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ON COMPOSITION-CLOSED CLASSES OF BOOLEAN FUNCTIONS

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Abstract. We determine all composition-closed equational classes of Boolean functions. These classes provide a natural generalization of clones and iterative algebras: they are closed under composition, permutation and identification (diagonalization) of variables and under introduction of inessential variables (cylindrification), but they do not necessarily contain projections. Thus the lattice formed by these classes is an extension of the Post lattice. The cardinality of this lattice is continuum, yet it is possible to describe its structure to some extent.

1. Introduction

The goal of this paper is to describe composition-closed equational classes of Boolean functions not necessarily containing projections, thereby generalizing Post's description of Boolean clones. First we recall the definition of a clone, and then we give an informal overview of the problem that we consider. For formal definitions and more background see Section 2 and [12, 16].

We define the *composition* of an n-ary function $f\colon A^n \to A$ by the k-ary functions $g_1, \ldots, g_n\colon A^k \to A$ as the k-ary function $f(g_1, \ldots, g_n)$ given by

$$f(g_1, \ldots, g_n)(\mathbf{a}) = f(g_1(\mathbf{a}), \ldots, g_n(\mathbf{a})) \text{ for all } \mathbf{a} \in A^k. \tag{1.1}$$

We say that f is the *outer function* of the composition, and $g_1, \ldots, g_n$ are the *inner functions*. A *clone* on the set A is a class $\mathcal{C} \subseteq \bigcup_{n\geq 1} A^{A^n}$ of finitary functions that is closed under composition and contains the *projections*

$$e_i^{(n)}\colon A^n \to A, (x_1, \ldots, x_n) \mapsto x_i \quad (n \in \mathbb{N}, 1 \leq i \leq n).$$

(Here, and in the rest of the paper, $\mathbb{N} = \{1, 2, \ldots\}$, i.e., we exclude 0 from the set of natural numbers.)

Although the above definition of composition is restrictive in the sense that the inner functions must have the same arity, by making use of projections one can see that clones are closed under compositions without restrictions on the arities. For example, let us suppose that f is a ternary function in a clone $\mathcal{C}$, and g_1, g_2, g_3 are unary, binary, ternary functions in $\mathcal{C}$, respectively. If we would like to build the composite function $h(x_1, x_2, x_3) = f(g_1(x_1), g_2(x_2, x_1), g_3(x_1, x_1, x_3))$ using only compositions of the form (1.1), then we could proceed as follows: first construct ternary functions $g_1', g_2', g_3' \in \mathcal{C}$ with the help of the projections:

$$g_1' = g_1(e_1^{(3)}),$$
$$g_2' = g_2(e_2^{(3)}, e_1^{(3)}),$$
$$g_3' = g_3(e_1^{(3)}, e_1^{(3)}, e_3^{(3)}),$$

and then form the composition $h = f(g_1', g_2', g_3') \in \mathcal{C}$.

As we can see from the above example, composing a function with projections allows us to add dummy variables to the function (see g_1'), to permute the variables of the function (see g_2') and to identify variables of the function (see g_3'). Function classes closed under the latter three operations are called *equational classes*, since they can be defined by functional equations (see Subsection 2.2). The above discussion shows

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that every clone is an equational class, but, as we shall see in Subsection 2.2, there are equational classes that are not clones.

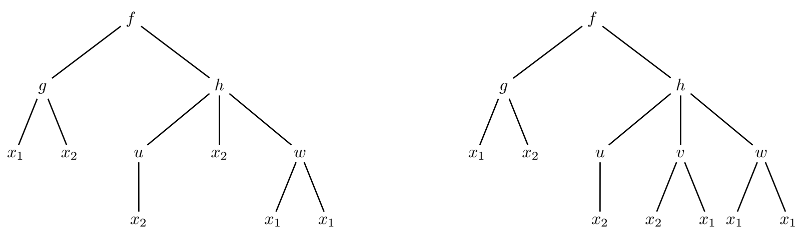
All clones on a given finite base set A form an algebraic lattice. This *clone lattice* has continuum cardinality if $|A| \geq 3$ (see [11]), and it seems to be a very hard problem to describe its structure. The case $|A| = 2$, i.e., the case of Boolean functions was settled by E. L. Post, who described all clones of Boolean functions in [15]. There are countably many such clones, and their lattice is known as the Post lattice (see Figure 2).

We will generalize the notion of a clone by considering function classes that are closed under composition (in the sense of (1.1)) but do not necessarily contain the projections. However, we would like to be able to identify and permute variables and introduce dummy variables, therefore we only consider composition-closed equational classes. These classes form a complete lattice that contains the clone lattice as the principal filter generated by the clone of projections (see Figure 6). The main result of this paper is a description of this lattice over a two-element base set. Although the clone lattice is countable in this case, we will see that the lattice of composition-closed equational classes of Boolean functions is uncountable.

Composition-closed equational classes subsume iterative algebras as well. A function class $\mathcal{K}$ is an *iterative algebra* if $f(g_1, \ldots, g_n) \in \mathcal{K}$, whenever $f \in \mathcal{K}$ and $g_i \in \mathcal{K} \cup \{\text{projections}\}$ for $i = 1, 2, \ldots, n$. Clearly, every iterative algebra is a composition-closed equational class, and an iterative algebra is a clone iff it contains the projections.

The difference between composition-closed equational classes and iterative algebras can be best understood by visualizing compositions as trees. As an example, let us consider the following two compositions:

$A = f(g(x_1, x_2), h(u(x_2), x_2, w(x_1, x_1)))$ $\qquad$ $B = f(g(x_1, x_2), h(u(x_2), v(x_2, x_1), w(x_1, x_1)))$



If $\mathcal{K}$ is an iterative algebra and $f, g, h, u, v, w \in \mathcal{K}$, then both A and B must belong to $\mathcal{K}$. However, if $\mathcal{K}$ is only assumed to be a composition-closed equational class, then A does not necessarily belong to $\mathcal{K}$, as it involves the composition $h(u(x_2), x_2, w(x_1, x_1))$, where one inner function is a projection. This problem does not arise with B, hence $B \in \mathcal{K}$ is guaranteed, whenever $\mathcal{K}$ is a composition-closed equational class.

In general, we can say that composition-closed equational classes are closed under compositions whose tree satisfies the following condition: for any internal node, either all or none of its children are leaves. As an exercise in handling such compositions, we invite the reader to verify the following fact: The clone generated by the addition operation of a field (or, more generally, of any additive commutative semigroup) consists of functions of the form $\sum a_i x_i$ $(a_i \in \mathbb{N})$. The iterative algebra generated by addition contains only those such functions where $\sum a_i \geq 2$, while the composition-closed equational class generated by addition contains only those where $\sum a_i$ is even.

The paper is organized as follows: In Section 2 we present the necessary background on equational classes, Boolean clones, and Galois connections between functions and relations. In Section 3 we make some basic observations about composition-closed equational classes of Boolean functions, and we outline a strategy for constructing all of them. In Section 4 we carry out this strategy for the easy cases, and then we deal with the harder cases in Sections 5 and 6. Finally, in Section 7 we put together all the information we found to get a picture about the lattice of composition-closed equational classes of Boolean functions.

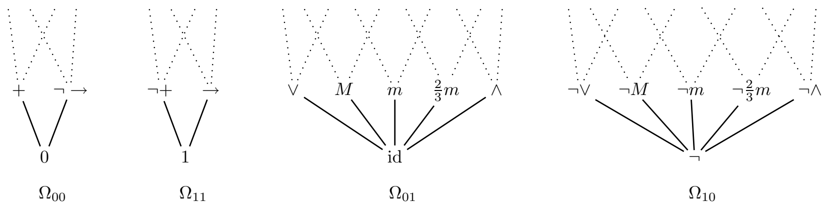


FIGURE 1. The subfunction quasiorder on Boolean functions

2. PRELIMINARIES

2.1. **Subfunctions.** Let f and g be operations on a set A of arity n and m, respectively. If there exists a map $\sigma\colon\ \{1,2,\ldots,n\}\to\{1,2,\ldots,m\}$ such that

$$g\left(x_1,\ldots,x_m\right)=f\left(x_{\sigma(1)},\ldots,x_{\sigma(n)}\right),$$

then we say that g is a *subfunction* (or identification minor or simple variable substitution) of f, and we denote this fact by $g\preceq f$. If σ is bijective, then g is obtained from f by permuting variables; if σ is not injective, then g is obtained from f by identifying variables; if σ is not surjective, then g is obtained from f by introducing inessential (dummy) variables.

The subfunction relation gives rise to a quasiorder on the set of all finitary functions on A (see [7]). The *equivalence* corresponding to this quasiorder is defined by $f\equiv g\iff f\preceq g$ and $g\preceq f$, and it is clear that f and g are equivalent iff they differ only in inessential variables and/or in the order of their variables. We will not distinguish between equivalent functions in the sequel. For example, we will denote the set of all constant zero functions (for any $0\in A$) simply by $\{0\}$, and $\{\mathrm{id}\}$ will stand for the set of all projections, as these are the functions equivalent to the identity function.

Let Ω denote the class of all Boolean functions, i.e., the class of finitary operations on $A=\{0,1\}$. The subfunction relation induces naturally a partial order on $\Omega/\equiv$; the bottom of this poset is shown in Figure 1. We can see (and it is easy to prove) that it has four connected components, namely $\Omega_{00},\Omega_{11},\Omega_{01},\Omega_{10}$, where

$$\Omega_{ab}=\{f\in\Omega : f\left(\mathbf{0}\right)=a,\ f\left(\mathbf{1}\right)=b\}\quad\left(a,b\in\{0,1\}\right).$$

Let us observe that Ω_{01} is nothing else but the clone of idempotent functions. For an arbitrary function class $\mathcal{K}$, we will abbreviate $\mathcal{K}\cap\Omega_{ab}$ by $\mathcal{K}_{ab}$, and we will later use the following (hopefully intuitive) notation as well:

$$\mathcal{K}_{0*}=\mathcal{K}_{00}\cup\mathcal{K}_{01},\ \mathcal{K}_{*1}=\mathcal{K}_{01}\cup\mathcal{K}_{11},\ \mathcal{K}_{=}=\mathcal{K}_{00}\cup\mathcal{K}_{11}.$$

The minimal elements of $(\Omega/\equiv;\preceq)$ are the unary functions: $0,1,\mathrm{id}$ and $\neg$ (negation). On the next level we can see the binary operations $+$ (addition modulo 2), $\rightarrow$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction) and the ternary functions M (majority operation), m (minority operation), $\frac{2}{3}m$ ($\frac{2}{3}$-minority operation, see [3]) together with their negations. Here negation is "taken from outside", e.g., $\neg\frac{2}{3}m$ is a shorthand notation for the function $\neg\frac{2}{3}m\left(x,y,z\right)=1+\frac{2}{3}m\left(x,y,z\right)=1+xy+yz+xz+x+z$.

2.2. **Equational classes.** A class $\mathcal{K}$ of operations on a set A is an *equational class* if it is an order ideal in the subfunction quasiorder, i.e., if $f\in\mathcal{K}$ and $g\preceq f$ imply $g\in\mathcal{K}$. The denomination is explained by the fact that these are exactly the classes that can be defined by functional equations [6, 8]. Clearly every clone is an equational class, but there are other equational classes as well; a natural example is the class of antimonotone (order reversing) Boolean functions, which can be defined by the functional equation $f\left(\mathbf{x}\wedge\mathbf{y}\right)\wedge f\left(\mathbf{x}\right)=f\left(\mathbf{x}\right)$. Another example is the class $\Omega_{=}$ (see Subsection 2.1), which can be defined by the equation $f\left(\mathbf{0}\right)=f\left(\mathbf{1}\right)$. Equational classes can be characterized by relational constraints as well; we will discuss this in more detail in Subsection 2.4.

The equational classes on a given set A form a lattice $\mathbf{E}_A$ with intersection and union as the lattice operations. This lattice has continuum cardinality already on the two-element set, and its structure is very complicated [7]. If $\mathcal{A}$ and $\mathcal{B}$ are classes of functions, then their *composition*, denoted by $\mathcal{A} \circ \mathcal{B}$, is the set of all compositions where the outer function belongs to $\mathcal{A}$ and the inner functions belong to $\mathcal{B}$:

$$\mathcal{A} \circ \mathcal{B} = \{f(g_1, \ldots, g_n) : f \in \mathcal{A},\ g_1, \ldots, g_n \in \mathcal{B}\}.$$

In general, associativity does not hold for function class composition, but it holds for equational classes [4, 5], hence we obtain a monoid $(\mathbf{E}_A; \circ)$ with the identity element $\{\mathrm{id}\}$.

We will mostly consider Boolean functions, and in this case we will drop the index A, and denote the set of equational classes simply by $\mathbf{E}$. A class $\mathcal{K}$ is *closed under composition* iff $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$. As we shall see in Proposition 3.3, this is equivalent to the formally stronger requirement that $\mathcal{K}$ is *idempotent*, i.e., $\mathcal{K} \circ \mathcal{K} = \mathcal{K}$ (cf. [4]). (Let us note that this is a distinguishing feature of Boolean functions: if A has at least three elements, then one can construct a class $\mathcal{K} \in \mathbf{E}_A$ such that $\mathcal{K} \circ \mathcal{K} \subsetneq \mathcal{K}$.) The goal of this paper is to describe the idempotent elements of $(\mathbf{E}; \circ)$.

The usual notation for the set of idempotents of a semigroup $\mathbf{S}$ is $E(\mathbf{S})$, but in our case this would lead to the somewhat awkward notation $E(\mathbf{E})$, therefore we will simply write $\mathbf{I}$ for the set of composition-closed equational classes over $\{0, 1\}$. Clearly $\mathbf{I}$ is closed under arbitrary intersections (we allow the empty class), hence it is a complete lattice. The lattice of clones appears in $\mathbf{I}$ as the principal filter generated by $\{\mathrm{id}\}$, and we will see that the rest of $\mathbf{I}$ is the principal ideal generated by $\Omega_=$ (see Figure 6). We will also see that the lattice $\mathbf{I}$ has continuum cardinality.

2.3. **The Post lattice.** There are countably many clones on the two-element set, and these have been described by E. L. Post in [15]. Figure 2 shows the clone lattice on $\{0, 1\}$, usually referred to as the *Post lattice*. The top element is Ω, the class of all Boolean functions, and the bottom element is $\{\mathrm{id}\}$, the clone consisting of projections only. The other clones labelled in the figure are the following:

- Ω_{0*} is the clone of 0-preserving functions;
- Ω_{*1} is the clone of 1-preserving functions;
- M is the clone of monotone (order preserving) functions;
- S is the clone of self-dual functions, i.e., functions satisfying $\neg f(\neg x_1, \ldots, \neg x_n) = f(x_1, \ldots, x_n)$;
- L is the clone of linear functions, i.e., functions of the form $x_1 + \cdots + x_n + c$ with $n \geq 0, c \in \{0, 1\}$;
- Λ consists of conjunctions $x_1 \wedge \cdots \wedge x_n$ $(n \in \mathbb{N})$ and the two constants $0, 1$;
- V consists of disjunctions $x_1 \vee \cdots \vee x_n$ $(n \in \mathbb{N})$ and the two constants $0, 1$;
- $\Omega^{(1)}$ is the clone of essentially at most unary functions;
- W^k is the clone of functions preserving the relation $\{0, 1\}^k \setminus \{\mathbf{0}\}$; it can be generated, e.g., by the function w_k of arity $k + 2$ defined by[1]
$$w_k(x_1, \ldots, x_{k+2}) = \begin{cases} 0, & \text{if } |\{i : x_i = 1\}| = 2 \text{ and } x_1 = 1; \\ 1, & \text{otherwise.} \end{cases}$$
- $W^\infty = W^2 \cap W^3 \cap \cdots$ is the clone generated by implication;
- U^k is the dual of W^k for $k = 2, 3, \ldots, \infty$.

All other clones can be obtained as intersections of these clones. The different types of nodes and edges in Figure 2 help the navigation in the Post lattice as follows:

- nodes representing clones of idempotent functions have a double outline (others have a single outline), and a double edge connects a clone $\mathcal{C}$ to $\mathcal{C} \cap \Omega_{01}$;
- nodes representing clones of monotone functions are filled (others have empty interior), and a thick edge connects a clone $\mathcal{C}$ to $\mathcal{C} \cap M$;

[1]The lower covers of W^k in the Post lattice are $W^k \cap M, W^k \cap \Omega_{01}$ and W^{k+1}. Therefore, in order to verify that w_k generates W^k, it suffices to check that $w_k \in W^k$ and w_k is neither monotone nor idempotent, nor does it belong to W^{k+1}.

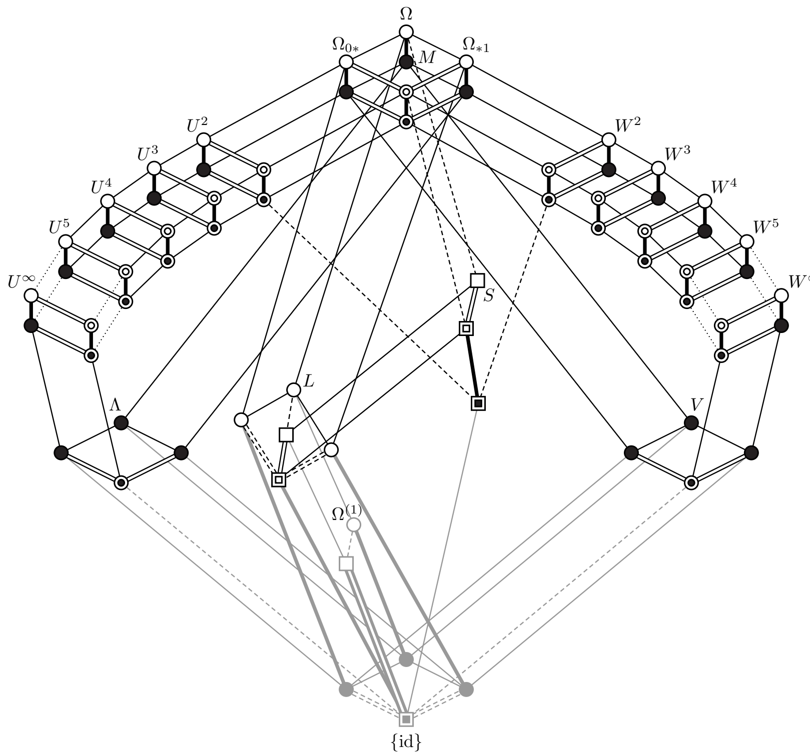


FIGURE 2. The Post lattice

- nodes representing clones of self-dual functions are squares (others are circles), and a dashed edge connects a clone $\mathcal{C}$ to $\mathcal{C} \cap S$;
- nodes representing clones of essentially at most unary functions are grey (others are black), and all edges incident with unary clones are grey.

2.4. **Relational constraints.** If $P \subseteq A^m$ is a relation of arity m, and $N \in A^{m \times n}$ is an $m \times n$ matrix such that each column of N belongs to P, then we say that N is a P*-matrix*. Applying an n-ary function f to the rows of N, we obtain the column vector $f(N) \in A^m$. A *relational constraint* of arity m is a pair (P, Q), where P and Q are m-ary relations. An n-ary function f *satisfies* the constraint (P, Q) if $f(N) \in Q$ for every P-matrix N of size $m \times n$. Satisfaction of relational constraints gives rise to a Galois connection that defines equational classes of functions.

Theorem 2.1 ([14]). *A class of functions on a finite set A is an equational class iff it can be defined by relational constraints.*

As an illustration of this theorem let us consider our two examples from Subsection 2.2: the class of antimonotone functions can be defined by $(\leq, \geq)$, and the class $\Omega_=$ can be defined by the constraint $(\{(0,1)\}, \{(0,0),(1,1)\})$.

Iterative algebras can be characterized by relational constraints as well. Let us note that iterative algebras are usually defined with the help of the five operations $\zeta, \tau, \Delta, \nabla, *$ introduced by Mal'cev [13], but using function class composition we can give a very compact definition: a function class $\mathcal{K}$ is an *iterative algebra* iff $\mathcal{K} \circ (\mathcal{K} \cup \{\mathrm{id}\}) \subseteq \mathcal{K}$.

Theorem 2.2 ([10]). *A class of functions on a finite set A is an iterative algebra iff it can be defined by relational constraints (P, Q) with $Q \subseteq P$.*

A function f *preserves* the relation P iff f satisfies the constraint (P, P). This induces the well-known Pol-Inv Galois connection between clones and relational clones.

Theorem 2.3 ([2, 9]). *A class of functions on a finite set A is a clone iff it can be defined by relations.*

Now we present another Galois connection that characterizes composition-closed equational classes. Let us say that a function f *strongly satisfies* the relational constraint (P, Q), if f satisfies both (P, Q) and (Q, Q) (i.e., f satisfies (P, Q) and preserves Q). The function class $\mathcal{K}$ is *strongly defined* by relational constraints if there exists a set $\{(P_i, Q_i) : i \in I\}$ of relational constraints such that a function belongs to $\mathcal{K}$ iff it strongly satisfies (P_i, Q_i) for all $i \in I$.

Theorem 2.4. *A class of functions on a finite set A is a composition-closed equational class iff it can be strongly defined by relational constraints.*

Proof. First let us suppose that $\mathcal{K}$ is strongly defined by a set $\{(P_i, Q_i) : i \in I\}$ of relational constraints, where P_i and Q_i are relations of arity m_i. By Theorem 2.1, $\mathcal{K}$ is an equational class. To verify that $\mathcal{K}$ is closed under composition, let us consider arbitrary functions $f, g_1, \ldots, g_n \in \mathcal{K}$, where the arity of f is n, and the arity of $g_1, \ldots, g_n$ is k. Then the composition $h = f(g_1, \ldots, g_n)$ is a k-ary function on A. Let $i \in I$, and let N be a P_i-matrix of size $m_i \times k$. Since $g_1, \ldots, g_n$ satisfy the constraint (P_i, Q_i), the $m_i \times n$ matrix N' formed by the column vectors $g_1(N), \ldots, g_n(N)$ is a Q_i-matrix. Therefore, $h(N) = f(N') \in Q_i$, as f preserves the relation Q_i. This shows that the composition h satisfies the constraint (P_i, Q_i). Noting that $f, g_1, \ldots, g_n$ all preserve the relation Q_i, we see that h preserves Q_i as well, hence h strongly satisfies (P_i, Q_i). This holds for all $i \in I$, thus $h \in \mathcal{K}$, as claimed.

For the other implication, let us assume that $\mathcal{K}$ is a composition-closed equational class. Let us write all elements of A^n below each other (as row vectors), and let O_n denote the resulting $|A|^n \times n$ matrix. Let P_n be the set of column vectors of O_n, and let Q_n be the set of all column vectors of the form $f(O_n)$, where $f \in \mathcal{K}$ is of arity n. Let $\mathcal{K}'$ be the class of functions strongly defined by $\{(P_n, Q_n) : n \in \mathbb{N}\}$. We will prove that $\mathcal{K}' = \mathcal{K}$.

If $f' \in \mathcal{K}'$ is a function of arity n, then $f'(O_n) \in Q_n$, hence, according to the definition of Q_n, there exists an n-ary function $f \in \mathcal{K}$ such that $f'(O_n) = f(O_n)$. Since the rows O_n contain every element of A^n, this implies that $f' = f$, thus $f' \in \mathcal{K}$, hence we can conclude that $\mathcal{K}' \subseteq \mathcal{K}$.

In order to prove that $\mathcal{K} \subseteq \mathcal{K}'$, we need to verify for an arbitrary n-ary function $f \in \mathcal{K}$ that f strongly satisfies the constraint (P_k, Q_k) for every $k \in \mathbb{N}$. If N_1 is a P_k-matrix of size $|A|^k \times n$, then $f(N_1) = f_1(O_k)$ for a suitable k-ary subfunction f_1 of f. Since $\mathcal{K}$ is an equational class, we have $f_1 \in \mathcal{K}$, hence $f_1(O_k) \in Q_k$. This shows that f satisfies the constraint (P_k, Q_k). We need to check yet that f preserves Q_k, i.e., that $f(N_2) \in Q_k$ for any Q_k-matrix N_2 of size $|A|^k \times n$. By the definition of Q_k, the list of columns of N_2 is of the form $g_1(O_k), \ldots, g_n(O_k)$ for some k-ary $g_1, \ldots, g_n \in \mathcal{K}$, thus $f(N_2) = h(O_k)$, where $h = f(g_1, \ldots, g_n)$. Since $\mathcal{K}$ is closed under composition, we have $h \in \mathcal{K}$, hence $f(N_2) = h(O_k) \in Q_k$, as claimed. $\square$

Concerning our two examples, let us observe that the class of antimonotone functions is not closed under composition, but $\Omega_=$ is closed. Indeed, $\Omega_=$ is strongly defined by the constraint $(\{(0,1)\}, \{(0,0), (1,1)\})$, since the relation $\{(0,0), (1,1)\}$ is just the equality relation, and it is preserved by every function.

3. Idempotents vs. clones

From now on we will restrict our attention to Boolean functions. In this section we make some basic observations about the relationship between a composition-closed equational class $\mathcal{K}$ and the clone $\mathcal{C}$ generated by $\mathcal{K}$. These observations will make it possible to construct all such classes for a given clone $\mathcal{C}$.

Let us first briefly discuss the unary case: There are 16 equational classes consisting of essentially at most unary functions, and the following 10 are closed under composition:

$$\emptyset,\ \{0\},\ \{1\},\ \{0,1\},$$
$$\{\mathrm{id}\},\ \{\mathrm{id},0\},\ \{\mathrm{id},1\},\ \{\mathrm{id},0,1\},\ \{\mathrm{id},\neg\},\ \{\mathrm{id},\neg,0,1\}.$$

As we shall see in the following lemma, disregarding these 10 trivial cases, we can always assume that $+, \neg+, \to$ or $\neg\to$ is present in our composition-closed equational class.

Lemma 3.1. *If $\mathcal{K}$ is a composition-closed equational class that is not a clone, then*

(1) $\mathcal{K} \subseteq \Omega_{=}$*;*
(2) *if* $\mathcal{K} \nsubseteq \Omega^{(1)}$*, then* $\mathcal{K} \cap \{+, \neg+, \to, \neg\to\} \neq \emptyset$*.*

Proof. If $\mathcal{K}$ is closed under composition, but $\mathcal{K}$ is not a clone, then id $\notin \mathcal{K}$. Since $\mathcal{K}$ is an equational class, for each $f \in \mathcal{K}$ the unary subfunction $\Delta_f(x) = f(x, \dots, x)$ of f belongs to $\mathcal{K}$. If there is a function $f \in \mathcal{K} \cap \Omega_{01}$, then we can conclude that $\mathrm{id} = \Delta_f \in \mathcal{K}$, a contradiction. If there is a function $f \in \mathcal{K} \cap \Omega_{10}$, then $\neg = \Delta_f \in \mathcal{K}$, and since $\mathcal{K}$ is closed under composition, we have $\mathrm{id} = \neg\neg \in \mathcal{K}$, which is a contradiction again. Thus $\mathcal{K} \cap \Omega_{01} = \emptyset$ and $\mathcal{K} \cap \Omega_{10} = \emptyset$, therefore $\mathcal{K} \subseteq \Omega_{=}$.

To prove the second statement of the lemma, let us observe that any equational class $\mathcal{K} \nsubseteq \Omega^{(1)}$ must contain at least one of the 14 functions shown on the "second level" of Figure 1. Since we have $\mathcal{K} \subseteq \Omega_{=}$, we see that at least one of the functions $+, \neg+, \to, \neg\to$ belongs to $\mathcal{K}$. □

Note that the first statement of the lemma shows that the non-clone composition-closed equational classes form a principal filter in **I** (see Figure 6). Hence the lattice **I** has six coatoms: the maximal clones $(\Omega_{0*}, \Omega_{*1}, M, S, L)$ and $\Omega_{=}$. Next we prove that every composition-closed equational class is idempotent, as mentioned in Subsection 2.2.

Lemma 3.2. *Let $\mathcal{K}_1, \mathcal{K}_2$ be composition-closed equational classes such that $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and $\mathcal{K}_1 \nsubseteq \Omega^{(1)}$. Then we have $\mathcal{K}_1 \circ \mathcal{K}_2 = \mathcal{K}_2$.*

Proof. One direction is obvious: $\mathcal{K}_1 \circ \mathcal{K}_2 \subseteq \mathcal{K}_2 \circ \mathcal{K}_2 \subseteq \mathcal{K}_2$. The containment $\mathcal{K}_2 \subseteq \mathcal{K}_1 \circ \mathcal{K}_2$ is also clear if id $\in \mathcal{K}_1$. If this is not the case, then one of the functions $+, \neg+, \to, \neg\to$ belongs to $\mathcal{K}_1$ by Lemma 3.1. If $f(x, y) = x + y \in \mathcal{K}_1$, then for any $g \in \mathcal{K}_2$ we have $g = f(f(x, x), g) \in \mathcal{K}_1 \circ \mathcal{K}_2$, thus $\mathcal{K}_2 \subseteq \mathcal{K}_1 \circ \mathcal{K}_2$. The same is true for $\neg+$ and $\to$, while for $f(x, y) = \neg(x \to y)$ we can use the composition $f(g, f(x, x))$. □

Proposition 3.3. *An equational class $\mathcal{K}$ is closed under composition iff it is idempotent.*

Proof. The "if" part is obvious; the "only if" part is also obvious if $\mathcal{K} \subseteq \Omega^{(1)}$, otherwise it follows from the above lemma with $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$. □

In light of the proposition above, in the following we will refer to a composition-closed equational class simply as an idempotent. In the next proposition we explore some relationships between an idempotent $\mathcal{K}$ and the clone $[\mathcal{K}]$ generated by $\mathcal{K}$ that will play a crucial role in the sequel.

Proposition 3.4. *Let $\mathcal{K} \nsubseteq \Omega^{(1)}$ be an idempotent, and let $\mathcal{C} = [\mathcal{K}]$. Then we have $\mathcal{C} \circ \mathcal{K} = \mathcal{K}$ and $\mathcal{K} \circ \mathcal{C} = \mathcal{C}$.*

Proof. The second equality follows from Lemma 3.2. Concerning the first equality, the containment $\mathcal{C} \circ \mathcal{K} \supseteq \mathcal{K}$ is clear, since id $\in \mathcal{C}$, so it remains to prove that $\mathcal{C} \circ \mathcal{K} \subseteq \mathcal{K}$. Every element of $\mathcal{C} \circ \mathcal{K}$ is of the form $h = f(g_1, \dots, g_n)$, where $f \in \mathcal{C}$ is n-ary, and $g_1, \dots, g_n \in \mathcal{K}$ are k-ary functions. Since $f \in \mathcal{C}$, it is a composition of elements of $\mathcal{K}$ and projections. We will prove $h \in \mathcal{K}$ by induction on the size (number of nodes) of the tree describing this composition (see Section 1).

If the tree has only one node, then $f \in \mathcal{K}$, and then $h \in \mathcal{K}$ follows since $\mathcal{K}$ is closed under composition. If the tree has at least two nodes, then f is of the form $f = u(v_1, \dots, v_r)$ with appropriate functions $u \in \mathcal{K}$ and $v_1, \dots, v_r \in \mathcal{C}$, where the composition tree of each v_i is smaller than the tree corresponding to f. Now we can write h as

$$h = (u(v_1, \dots, v_r))(g_1, \dots, g_n) = u(v_1(g_1, \dots, g_n), \dots, v_r(g_1, \dots, g_n)).$$

By the induction hypothesis, every one of the functions $v_i(g_1, \ldots, g_n)$ belongs to $\mathcal{K}$, therefore $h \in \mathcal{K} \circ \mathcal{K} = \mathcal{K}$. □

For any equational class $\mathcal{K}$ let us write $\lfloor \mathcal{K} \rfloor$ for the smallest idempotent containing $\mathcal{K}$, and recall that $[\mathcal{K}]$ denotes the smallest clone containing $\mathcal{K}$. The next proposition shows a relationship between these two closure operators.

Proposition 3.5. *For any equational class $\mathcal{K}$ we have $[\mathcal{K}] = \lfloor \mathcal{K} \cup \{\mathrm{id}\} \rfloor$ and $\lfloor \mathcal{K} \rfloor = [\mathcal{K}] \circ \mathcal{K}$.*

Proof. The first equality expresses the obvious fact that the clone generated by $\mathcal{K}$ consists of those functions that can be obtained from elements of $\mathcal{K}$ and from projections by means of composition. Introducing the notation $\mathcal{C} = [\mathcal{K}]$, the second equality takes the form $\lfloor \mathcal{K} \rfloor = \mathcal{C} \circ \mathcal{K}$. The right hand side contains $\mathcal{K}$, since $\mathrm{id} \in \mathcal{C}$, and it is closed under composition (hence idempotent):

$$(\mathcal{C} \circ \mathcal{K}) \circ (\mathcal{C} \circ \mathcal{K}) \subseteq \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \mathcal{K} = \mathcal{C} \circ \mathcal{K}.$$

To prove that $\mathcal{C} \circ \mathcal{K}$ is indeed the smallest idempotent containing $\mathcal{K}$, we have to show that $\mathcal{C} \circ \mathcal{K} \subseteq \lfloor \mathcal{K} \rfloor$. The case $\mathcal{K} \subseteq \Omega^{(1)}$ is trivial, otherwise we can apply Proposition 3.4 to the idempotent $\lfloor \mathcal{K} \rfloor$, and we get

$$\mathcal{C} \circ \lfloor \mathcal{K} \rfloor = [\lfloor \mathcal{K} \rfloor] \circ \lfloor \mathcal{K} \rfloor = \lfloor \mathcal{K} \rfloor .$$

Therefore, $\mathcal{C} \circ \mathcal{K} \subseteq \mathcal{C} \circ \lfloor \mathcal{K} \rfloor = \lfloor \mathcal{K} \rfloor$, and this proves the proposition. □

Example 3.6. Let us give some simple examples of idempotents together with relational constraints strongly defining them.

- $\Omega_{=} = \{f \in \Omega : f(\mathbf{0}) = f(\mathbf{1})\}$
  strongly defined by $(\{(0,1)\}, \{(0,0), (1,1)\})$
- $\Omega_{00} = \{f \in \Omega : f(\mathbf{0}) = f(\mathbf{1}) = 0\}$
  strongly defined by $(\{(0,1)\}, \{(0,0)\})$
- $\Omega_{11} = \{f \in \Omega : f(\mathbf{0}) = f(\mathbf{1}) = 1\}$
  strongly defined by $(\{(0,1)\}, \{(1,1)\})$
- $\mathcal{R} = \{f \in \Omega : f(\neg x_1, \ldots, \neg x_n) = f(x_1, \ldots, x_n)\}$
  strongly defined by $(\{(0,1), (1,0)\}, \{(0,0), (1,1)\})$

Members of the class $\mathcal{R}$ are called *reflexive functions*.

Example 3.7. One can determine $\lfloor + \rfloor$ using our observations about the composition-closed equational class generated by the addition operation of a field (see Section 1). Alternatively, we can use Proposition 3.5: the elements of $\lfloor + \rfloor$ are exactly the functions of the form

$$f(x_{i_1} + x_{j_1}, \ldots, x_{i_n} + x_{j_n}),$$

where $f \in [+] = L_{0*}$ and x_{i_k}, x_{j_k} are arbitrary (not necessarily distinct) variables. Such a function is a sum of an even number of variables, hence

$$\lfloor + \rfloor = \{x_1 + \cdots + x_n : n \text{ is even}\} = L_{00}.$$

Similarly, we have

$$\lfloor \neg + \rfloor = \{x_1 + \cdots + x_n + 1 : n \text{ is even}\} = L_{11}.$$

Example 3.8. Proposition 3.5 gives the following general form for the elements of $\lfloor \rightarrow \rfloor$:

$$f(x_{i_1} \rightarrow x_{j_1}, \ldots, x_{i_n} \rightarrow x_{j_n}),$$

where $f \in [\rightarrow] = W^{\infty}$, and x_{i_k}, x_{j_k} are arbitrary (not necessarily distinct) variables. Here, the lack of associativity makes it harder to decide whether a given function is of this form. For example, the function $f(x,y,z) = x \rightarrow (y \rightarrow z)$ seems not to have the required form, but we may rewrite it as $(x \rightarrow y) \rightarrow (x \rightarrow z)$, therefore $f \in \lfloor \rightarrow \rfloor$. On the other hand, $g(x,y,z) = (x \rightarrow y) \rightarrow z$ does not belong to $\lfloor \rightarrow \rfloor$. To verify this, let us observe that $\rightarrow \in \Omega_{11}$, hence $\lfloor \rightarrow \rfloor \subseteq \Omega_{11}$. However, $g \notin \Omega_{11}$ as $g(0,0,0) = 0$, therefore $g \notin \lfloor \rightarrow \rfloor$. In Proposition 5.4 we will describe $\lfloor \rightarrow \rfloor$ by relational constraints; this description makes it easy to decide membership for $\lfloor \rightarrow \rfloor$.

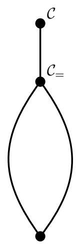


FIGURE 3. A typical nontrivial interval $\mathbf{I}(\mathcal{C})$

Proposition 3.5 suggests the following strategy for finding all idempotents: Let us fix a clone $\mathcal{C}$, and form all possible compositions $\mathcal{C} \circ \mathcal{K}$, where $\mathcal{K}$ is an equational class, such that $[\mathcal{K}] = \mathcal{C}$. This way we obtain all those idempotents that generate the clone $\mathcal{C}$. Let us denote the set of these idempotents by $\mathbf{I}(\mathcal{C})$:

$$\mathbf{I}(\mathcal{C}) = \{\mathcal{K} \in \mathbf{I} : [\mathcal{K}] = \mathcal{C}\}.$$

Clearly these sets form a partition of $\mathbf{I}$. For unary clones $\mathcal{C}$ it is an easy task to determine $\mathbf{I}(\mathcal{C})$: for $\mathcal{C} \subseteq \{0, 1, \mathrm{id}\}$ we have $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}, \mathcal{C}_=\}$; for the other unary clones we have $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}\}$.

Theorem 3.9. *For any clone $\mathcal{C}$, the set $\mathbf{I}(\mathcal{C})$ is an interval in the lattice of idempotents. If $\mathbf{I}(\mathcal{C})$ has more than one element, then $\mathcal{C}_=$ is its only coatom.*

Proof. We have settled the unary case above, so now let us suppose that $\mathcal{C} \nsubseteq \Omega^{(1)}$. It is clear that the largest element of $\mathbf{I}(\mathcal{C})$ is $\mathcal{C}$ itself, and it is also clear that $\mathbf{I}(\mathcal{C})$ is convex. Therefore, in order to prove that it is indeed an interval, it suffices to show that $\mathbf{I}(\mathcal{C})$ is closed under arbitrary intersections, hence it has a least element.

Let $\mathcal{K}_i \in \mathbf{I}(\mathcal{C})$ $(i \in I)$, and let $\mathcal{K} = \bigcap \mathcal{K}_i$. Then $\mathcal{C} \circ \mathcal{K} \subseteq \bigcap (\mathcal{C} \circ \mathcal{K}_i) = \bigcap \mathcal{K}_i = \mathcal{K}$ by Proposition 3.4. On the other hand, from $\mathrm{id} \in \mathcal{C}$ it follows that $\mathcal{C} \circ \mathcal{K} \supseteq \mathcal{K}$, so we have $\mathcal{C} \circ \mathcal{K} = \mathcal{K}$. Composing by $[\mathcal{K}]$ on the right, we get $\mathcal{C} \circ \mathcal{K} \circ [\mathcal{K}] = \mathcal{K} \circ [\mathcal{K}]$, and using Proposition 3.4 once more, we can conclude that $\mathcal{C} \circ [\mathcal{K}] = [\mathcal{K}]$. The left hand side contains $\mathcal{C}$, since $\mathrm{id} \in [\mathcal{K}]$, so we have $\mathcal{C} \subseteq [\mathcal{K}]$. The containment $[\mathcal{K}] \subseteq \mathcal{C}$ is obvious, so we see that $[\mathcal{K}] = \mathcal{C}$, and therefore $\mathcal{K} \in \mathbf{I}(\mathcal{C})$, as claimed. (Note that what we have proved is rather surprising: if each $\mathcal{K}_i$ contains a generating set of the clone $\mathcal{C}$, then so does $\bigcap \mathcal{K}_i$.)

The statement about the unique coatom follows from Lemma 3.1: if $\mathcal{K}$ is an element of $\mathbf{I}(\mathcal{C})$ that is different from $\mathcal{C}$, then $\mathcal{K}$ is a composition-closed equational class that is not a clone, hence $\mathcal{K} \subseteq \mathcal{C} \cap \Omega_= = \mathcal{C}_=$. ☐

The contents of the above theorem are represented in Figure 3, which shows a picture of a typical nontrivial interval $\mathbf{I}(\mathcal{C})$. As the next corollary shows, Theorem 3.9 allows us to prove that $\mathbf{I}(\mathcal{C})$ is trivial for “many” Boolean clones.

Corollary 3.10. *If $\mathcal{C}$ is a clone different from $\{\mathrm{id}\}, \{0, \mathrm{id}\}, \{1, \mathrm{id}\}, \{0, 1, \mathrm{id}\}, \Omega, \Omega_{0*}, \Omega_{*1}, L, L_{0*}, L_{*1}, W^2, \ldots, W^\infty, U^2, \ldots, U^\infty$, then $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}\}$.*

Proof. If $\mathbf{I}(\mathcal{C})$ has at least two elements, then $\mathcal{C}_= \in \mathbf{I}(\mathcal{C})$ according to Theorem 3.9. This implies that $\mathcal{C}_=$ generates the clone $\mathcal{C}$. However, for many clones, $\mathcal{C}_= = \mathcal{C} \cap \Omega_=$ is too small to generate $\mathcal{C}$. Indeed, $S \cap \Omega_= = \Omega_{01} \cap \Omega_= = \emptyset$ and $M \cap \Omega_= = \{0, 1\}$, therefore (ignoring the unary clones) $\mathbf{I}(\mathcal{C})$ is trivial unless $\mathcal{C}$ is one of the clones represented in Figure 2 by a black circle with a single outline and an empty (non-filled) interior. ☐

Up to duality, only the cases $\mathcal{C} = \Omega, \Omega_{*1}, L, L_{0*}, W^2, \ldots, W^\infty$ remain. The first four cases are treated in Section 4; we will see that in these cases $\mathbf{I}(\mathcal{C})$ has at most 3 elements. In Section 5 we give a characterization of the elements of $\mathbf{I}(W^\infty)$, and we will prove that this interval is uncountable. We consider $\mathbf{I}\left(W^k\right)$ for finite k in Section 6, and we will show that these intervals are finite, but their cardinalities do not have a common upper bound.

4. Idempotents corresponding to $\Omega, \Omega_{*1}, L, L_{0*}$

As explained in the previous section, $\mathbf{I}(\mathcal{C})$ can be determined by computing all possible compositions of the form $\mathcal{C} \circ \mathcal{K}$, where $\mathcal{K}$ is an equational class generating the clone $\mathcal{C}$. In the following two lemmas we compute six such compositions for $\mathcal{C} = \Omega$ and $\mathcal{C} = \Omega_{*1}$, and then we will show that these suffice to determine the intervals $\mathbf{I}(\Omega)$ and $\mathbf{I}(\Omega_{*1})$.

Lemma 4.1. *The following equalities hold:*

(1) $\Omega \circ \{\rightarrow\} = \Omega \circ \{\neg \rightarrow\} = \Omega_{=}$;
(2) $\Omega_{*1} \circ \{\rightarrow\} = \Omega_{11}$.

Proof. (1) We prove only that $\Omega \circ \{\rightarrow\} = \Omega_{=}$; the other case is similar. The functions in $\Omega \circ \{\rightarrow\}$ are of the following form:

$$h(x_1, \dots, x_k) = f(x_{i_1} \rightarrow x_{j_1}, \dots, x_{i_n} \rightarrow x_{j_n}), \tag{4.1}$$

where $f \in \Omega$ and $i_1, j_1, \dots, i_n, j_n \in \{1, \dots, k\}$. Since $0 \rightarrow 0 = 1 \rightarrow 1$, every function of this form belongs to $\Omega_{=}$. Conversely, for any given $h \in \Omega_{=}$ we will construct a suitable f so that (4.1) holds. Let us choose $n = k^2$, and let $\{(i_1, j_1), \dots, (i_n, j_n)\} = \{1, \dots, k\} \times \{1, \dots, k\}$. Then we can rewrite (4.1) as $h(\mathbf{a}) = f(\widehat{\mathbf{a}})$ for all $\mathbf{a} = (a_1, \dots, a_k) \in \{0,1\}^k$, where $\widehat{\mathbf{a}}$ denotes the vector formed from all possible implications between entries of $\mathbf{a}$. Now we can treat this as the definition of f:

$$f(\mathbf{b}) := \begin{cases} h(\mathbf{a}), & \text{if } \mathbf{b} = \widehat{\mathbf{a}}; \\ 0, & \text{if } \nexists \mathbf{a} \in \{0,1\}^k : \mathbf{b} = \widehat{\mathbf{a}}. \end{cases}$$

All we need to show is that f is well-defined, i.e., that $\mathbf{b} = \widehat{\mathbf{a}}$ determines $h(\mathbf{a})$ uniquely.

If $\widehat{\mathbf{a}} = (1, \dots, 1)$, then $a_1 = \cdots = a_k$, but we cannot tell whether $\mathbf{a} = \mathbf{0}$ or $\mathbf{a} = \mathbf{1}$. However, since $h \in \Omega_{=}$, the value of $h(\mathbf{a})$ is the same in both cases. Now let us suppose that at least one entry of $\widehat{\mathbf{a}}$ is 0, say, $a_1 \rightarrow a_2 = 0$. Then we can infer immediately that $a_1 = 1$ and $a_2 = 0$. Using this information we can recover the vector $\mathbf{a}$, since $a_i \rightarrow a_2 = \neg a_i$ for all $i \in \{1, \dots, k\}$. Thus in this case $\widehat{\mathbf{a}}$ uniquely determines $\mathbf{a}$, hence $h(\mathbf{a})$ is uniquely determined as well.

(2) We can proceed similarly as above, with $h \in \Omega_{11}$ and $f \in \Omega_{*1}$. □

Lemma 4.2. *The following equalities hold:*

(1) $\Omega \circ \{+\} = \Omega \circ \{\neg +\} = \mathcal{R}$;
(2) $\Omega_{*1} \circ \{\neg +\} = \mathcal{R}_{11}$.

Proof. We can use the same argument as in the previous proposition. For example, considering $\Omega \circ \{+\} = \mathcal{R}$, we have

$$h(x_1, \dots, x_k) = f(x_{i_1} + x_{j_1}, \dots, x_{i_n} + x_{j_n}),$$

in place of (4.1), and we define $\widehat{\mathbf{a}} \in \{0,1\}^{k^2}$ to be the vector formed from all sums of two entries of $\mathbf{a}$. Now $\widehat{\mathbf{a}}$ tells us which entries of $\mathbf{a}$ are the same, but does not tell us which one is 0 and which one is 1. Therefore $\widehat{\mathbf{a}}$ determines $\mathbf{a}$ up to negation, and this is equivalent to the condition $h \in \mathcal{R}$. □

Theorem 4.3. *If $\mathcal{C} = \Omega$ or $\mathcal{C} = \Omega_{*1}$, then $\mathbf{I}(\mathcal{C})$ is a three-element chain: $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}, \mathcal{C}_{=}, \mathcal{C} \cap \mathcal{R}\}$.*

Proof. Let us consider first $\mathcal{C} = \Omega$. If $\mathcal{K} \in \mathbf{I}(\Omega)$, then $\Omega \circ \mathcal{K} = \mathcal{K}$ by Proposition 3.4. If $\mathcal{K}$ is a clone, then clearly we have $\mathcal{K} = \Omega$. If this is not the case, then $\mathcal{K} \subseteq \Omega_{=}$, and at least one of the operations $+, \neg \rightarrow, \neg +, \rightarrow$ belongs to $\mathcal{K}$ by Lemma 3.1. If $\rightarrow \in \mathcal{K}$ or $\neg \rightarrow \in \mathcal{K}$, then $\mathcal{K} = \Omega \circ \mathcal{K} \supseteq \Omega_{=}$ by Lemma 4.1, hence $\mathcal{K} = \Omega_{=}$. In the remaining cases the binary operations in $\mathcal{K}$ are $+$ and/or $\neg +$, hence $\mathcal{K} = \Omega \circ \mathcal{K} \supseteq \mathcal{R}$ by Lemma 4.2. On the other hand, in these cases every binary operation in $\mathcal{K}$ is reflexive. We will show that this implies $\mathcal{K} \subseteq \mathcal{R}$.

If $f(x_1, \dots, x_n) \in \mathcal{K}$ and $\mathbf{a} \in \{0,1\}^n$, then $f(\mathbf{a}) = g(0,1)$, where $g(x,y)$ is obtained from $f(x_1, \dots, x_n)$ by replacing x_i by x or y depending on whether $a_i = 0$ or $a_i = 1$.

Since g is a subfunction of f, we have $g \in \mathcal{K}$, and then the reflexivity of g implies $f(\neg \mathbf{a}) = g(1,0) = g(0,1) = f(\mathbf{a})$. This is true for all $\mathbf{a} \in \{0,1\}^n$, thus f is reflexive, as claimed.

The above considerations show that the only possible elements of $\mathbf{I}(\Omega)$ are $\Omega, \Omega_=$ and $\mathcal{R}$. It is easily verified that each of these three classes indeed generate the clone Ω, hence $\mathbf{I}(\Omega) = \{\Omega, \Omega_=, \mathcal{R}\}$.

If $\mathcal{C} = \Omega_{*1}$, then $+, \neg \to$ cannot belong to $\mathcal{K}$. A similar argument as the above one shows that $\mathcal{K} = \Omega_{*1}$ if $\mathrm{id} \in \mathcal{K}$, $\mathcal{K} = \Omega_{*1} \cap \Omega_= = \Omega_{11}$ if $\mathrm{id} \notin \mathcal{K}$ but $\to \in \mathcal{K}$, and $\mathcal{K} = \Omega_{*1} \cap \mathcal{R} = \mathcal{R}_{11}$ otherwise. Thus we have $\mathbf{I}(\Omega_{*1}) = \{\Omega_{*1}, \Omega_{11}, \mathcal{R}_{11}\}$. □

As one can expect, the case of linear functions is considerably simpler.

Theorem 4.4. *If $\mathcal{C} = L$ or $\mathcal{C} = L_{0*}$ then $\mathbf{I}(\mathcal{C})$ is a two-element chain: $\mathbf{I}(\mathcal{C}) = \{\mathcal{C}, \mathcal{C}_=\}$.*

Proof. If $\mathcal{K} \in \mathbf{I}(L_{0*}) \setminus \{L_{0*}\}$, then $\mathcal{K} \subseteq L_{0*} \cap \Omega_=$ by Theorem 3.9. However, $L_{0*} \cap \Omega_= = L_{00} = \lfloor + \rfloor$ (see Example 3.7), so we can conclude that $\mathcal{K} \subseteq \lfloor + \rfloor$. On the other hand, since $\mathcal{K} \subseteq \Omega_{00}$, it contains at least one of the functions $+, \neg \to$ by Lemma 3.1. Clearly $\neg \to \notin \mathcal{K}$, so we must have $+ \in \mathcal{K}$, and then $\lfloor + \rfloor \subseteq \mathcal{K}$. Thus $\mathcal{K} = \lfloor + \rfloor = L_{00}$, hence $\mathbf{I}(L_{0*}) = \{L_{0*}, L_{00}\}$.

Now let us suppose that $\mathcal{K} \in \mathbf{I}(L) \setminus \{L\}$. Then we have $\mathcal{K} \subseteq L \cap \Omega_= = L_{00} \cup L_{11} = \lfloor + \rfloor \cup \lfloor \neg + \rfloor$. Similarly to the previous case we see that $\lfloor + \rfloor \subseteq \mathcal{K}$ or $\lfloor \neg + \rfloor \subseteq \mathcal{K}$. If only one of these held, then $\mathcal{K}$ would be a subset of either Ω_{00} or Ω_{11}, contradicting $[\mathcal{K}] = L$. So we must have $\mathcal{K} = \lfloor + \rfloor \cup \lfloor \neg + \rfloor = L_=$, hence $\mathbf{I}(L) = \{L, L_=\}$. □

5. Idempotents corresponding to W^∞

In the above cases for each given clone $\mathcal{C}$, the interval $\mathbf{I}(\mathcal{C})$ was finite. For $\mathcal{C} = W^\infty$ the situation is more complicated: we will prove in this section that there are continuously many idempotents generating the clone W^∞. In the next section we will see that the intervals $\mathbf{I}(W^k)$ are finite, but their sizes tend to infinity as $k \to \infty$. In these two sections we will often work with the set of zeros (set of "false points") of functions, so let us set up some notation for this.

For an n-ary Boolean function f, let $f^{-1}(0) = \{\mathbf{a} \in \{0,1\}^n : f(\mathbf{a}) = 0\}$. We will often treat this set as a matrix: writing the elements of $f^{-1}(0)$ as row vectors below each other, we obtain an $m \times n$ matrix, where $m = \left|f^{-1}(0)\right|$. The order of the rows is irrelevant, moreover, since we do not need to distinguish between equivalent functions, we may rearrange the columns as well (by permuting variables). Let us observe that $f \in W^\infty$ iff $f^{-1}(0)$ has a constant 0 column, and $f \in W^k$ iff every matrix formed from at most k rows of $f^{-1}(0)$ has a constant 0 column.

Next we introduce some operators on function classes that deal with zeros of functions. For $k \geq 2$ and $\mathcal{K} \subseteq \Omega$ let $Z_k\mathcal{K}$ denote the set of functions $f \in \Omega$ such that for every at most k-element subset $H \subseteq f^{-1}(0)$ there exists a function $g \in \mathcal{K}$ of the same arity as f with $H \subseteq g^{-1}(0)$. Furthermore, let $Z_\infty\mathcal{K}$ be the set of functions $f \in \Omega$ such that there exists a function $g \in \mathcal{K}$ of the same arity as f with $f^{-1}(0) \subseteq g^{-1}(0)$. Clearly, $Z_2, Z_3, \ldots, Z_\infty$ are closure operators on Ω, and for any $\mathcal{K} \subseteq \Omega$ we have

$$Z_2\mathcal{K} \supseteq Z_3\mathcal{K} \supseteq \cdots \supseteq Z_\infty\mathcal{K} = \bigcap_{k \geq 2} (Z_k\mathcal{K}). \tag{5.1}$$

Proposition 5.2 below describes $\mathbf{I}(W^\infty)$ with the help of the operator Z_∞.

Lemma 5.1. *If $\mathcal{K}$ is an idempotent equational class such that $[\mathcal{K}] = W^k$ for some $k \in \{2, 3, \ldots, \infty\}$, then $Z_\infty\mathcal{K} = \mathcal{K}$.*

Proof. For $\mathcal{K} = W^k$ the statement is clear, so let us suppose that $\mathcal{K} \in \mathbf{I}(W^k) \setminus \{W^k\}$. Let $f \in \Omega, g \in \mathcal{K}$ be n-ary functions such that $f^{-1}(0) \subseteq g^{-1}(0)$. We have to show that $f \in \mathcal{K}$. We can suppose without loss of generality that $\left|g^{-1}(0) \setminus f^{-1}(0)\right| = 1$, i.e., f and g differ only at one position $\mathbf{a} \in \{0,1\}^n$. (Otherwise we apply this several times changing the values of g one by one until we reach the desired function f.) For notational simplicity let us also assume that $a_1 = \cdots = a_l = 0$ and $a_{l+1} = \cdots = a_n =$

1. Then $g(0,\dots,0,1,\dots,1)=0$ and $f(0,\dots,0,1,\dots,1)=1$. (Here, and in the rest of the proof all n-tuples are split into two parts of size l and $n-l$.)

From Proposition 3.4 we know that $\mathcal{K}=[\mathcal{K}]\circ\mathcal{K}=W^k\circ\mathcal{K}\supseteq W^\infty\circ\mathcal{K}$, so it suffices to show that $f\in W^\infty\circ\mathcal{K}$. We consider the following function of arity $N+1$ with $N=l(n-l)$, which clearly belongs to W^∞:

$$h(z_1,\dots,z_N,w)=\Big(\bigvee_{1\le k\le N} z_k\Big)\to w.$$

Let us construct the following function $f'\in W^\infty\circ\mathcal{K}$, which is a composition of h (as outer function) and several subfunctions of g (as inner functions):

$$\begin{aligned} &f'(x_1,\dots,x_l,y_1,\dots,y_{n-l})\\ &\qquad=\Big(\bigvee_{\substack{1\le i\le l\\ 1\le j\le n-l}} g(x_i,\dots,x_i,y_j,\dots,y_j)\Big)\to g(x_1,\dots,x_l,y_1,\dots,y_{n-l}).\end{aligned}$$

We claim that $f'=f$. In order to verify this fact, let us observe that since $g(0,\dots,0,1,\dots,1)=0$, we have $g(1,\dots,1,0,\dots,0)=1$, as $g\in\mathcal{K}\subseteq W^2$. We also have $g(1,\dots,1,1,\dots,1)=1$, and then we must have $g(0,\dots,0,0,\dots,0)=1$, since otherwise id would be a subfunction of g, implying $\mathcal{K}=[\mathcal{K}]=W^k$, contrary to our assumption. Therefore $g(x_i,\dots,x_i,y_j,\dots,y_j)=0$ iff $x_i=0$ and $y_j=1$, hence the disjunction in the definition of f' equals 0 iff $x_1=\dots=x_l=0$ and $y_1=\dots=y_{n-l}=1$. Thus f' differs from g only at the position $\mathbf{a}$, where $f'(\mathbf{a})=1$ and $g(\mathbf{a})=0$. Hence $f=f'\in W^\infty\circ\mathcal{K}\subseteq\mathcal{K}$, as claimed. □

Proposition 5.2. *Let $\mathcal{K}$ be an equational class such that $[\mathcal{K}]=W^\infty$. Then*

$$\mathcal{K}\in\mathbf{I}(W^\infty)\iff Z_\infty\mathcal{K}=\mathcal{K}.$$

Proof. "$\Longrightarrow$": Follows from the previous lemma with $k=\infty$.

"$\Longleftarrow$": Let us suppose that $[\mathcal{K}]=W^\infty$ and $Z_\infty\mathcal{K}=\mathcal{K}$. Let $f\in\mathcal{K}$ be n-ary and $g_1,\dots,g_n\in\mathcal{K}$ be k-ary functions. We have to prove that $h=f(g_1,\dots,g_n)$ belongs to $\mathcal{K}$. Since $f\in W^\infty$, there exists an index i such that for all $(a_1,\dots,a_n)\in f^{-1}(0)$ we have $a_i=0$. Therefore, for all $(b_1,\dots,b_k)\in h^{-1}(0)$ we have $g_i(b_1,\dots,b_k)=0$, i.e., $h^{-1}(0)\subseteq g_i^{-1}(0)$. This proves that $h\in Z_\infty\mathcal{K}=\mathcal{K}$. □

Remark 5.3. Let us note that $Z_\infty\mathcal{K}=\mathcal{K}$ iff $\mathcal{K}$ is a filter in the usual pointwise ordering $\le$ of Boolean functions. Let $\sqsubseteq$ be the transitive closure of $\preceq\cup\ge$. Then $\sqsubseteq$ is a quasiorder on Ω, and a class $\mathcal{K}$ is an ideal with respect to this quasiorder iff $\mathcal{K}$ is an ideal w.r.t. $\preceq$ (i.e., an equational class) and a filter w.r.t. $\le$ (i.e., satisfies $Z_\infty\mathcal{K}=\mathcal{K}$). Thus we can reformulate the above proposition as follows: $\mathcal{K}\in\mathbf{I}(W^\infty)$ iff $[\mathcal{K}]=W^\infty$ and $\mathcal{K}$ is an ideal w.r.t. $\sqsubseteq$. This implies the somewhat surprising fact that a union of idempotents is idempotent in this case, i.e., the lattice operations in the interval $\mathbf{I}(W^\infty)$ coincide with the set operations $\cap$ and $\cup$. Consequently, $\mathbf{I}(W^\infty)$ is a distributive lattice.

In the next proposition we describe $\bigcap\mathbf{I}(W^\infty)$, the bottom element of the interval $\mathbf{I}(W^\infty)$.

Proposition 5.4. *The bottom element of the interval $\mathbf{I}(W^\infty)$ is $\lfloor\to\rfloor$, which is strongly defined by the relational constraints*

$$\left(\{0,1\}^k\setminus\{\mathbf{1}\},\{0,1\}^k\setminus\{\mathbf{0}\}\right)\quad(k\in\mathbb{N}).$$

Proof. Let $\mathcal{B}^\infty$ denote the function class strongly defined by the above constraints. We will prove that

$$\bigcap\mathbf{I}(W^\infty)\subseteq\lfloor\to\rfloor\subseteq\mathcal{B}^\infty\subseteq\bigcap\mathbf{I}(W^\infty).$$

The first containment follows immediately from the fact that $\lfloor\to\rfloor$ belongs to $\mathbf{I}(W^\infty)$, and for the second one it suffices to verify that $\to\in\mathcal{B}^\infty$, since $\mathcal{B}^\infty$ is an idempotent according to Theorem 2.4.

For the third containment, let f be an arbitrary n-ary function in $\mathcal{B}^{\infty}$, and let $k = \left|f^{-1}(0)\right|$. Since f preserves the relation $\{0,1\}^{k} \setminus \{\mathbf{0}\}$, the matrix $f^{-1}(0)$ has a constant 0 column. Moreover, since f satisfies the relational constraint $(\{0,1\}^{k} \setminus \{\mathbf{1}\}, \{0,1\}^{k} \setminus \{\mathbf{0}\})$, the matrix $f^{-1}(0)$ has a constant 1 column as well. We can suppose without loss of generality that the first column of $f^{-1}(0)$ is constant 1 and the second column is constant 0.

Since $[\bigcap \mathbf{I}(W^{\infty})] = W^{\infty}$, there is at least one nonmonotone function $g_1 \in \bigcap \mathbf{I}(W^{\infty})$. For such a function, $g_1^{-1}(0) \nsubseteq \{\mathbf{0},\mathbf{1}\}$, thus, permuting variables, we may suppose that $g_1^{-1}(0)$ contains a vector of the form $(1,\ldots,1,0,\ldots,0)$. Applying the operator Z_{∞}, we can obtain a function $g_2 \in Z_{\infty} \bigcap \mathbf{I}(W^{\infty}) = \bigcap \mathbf{I}(W^{\infty})$ of the same arity as g_1 with $g_2^{-1}(0) = \{(1,\ldots,1,0,\ldots,0)\}$. Identifying appropriately the variables of g_2, we get a binary function $g_3 \in \bigcap \mathbf{I}(W^{\infty})$ with $g_3^{-1}(0) = \{(1,0)\}$ (g_3 is nothing else but the implication). Adding $n-2$ dummy variables, we can obtain a function $g_4 \in \bigcap \mathbf{I}(W^{\infty})$ of arity n, such that $g_4^{-1}(0) = \{1\} \times \{0\} \times \{0,1\}^{n-2}$, i.e., *every* n-tuple of the form $(1,0,\ldots)$ belongs to $g_4^{-1}(0)$. Since $f^{-1}(0)$ consists of *some* of these tuples, we have $f \in Z_{\infty}\{g_4\} \subseteq Z_{\infty} \bigcap \mathbf{I}(W^{\infty}) = \bigcap \mathbf{I}(W^{\infty})$. □

Combining the previous two propositions and Remark 5.3, we get the following characterizations of the idempotents corresponding to W^{∞}.

Theorem 5.5. *For any class $\mathcal{K}$ of Boolean functions the following conditions are equivalent:*

(1) $\mathcal{K} \in \mathbf{I}(W^{\infty})$*;*
(2) $\rightarrow \in \mathcal{K} \subseteq W^{\infty}$*, and $\mathcal{K}$ is an equational class satisfying $Z_{\infty}\mathcal{K} = \mathcal{K}$;*
(3) $\rightarrow \in \mathcal{K} \subseteq W^{\infty}$*, and $\mathcal{K}$ is an ideal with respect to the quasiorder $\sqsubseteq$.*

We conclude this section by proving that the interval $\mathbf{I}(W^{\infty})$ is uncountable.

Theorem 5.6. *The interval $\mathbf{I}(W^{\infty})$ has continuum cardinality.*

Proof. Let J_n be the following $n \times (n+1)$ matrix over $\{0,1\}$:

$$J_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix},$$

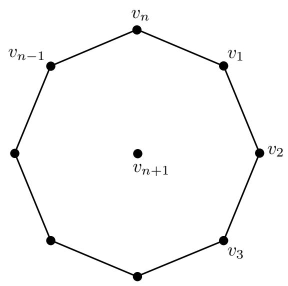
and let f_n be the $(n+1)$-ary Boolean function such that $f_n^{-1}(0)$ consists of the rows of J_n, and let P_n be the n-ary relation consisting of the columns of J_n. It is straightforward to verify that $[f_n] = W^{\infty}$, hence $\lfloor f_n \rfloor \in \mathbf{I}(W^{\infty})$. We claim that for all natural numbers m, n,

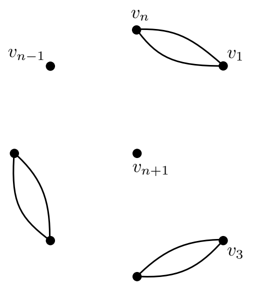
$$\text{if } m \text{ is odd, then } f_n \text{ strongly satisfies } (P_m, \{0,1\}^{m} \setminus \{\mathbf{0}\}) \text{ iff } m \neq n.$$

It is clear that f_n does not satisfy $(P_n, \{0,1\}^{n} \setminus \{\mathbf{0}\})$, since $f_n(J_n) = \mathbf{0}$. Now let us assume that m is odd and $m \neq n$. The matrix J_n has a constant 0 column, therefore f_n preserves $\{0,1\}^{m} \setminus \{\mathbf{0}\}$. Suppose for contradiction that f_n does not satisfy $(P_m, \{0,1\}^{m} \setminus \{\mathbf{0}\})$, i.e., there exists a P_m-matrix N of size $m \times (n+1)$ such that $f(N) = \mathbf{0}$. This means that every column of N is a column of J_m, and every row of N is a row of J_n.

Let us interpret the matrix N as the incidence matrix of a graph G (with possibly multiple edges): the rows of N correspond to the m edges of G, and the columns of N correspond to the vertices $v_1, \ldots, v_{n+1}$ of G. Since each row of N is a row of J_n, the vertex v_{n+1} is isolated, and every edge connects two consecutive vertices in the cyclical ordering of $v_1, \ldots, v_n$. Since every column of N is a column of J_m, every vertex has degree 0 or 2. These properties imply that G is either a cycle on the vertices $v_1, \ldots, v_n$

(together with the isolated vertex v_{n+1}), or the components of G are 2-cycles (double edges) and isolated vertices:





Both cases lead to a contradiction: in the former case we have $m = n$, while in the latter case we can conclude that m is even. This contradiction proves our claim.

To finish the proof of the theorem it suffices to observe that our claim implies that $I \mapsto \lfloor \{f_i : i \in I\} \rfloor$ embeds the power set of the set of odd natural numbers into $\mathbf{I}(W^\infty)$. $\square$

Remark 5.7. We have shown that the power set of a countably infinite set embeds into $\mathbf{I}(W^\infty)$, and it is obvious that $\mathbf{I}(W^\infty)$ embeds into the power set of Ω. Thus, $\mathbf{I}(W^\infty)$ is equimorphic to the power set of a countably infinite set.

6. Idempotents corresponding to W^k

We will prove that the analogue of Proposition 5.2 holds for finite values of k as well. First we need a technical lemma.

Lemma 6.1. *For any $\mathcal{K} \subseteq \Omega$ and $l \geq 2$ we have $Z_l\mathcal{K} \subseteq Z_{l+1}\left(W^l \circ \mathcal{K}\right)$.*

Proof. Let f be an n-ary function in $Z_l\mathcal{K}$, and let $\mathbf{a}_1, \dots, \mathbf{a}_{l+1} \in f^{-1}(0)$. We need to show that there exists an n-ary function $u \in W^l \circ \mathcal{K}$ such that $\mathbf{a}_1, \dots, \mathbf{a}_{l+1} \in u^{-1}(0)$. Since $f \in Z_l\mathcal{K}$, for every $j = 1, \dots, l+1$ we can find a function $g_j \in \mathcal{K}$ such that $\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \dots, \mathbf{a}_{l+1} \in g_j^{-1}(0)$. Now let us consider the following function h of arity $l+1$:

$$h(x_1, \dots, x_{l+1}) = \begin{cases} 0, & \text{if } |\{i : x_i = 1\}| \leq 1; \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $h \in W^l$, therefore $u = h(g_1, \dots, g_{l+1}) \in W^l \circ \mathcal{K}$, and $\mathbf{a}_1, \dots, \mathbf{a}_{l+1} \in u^{-1}(0)$. This proves that $f \in Z_{l+1}\left(W^l \circ \mathcal{K}\right)$. $\square$

Proposition 6.2. *Let $\mathcal{K}$ be an equational class such that $[\mathcal{K}] = W^k$. Then*

$$\mathcal{K} \in \mathbf{I}\left(W^k\right) \iff Z_k\mathcal{K} = \mathcal{K}.$$

Proof. "$\Longrightarrow$": Let $\mathcal{K}$ be an idempotent such that $[\mathcal{K}] = W^k$. Lemma 5.1 shows that $Z_\infty\mathcal{K} = \mathcal{K}$. For any $l \geq k$ we have $W^l \circ \mathcal{K} \subseteq W^k \circ \mathcal{K} = \mathcal{K}$ according to Proposition 3.4, and using the previous lemma we get $Z_l\mathcal{K} \subseteq Z_{l+1}\left(W^l \circ \mathcal{K}\right) \subseteq Z_{l+1}\mathcal{K}$. The reversed containment $Z_l\mathcal{K} \supseteq Z_{l+1}\mathcal{K}$ is obvious, so we have $Z_k\mathcal{K} = Z_{k+1}\mathcal{K} = Z_{k+2}\mathcal{K} = \cdots = Z_\infty\mathcal{K} = \mathcal{K}$ in light of (5.1).

"$\Longleftarrow$": We just need to modify slightly the second part of the proof of Proposition 5.2. Let us suppose that $\mathcal{K}$ is an equational class such that $[\mathcal{K}] = W^k$ and $\mathcal{K}$ is closed under the operator Z_k. Let $f \in \mathcal{K}$ be n-ary, and $g_1, \dots, g_n \in \mathcal{K}$ be m-ary functions. We have to prove that $h = f(g_1, \dots, g_n)$ belongs to $\mathcal{K} = Z_k\mathcal{K}$. If $\mathbf{a}_1, \dots, \mathbf{a}_k \in h^{-1}(0)$, then the vectors $(g_1(\mathbf{a}_j), \dots, g_n(\mathbf{a}_j))$ belong to $f^{-1}(0)$ for $j = 1, \dots, k$. Since $f \in W^k$, there exists an index i such that $g_i(\mathbf{a}_j) = 0$ for $j = 1, \dots, k$, i.e., $\mathbf{a}_1, \dots, \mathbf{a}_k \in g_i^{-1}(0)$. This shows that $h \in Z_k\mathcal{K} = \mathcal{K}$. $\square$

Our next task is, just like in the previous section, to describe $\bigcap \mathbf{I}\left(W^k\right)$, the bottom element of the interval $\mathbf{I}\left(W^k\right)$. The proof is very similar to the proof of Proposition 5.4.

Proposition 6.3. *The bottom element of the interval* $\mathbf{I}\left(W^k\right)$ *is* $\lfloor w_k \rfloor$*, which is strongly defined by the relational constraint*

$$\left(\{0,1\}^k \setminus \{\mathbf{1}\}, \{0,1\}^k \setminus \{\mathbf{0}\}\right).$$

Proof. Let $\mathcal{B}^k$ denote the function class strongly defined by the above constraint. We will prove that

$$\bigcap \mathbf{I}\left(W^k\right) \subseteq \lfloor w_k \rfloor \subseteq \mathcal{B}^k \subseteq \bigcap \mathbf{I}\left(W^k\right).$$

The first containment follows immediately from the fact that $\lfloor w_k \rfloor \in \mathbf{I}\left(W^k\right)$, and for the second one it suffices to verify that $w_k \in \mathcal{B}^k$, since $\mathcal{B}^k$ is an idempotent according to Theorem 2.4.

For the third containment, let f be an arbitrary n-ary function in $\mathcal{B}^k$, and N be an at most k-element subset of $f^{-1}(0)$. Since f preserves the relation $\{0,1\}^k \setminus \{\mathbf{0}\}$, viewing N as a matrix, it has a constant 0 column. Moreover, since f satisfies the constraint $(\{0,1\}^k \setminus \{\mathbf{1}\}, \{0,1\}^k \setminus \{\mathbf{0}\})$, the matrix N has a constant 1 column as well. We can suppose without loss of generality that the first column of N is constant 1 and the second column is constant 0.

Since $\left[\bigcap \mathbf{I}\left(W^k\right)\right] = W^k$, there is at least one nonmonotone function in $\bigcap \mathbf{I}\left(W^k\right)$, and taking into account that $\bigcap \mathbf{I}\left(W^k\right)$ is closed under the operator Z_∞, we can apply the same argument as in the proof of Proposition 5.4 to construct a function $g_4 \in \bigcap \mathbf{I}\left(W^k\right)$ of arity n, such that $g_4^{-1}(0) = \{1\} \times \{0\} \times \{0,1\}^{n-2}$. Then we have $N \subseteq g_4^{-1}(0)$, and since we can construct such a function g_4 for any at most k-element subset N of $f^{-1}(0)$, we can conclude that $f \in Z_k \bigcap \mathbf{I}\left(W^k\right) = \bigcap \mathbf{I}\left(W^k\right)$. □

The previous two propositions yield the following characterization of the idempotents corresponding to W^k.

Theorem 6.4. *For any class* $\mathcal{K}$ *of Boolean functions the following conditions are equivalent:*

1. $\mathcal{K} \in \mathbf{I}\left(W^k\right)$*;*
2. $w_k \in \mathcal{K} \subseteq W^k$*, and* $\mathcal{K}$ *is an equational class satisfying* $Z_k\mathcal{K} = \mathcal{K}$*.*

Finally, we prove that the intervals $\mathbf{I}\left(W^k\right)$ are finite, but their sizes do not have a common upper bound.

Theorem 6.5. *For any* $k \geq 2$*, the interval* $\mathbf{I}\left(W^k\right)$ *is finite and has at least* $k+1$ *elements.*

Proof. To obtain the lower bound, let us observe that for any $j, k \geq 2$ we have $\mathcal{B}^j \cap W^k \supseteq \mathcal{B}^{j+1} \cap W^k$, and this containment is proper, since $v_j \in \mathcal{B}^j \cap W^k$ and $v_j \notin \mathcal{B}^{j+1} \cap W^k$, where $v_j(x_1, \ldots, x_{j+2}) = w_j(\neg x_1, \ldots, \neg x_{j+2})$. Thus we have the following chain of length $k+1$ in $\mathbf{I}\left(W^k\right)$ (see Figure 4):

$$W^k \supset W^k_= \supset \mathcal{B}^2 \cap W^k \supset \mathcal{B}^3 \cap W^k \supset \cdots \supset \mathcal{B}^k \cap W^k = \mathcal{B}^k.$$

In order to prove the finiteness of $\mathbf{I}\left(W^k\right)$, we define the *skeleton* of an idempotent class $\mathcal{K} \in \mathbf{I}\left(W^k\right)$ as follows. For every $f \in \mathcal{K}$, let us construct all matrices formed by at most k rows of the matrix $f^{-1}(0)$, and delete repeated columns, if there are any. The skeleton of $\mathcal{K}$ is the collection of all such matrices. Every matrix in the skeleton has at most k rows, and at most 2^k columns, since there are no repeated columns. There are only finitely many such matrices, hence there are only finitely many possible skeletons. Therefore, it suffices to prove that different idempotents in $\mathbf{I}\left(W^k\right)$ have different skeletons.

So let us suppose that $\mathcal{K}_1, \mathcal{K}_2 \in \mathbf{I}\left(W^k\right)$ have the same skeleton $\mathcal{S}$. Let f_1 be any function in $\mathcal{K}_1$, and let $N_1 \subseteq f_1^{-1}(0)$ be any set with at most k elements. Deleting repeated columns of the matrix N_1, we obtain a matrix $N' \in \mathcal{S}$. Since $\mathcal{S}$ is the skeleton of $\mathcal{K}_2$ as well, there exists a function $f_2 \in \mathcal{K}_2$ and a matrix N_2 formed by at most k rows of $f_2^{-1}(0)$, such that deleting repeated columns of N_2 we obtain the same matrix N'. Identifying variables of f_2 we can construct a function $g_2 \in \mathcal{K}_2$ such

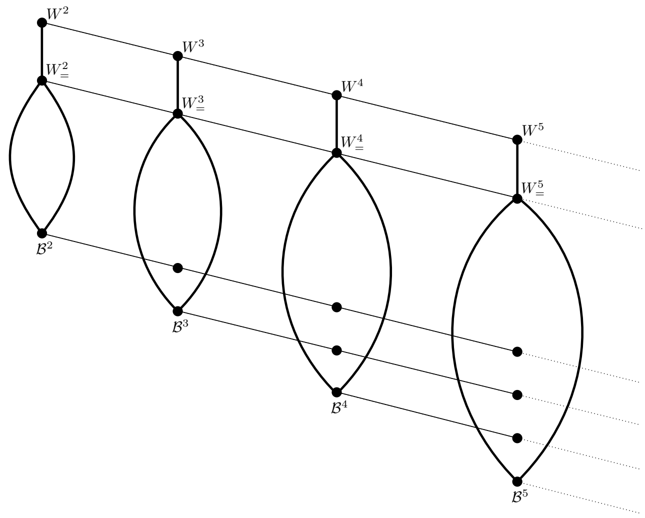


FIGURE 4. The intervals $\mathbf{I}\left(W^k\right)$

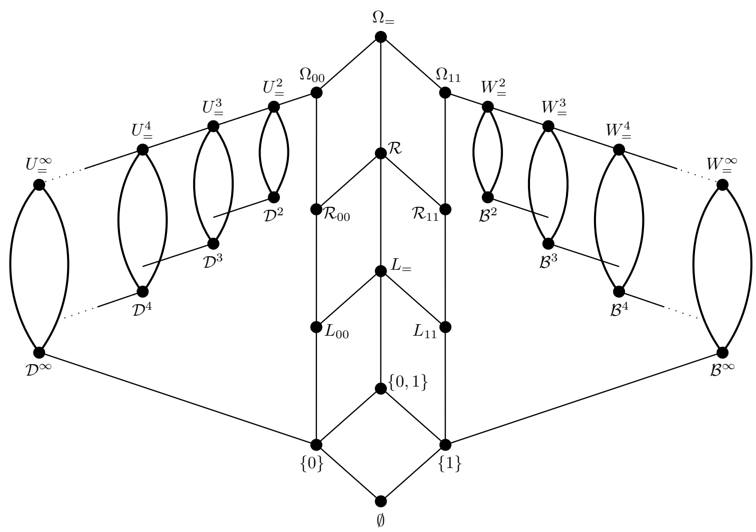


FIGURE 5. The lattice of closed classes without projections

that $g_2^{-1}(0) \supseteq N'$. Now adding dummy variables to g_2, we can construct a function $h_2 \in \mathcal{K}_2$ such that $h_2^{-1}(0) \supseteq N_1$. Since we can do this for any at most k-element subset N_1 of $f_1^{-1}(0)$, we can conclude that $f_1 \in Z_k\mathcal{K}_2 = \mathcal{K}_2$. This proves that $\mathcal{K}_1 \subseteq \mathcal{K}_2$, and a similar argument yields $\mathcal{K}_1 \supseteq \mathcal{K}_2$, thus we have $\mathcal{K}_1 = \mathcal{K}_2$. $\square$

7. CONCLUDING REMARKS

Assembling the results of the previous sections, we can draw the lattice of projection-free idempotents as shown in Figure 5. The bottom of the interval $\mathbf{I}\left(U^k\right)$ is denoted by $\mathcal{D}^k$; it is the dual of $\mathcal{B}^k$, hence it can be strongly defined by the relational constraint $\left(\{0,1\}^k \setminus \{\mathbf{0}\}, \{0,1\}^k \setminus \{\mathbf{1}\}\right)$ for finite k, and by all of these constraints for $k=\infty$. To obtain the whole lattice $\mathbf{I}$, we have to put together this lattice with the Post lattice, as shown schematically in Figure 6. For the "real picture" we would have to connect $\mathcal{C}_=$ to $\mathcal{C}$ for each nontrivial interval $\mathbf{I}(\mathcal{C})$, but this would make the figure incomprehensibly complex.

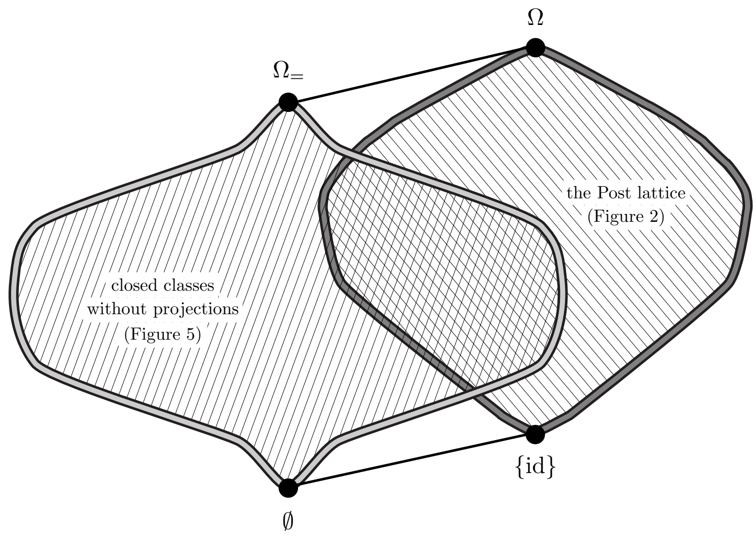


FIGURE 6. The lattice $\mathbf{I}$ of composition-closed equational classes of Boolean functions

Finally, let us mention a few directions for further investigations. Our characterization of the intervals $\mathbf{I}(W^{\infty})$ and $\mathbf{I}\left(W^{k}\right)$ is not explicit, hence a more concrete description would be desirable. In particular, it would be interesting to determine (at least asymptotically) the size of $\mathbf{I}\left(W^{k}\right)$. To better understand the structure of $\mathbf{I}(W^{\infty})$, the quasiorder $\sqsubseteq$ defined in Remark 5.3 should be studied. Although the lattice of clones over a base set with at least three elements is not fully described, it may be possible to get some results about composition-closed equational classes over arbitrary finite domains, e.g., determine minimal and maximal closed classes. The description of $\mathbf{I}$ obtained in this paper can be regarded as a first step in the study of the semigroup $(\mathbf{E};\circ)$; for further results in this direction see [1].

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