has p-permutable but not $(p+1)$-permutable lattice derivatives.

5. Open problems and concluding remarks

We proposed relaxations of monotonicity, namely p-local monotonicity, and we presented characterizations of each in terms of "forbidden" sections. Also, for each p, we observed that p-locally monotone functions have the property that any p of their lattice derivatives permute, and showed that the converse also holds in the special case of symmetric functions. The classes of 2-locally monotone functions, and of functions having 2-permutable lattice derivatives were explicitly described. However, similar descriptions elude us for $p \geq 3$. Hence we are left with the following problems.

Problem 1. For $p \geq 3$, describe the class of p-locally monotone functions and that of functions having p-permutable lattice derivatives.

Problem 2. For $p \geq 3$, determine necessary and sufficient conditions on functions for the equivalence between p-local monotonicity and p-permutability of lattice derivatives.

Acknowledgments

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