MINIMAL CLONES WITH FEW MAJORITY OPERATIONS

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Dedicated to Béla Csákány on his seventy-fifth birthday

ABSTRACT. We characterize minimal clones generated by a majority function containing at most seven ternary operations.

1. INTRODUCTION

A set $\mathcal{C}$ of finitary operations on a set A is a *(concrete) clone*, if it is closed under composition of functions and contains all projections. If $\mathbb{A} = (A; F)$ is an algebra, then the set of its term functions, denoted by $\operatorname{Clo} \mathbb{A}$, is a clone on A, called the *clone of the algebra* $\mathbb{A}$. This is the smallest clone containing F, therefore we say that F *generates* $\operatorname{Clo} \mathbb{A}$, and we write $[F] = \operatorname{Clo} \mathbb{A}$. Clearly, every clone arises as the clone of an algebra: we just need to pick a generating set for the clone, and let these be the basic operations of the algebra.

An *(abstract) clone* is a heterogeneous algebra that captures the compositional structure of concrete clones [BL, Ta]. More precisely, an abstract clone $\mathcal{C}$ is given by a family $\mathcal{C}^{(n)}$ $(n \geq 1)$ of sets with distinguished elements $e_i^{(n)} \in \mathcal{C}^{(n)}$ $(1 \leq i \leq n)$ and mappings

$$F_k^n : \mathcal{C}^{(n)} \times \left(\mathcal{C}^{(k)} \right)^n \to \mathcal{C}^{(k)}, (f, g_1, \ldots, g_n) \mapsto f(g_1, \ldots, g_n) \quad (n, k \geq 1),$$

such that the following three axioms are satisfied for all $f \in \mathcal{C}^{(n)}$, $g_1, \ldots, g_n \in \mathcal{C}^{(k)}$, $h_1, \ldots, h_k \in \mathcal{C}^{(l)}$ $(n, k, l \geq 1)$:

$$e_i^{(n)}(g_1, \ldots, g_n) = g_i \quad (i = 1, \ldots, n),$$
$$f\left(e_1^{(n)}, \ldots, e_n^{(n)}\right) = f,$$
$$f(g_1, \ldots, g_n)(h_1, \ldots, h_k) = f(g_1(h_1, \ldots, h_k), \ldots, g_n(h_1, \ldots, h_k)).$$

The notion of a subclone, clone homomorphism and factor clone can be defined in a natural way, and the isomorphism theorems can be proved for abstract clones.

Every concrete clone can be regarded as an abstract clone if we let $e_i^{(n)}$ be the i-th n-ary projection, and $F_k^n(f, g_1, \ldots, g_n)$ be the composition of f by $g_1, \ldots, g_n$, as we have already indicated it in the notation. We will call the elements $e_i^{(n)}$ *projections*, the mappings F_k^n *composition* operations, and $\mathcal{C}^{(n)}$ the n-ary part of $\mathcal{C}$, even if the elements of the abstract clone are not functions. However, every abstract clone is isomorphic to a concrete clone, so we can always assume that the elements of the clone are actually functions.

There is a close relationship between abstract clones and varieties; roughly speaking, abstract clones are the same as varieties up to term-equivalence [Ke, LP]. To explain this more explicitly, let us fix an abstract clone $\mathcal{C}$, and a generating set F of $\mathcal{C}$. For any clone homomorphism φ from $\mathcal{C}$ to a concrete clone on some set A we can construct the algebra $\mathbb{A} = (A; \varphi(F))$ whose clone is $\varphi(\mathcal{C})$. It is not hard to see that the algebras arising this way form a variety. If we choose another set of generators, then we get another variety which is term-equivalent to the previous one. Conversely,

a clone can be assigned to every variety, namely the clone of the countably generated free algebra of the variety, and these two correspondences between clones and varieties are inverses of each other (up to isomorphism of clones and term-equivalence of varieties).

If $\mathcal{C}$ and $\mathcal{V}$ correspond to each other, then subvarieties of $\mathcal{V}$ correspond to factor clones of $\mathcal{C}$, and the congruence lattice of $\mathcal{C}$ is dually isomorphic to the subvariety lattice of $\mathcal{V}$. If $\mathcal{V}$ is generated by $\mathbb{A}$, then $\mathcal{C} \cong \operatorname{Clo}\mathbb{A}$, and an algebra $\mathbb{B}$ (of the appropriate type) belongs to $\mathcal{V}$ iff $\operatorname{Clo}\mathbb{B}$ is a homomorphic image of $\operatorname{Clo}\mathbb{A}$. An important special case is when $\mathbb{B}$ is a subalgebra of $\mathbb{A}$. In this case the restriction map $\operatorname{Clo}\mathbb{A} \to \operatorname{Clo}\mathbb{B}$, $f \mapsto f|_B$ is a surjective clone homomorphism.

All clones on a given set A form a lattice with respect to inclusion; the smallest element of this lattice is the *trivial clone*, the clone of all projections on A, while the greatest element is the clone of all finitary operations on A. These clones will be denoted by $\mathcal{I}_A$ and $\mathcal{O}_A$ respectively. The elements of the trivial clone (the projections) will be referred to as *trivial functions*. An abstract clone $\mathcal{C}$ is called trivial if $\mathcal{C}^{(n)} = \left\{ e_1^{(n)}, \ldots, e_n^{(n)} \right\}$ for all $n \geq 1$.

We say that $\mathcal{C}$ is a *minimal clone*, if it has exactly two subclones: the trivial clone and $\mathcal{C}$ itself. In the case of concrete clones on a given set A, we can identify minimal clones as the atoms of the clone lattice. On finite sets there are finitely many minimal clones, and every clone contains a minimal one (cf. [PK, Qu2, Sz]).

Clearly, a nontrivial clone is minimal iff it is generated by any of its nontrivial elements. Therefore all minimal clones are one-generated, thus they arise as clones of algebras with a single basic operation. It is convenient to choose a function of the least possible arity as a generator of a minimal clone. These generators are called *minimal functions*: f is a minimal function iff $[f]$ is a minimal clone and there is no nontrivial function in $[f]$ whose arity is less than the arity of f. According to Świerczkowski's lemma [Sw], a minimal function must be either a unary operation, a binary idempotent operation, a ternary majority or minority operation or a semiprojection. Rosenberg's theorem [Ro, Sz] characterizes minimal unary and ternary minority operations, but for the other three types a general description of minimal functions (or clones) seems to be far beyond reach.

There are numerous partial results that describe minimal clones or minimal functions under certain assumptions, and the goal of this paper is to prove a new theorem of this kind. In the next section we recall only a few facts about minimal clones that we will need in the sequel; for a survey of minimal clones see [Cs3] and [Qu2]. In Section 3 we prove a theorem about the possible symmetries of majority functions in a minimal clone (Theorem 3.3), and in Section 4 we use this theorem to obtain a characterization of those minimal clones generated by a majority operation which contain at most seven ternary operations (Theorem 4.1).

2. Preliminaries

For brevity we will say that $\mathcal{C}$ is a *majority clone* if $\mathcal{C} = [f]$ where f is a *majority operation*, i.e. f satisfies the identities

$$f(x,x,y) = f(x,y,x) = f(y,x,x) = x.$$

First we state and prove a very special property of majority clones. This fact seems to be folklore; usually it is derived from Rosenberg's theorem or from Świerczkowski's lemma. Here we give an almost self-contained proof.

Theorem 2.1 ([Cs2])**.** *Let $\mathcal{C}$ be a clone generated by a majority operation f. If every majority operation in $\mathcal{C}$ generates f, then $\mathcal{C}$ is a minimal clone.*

Proof. The key is the following observation, which can be proved by a simple induction argument [Cs2]. If g is a nontrivial operation in a clone generated by a majority function, then g is a so-called *near-unanimity operation*, i.e. it satisfies the identities

$$g(y,x,x,\ldots,x,x) = g(x,y,x,\ldots,x,x) = \cdots = g(x,x,x,\ldots,x,y) = x.$$

We show that any near-unanimity function g of arity $n \geq 4$ produces a non-trivial function of arity $n - 1$ by a suitable identification of its variables. Let us suppose that $g(x, x, x_3, \ldots, x_n)$ is a projection. Identifying all the x_is except for x_n with x, we get a projection onto x by the near-unanimity property, therefore $g(x, x, x_3, \ldots, x_n)$ cannot be a projection onto x_n. This can be done for any x_i instead of x_n, thus $g(x, x, x_3, \ldots, x_n)$ has to be a projection onto x. If we suppose that $g(x_1, x_2, y, y, x_5, \ldots, x_n)$ is also a projection, then a similar argument shows that it must be a projection onto y. Now we have a contradiction, because $g(x, x, y, y, x_5, \ldots, x_n)$ is a projection to x and y at the same time (this is where we use that $n \geq 4$). Thus we have proved that either $g(x, x, x_3, \ldots, x_n)$ or $g(x_1, x_2, y, y, x_5, \ldots, x_n)$ is nontrivial.

Now if g is an at least quaternary near-unanimity function in the clone $\mathcal{C}$, then it produces a nontrivial function of arity one less, which is again a near-unanimity function, since it is still generated by f. Hence if this new function is still of arity at least 4, then it produces a near-unanimity function of lesser arity too, and we can continue this way until we end up with a near-unanimity function of arity 3, i.e. a majority operation. Since it was supposed that every majority operation in $\mathcal{C}$ generates f, we have $f \in [g]$, hence $\mathcal{C} = [g]$, and this shows that $\mathcal{C}$ is a minimal clone. $\qquad\square$

The advantage of this property is that in order to prove the minimality of a majority clone it suffices to consider the ternary part of the clone. On a finite set this means a finite number of functions, while in the binary and semiprojection case one has to consider infinitely many functions.

Restricting our attention to the ternary operations of an abstract clone we get the algebra $\left(\mathcal{C}^{(3)}; F_3^3, e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\right)$. We will refer to this algebra as the ternary part of $\mathcal{C}$, and denote it briefly by $\mathcal{C}^{(3)}$. This is an algebra with one quaternary and three nullary operations satisfying the following identities.

$$F_3^3\left(e_i^{(3)}, f_1, f_2, f_3\right) = f_i \quad (i = 1, 2, 3)$$

$$F_3^3\left(f, e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\right) = f$$

$$F_3^3\left(F_3^3(f, g_1, g_2, g_3), h_1, h_2, h_3\right) =$$
$$F_3^3(f, F_3^3(g_1, h_1, h_2, h_3), F_3^3(g_2, h_1, h_2, h_3), F_3^3(g_3, h_1, h_2, h_3))$$

Now Theorem 2.1 can be formulated in the following way: A majority clone $\mathcal{C}$ is minimal iff $\left\{e_1^{(3)}, e_2^{(3)}, e_3^{(3)}\right\}$ is the only proper subalgebra of $\mathcal{C}^{(3)}$.

As opposed to the case of binary operations and semiprojections, there are not many examples of minimal majority functions. The simplest ones are the median function $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ on any lattice [PK], and the dual discriminator function on any set [CsG, FP]. The description of minimal clones on the three-element set given by B. Csákány [Cs1] yields some more examples.

Theorem 2.2 ([Cs1]). *If f is a minimal majority function on a three-element set, then f is isomorphic to one of the twelve majority functions shown in Table 1. These functions belong to three minimal clones containing $1, 3$ and 8 majority operations respectively, as shown in the table.*

Note that we have omitted those triples in the table where the majority rule determines the value of the functions. Let us also observe that m_1 can be defined as the median function of the three-element chain (with the unusual order $2 < 1 < 3$ or $3 < 1 < 2$), and m_2 is nothing else but the dual discriminator, up to a permutation of variables (the third function in $[m_2]$ is actually the dual discriminator).

Based on this theorem, B. Csákány obtained a characterization of minimal majority operations which are *conservative* [Cs2]. A function is *conservative* if it preserves every subset of the underlying set (cf. [Qu1]). It is clear that if f is a conservative minimal majority function on a set A, and $B \subseteq A$ is a three-element subset, then

	m_1	m_2			m_3							
$(1,2,3)$	1	1	2	3	3	3	1	3	1	1	3	1
$(2,3,1)$	1	2	3	1	3	1	3	3	1	3	1	1
$(3,1,2)$	1	3	1	2	3	3	3	1	1	1	1	3
$(2,1,3)$	1	2	1	3	1	3	1	1	3	1	3	3
$(1,3,2)$	1	1	3	2	1	1	1	3	3	3	3	1
$(3,2,1)$	1	3	2	1	1	1	3	1	3	3	1	3

TABLE 1. Minimal majority functions on the three-element set

$f|_B$ is a minimal majority function on B. Thus $f|_B$ is isomorphic to one of the above twelve functions. These restrictions determine f, so we can say that f is somehow glued together from copies of the functions listed in Table 1.

We do not quote the result here, but let us note that from this description it follows that there are only four conservative minimal majority clones up to isomorphism of the ternary part of the clone (but not up to isomorphism of the whole clone; see the example in Section 3). For three of them the ternary part is isomorphic to $[m_1]^{(3)}, [m_2]^{(3)}, [m_3]^{(3)}$ respectively, hence they contain $1, 3$ and 8 majority operations, while the fourth one contains 24 majority operations.

As the next theorem shows, the nonconservative minimal majority functions on the four-element set are quite similar to those on the three-element set.

Theorem 2.3 ([Wa]). *If f is a minimal majority function on a four-element set, then f is either conservative, or isomorphic to one of the twelve majority functions shown in Table 2. These functions belong to three minimal clones containing $1, 3$ and 8 majority operations respectively, as shown in the table. (The middle two rows mean that if $\{a, b, c\}$ equals $\{1, 2, 4\}$ or $\{1, 3, 4\}$, then the value of the functions on (a, b, c) is 4.) Moreover, the clone generated by M_i is isomorphic to $[m_i]$.*

	M_1	M_2			M_3							
$(1,2,3)$	4	4	2	3	3	3	4	3	4	4	3	4
$(2,3,1)$	4	2	3	4	3	4	3	3	4	3	4	4
$(3,1,2)$	4	3	4	2	3	3	3	4	4	4	4	3
$(2,1,3)$	4	2	4	3	4	3	4	4	3	4	3	3
$(1,3,2)$	4	4	3	2	4	4	4	3	3	3	3	4
$(3,2,1)$	4	3	2	4	4	4	3	4	3	3	4	3
$\{1,2,4\}$	4	4	4	4	4	4	4	4	4	4	4	4
$\{1,3,4\}$	4	4	4	4	4	4	4	4	4	4	4	4
$(4,2,3)$	4	4	2	3	3	3	4	3	4	4	3	4
$(2,3,4)$	4	2	3	4	3	4	3	3	4	3	4	4
$(3,4,2)$	4	3	4	2	3	3	3	4	4	4	4	3
$(2,4,3)$	4	2	4	3	4	3	4	4	3	4	3	3
$(4,3,2)$	4	4	3	2	4	4	4	3	3	3	3	4
$(3,2,4)$	4	3	2	4	4	4	3	4	3	3	4	3

TABLE 2. Nonconservative minimal majority functions on the four-element set

In the above examples, and actually for all known examples of minimal majority clones, the ternary part of the clone is isomorphic to the ternary part of a conservative clone. Thus we have only four examples for minimal majority clones up to isomorphism of the ternary part of the clones, so it is natural to ask if there are other examples at all. We investigate this question by describing minimal majority clones with few (at most seven) ternary operations. It turns out that if $\mathcal{C}$ is such a

clone, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$ or $[m_2]^{(3)}$. Let us note that the binary case is studied in [LP], where binary minimal clones with at most seven binary operations are determined.

3. Symmetries of minimal majority functions

For any abstract clone $\mathcal{C}$, the symmetric group S_n acts naturally on $\mathcal{C}^{(n)}$: applying a permutation $\pi \in S_n$ to $f \in \mathcal{C}^{(n)}$ we get

$$(3.1) \qquad f\left(e_{\pi(1)}^{(n)}, e_{\pi(2)}^{(n)}, \ldots, e_{\pi(n)}^{(n)}\right).$$

In the case of concrete clones this means that we permute the variables of f, and we will adopt this terminology to the abstract case, even though we cannot speak about variables here. If f is a nontrivial operation, then so are the operations of the form (3.1), hence S_n acts on $\mathcal{C}^{(3)} \setminus \mathcal{I}$, too. Let us denote by $\sigma(f)$ the stabilizer of f, i.e. the group of permutations leaving f invariant.

If f is a majority operation, then $\sigma(f)$ is a subgroup of S_3, therefore it has $1, 2, 3$ or 6 elements. If $\sigma(f) \supseteq A_3$, then we say that f is *cyclically symmetric*, and if $\sigma(f) = S_3$, then we say that f is *totally symmetric*.

If $\mathcal{C}$ is a majority clone with just one majority operation, then the majority rule and the clone axioms completely determine the structure of $\mathcal{C}^{(3)}$, and it is clear from Theorem 2.1 that in this case $\mathcal{C}$ is minimal. For example, $[m_1]$ is such a clone, so we have the following theorem.

Theorem 3.1. *If $\mathcal{C}$ is a minimal clone with one majority operation, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_1]^{(3)}$.*

If f is the unique majority operation in such a clone, then every nontrivial ternary superposition of f yields f itself. In particular, f is totally symmetric, and satisfies $f(f(x,y,z),y,z) = f(x,y,z)$. It is easy to check that this identity together with the total symmetry ensures that f does not generate any nontrivial ternary operation other than f. Thus the clones described in the above theorem are exactly the factor clones of the clone of the variety $\mathcal{M}_1$ defined by the following identities:

$$(3.2) \qquad f(x,y,z) = f(y,z,x) = f(y,x,z) = f(f(x,y,z),y,z), \ f(x,x,y) = x.$$

This variety has infinitely many subvarieties, therefore there are infinitely many nonisomorphic minimal clones with just one majority operation. To show this, we will construct a subdirectly irreducible (in fact, simple) algebra $\mathbb{A}_n \in \mathcal{M}_1$ of size n for every $n > 6$. Since $\mathcal{M}_1$ is congruence distributive, $\mathbb{A}_m \notin \mathrm{HSP}(\mathbb{A}_n)$ if $m > n$ by Jónsson's lemma, hence the subvarieties $\mathrm{HSP}(\mathbb{A}_n)$ are all different, and the clones $\mathrm{Clo}\,\mathbb{A}_n$ are pairwise nonisomorphic.

Example. Let $\mathbb{A}_n = (\{1, 2, \ldots, n\}; f)$, where f is a totally symmetric majority operation defined for $1 \le a < b < c \le n$ by

$$f(a,b,c) = \begin{cases} a & \text{if } \lceil \frac{a+c}{2} \rceil < b; \\ b & \text{if } b = \lfloor \frac{a+c}{2} \rfloor \text{ or } b = \lceil \frac{a+c}{2} \rceil; \\ c & \text{if } b < \lfloor \frac{a+c}{2} \rfloor. \end{cases}$$

Note that it suffices to define $f(a,b,c)$ for $a < b < c$ since f is a totally symmetric majority function. Let us consider the elements of $\mathbb{A}_n$ as points on the real line. We will call the points $\lfloor \frac{a+c}{2} \rfloor$ and $\lceil \frac{a+c}{2} \rceil$ the midpoints of the segment between a and c. (Segments of even length have one midpoint, but segments of odd length have two midpoints!) If $a < b < c$ and b is a midpoint of the segment between a and c, then $f(a,b,c) = b$, otherwise $f(a,b,c)$ is that endpoint of this segment which is farther from b.

It is easy to check that $\mathbb{A}_n \in \mathcal{M}_1$ (note that f is conservative), and we claim that $\mathbb{A}_n$ is simple if $n > 6$. Let us first observe that since f is a majority operation, any congruence class I has the following property: if at least two of a, b, c belong to I, then $f(a,b,c) \in I$. Let us call such subsets ideals of $\mathbb{A}_n$. If I is an ideal and $a, c \in I$,

then I contains the midpoints of the segment between a and c. Successively taking midpoints we can reach any point between a and c, therefore this whole segment belongs to I, i.e. ideals are convex.

Let ϑ be a nontrivial congruence of $\mathbb{A}_n$, and let a be the least element of $\mathbb{A}_n$ that belongs to a non-singleton block I of ϑ. Since a is the smallest element of I, which is a convex set with at least two elements, we must have $a + 1 \in I$. If $a \geq 4$, then $f(1, a, a+1) = 1$, and by the ideal property $f(1, a, a+1) \in I$. Now $2 \in I$ follows by convexity, and then $n = f(1, 2, n) \in I$ (here we need that $n \geq 5$). As both 1 and n belong to I, we have $I = \{1, 2, \ldots, n\}$, i.e. ϑ is the total relation on $\mathbb{A}_n$.

If $a + 1 \leq n - 3$, then a similar argument works: $n = f(a, a+1, n) \in I$, and then $1 = f(1, n-1, n) \in I$, therefore ϑ is the total relation again. The assumption $n > 6$ ensures that at least one of $a \geq 4$ and $a + 1 \leq n - 3$ holds, hence $\mathbb{A}_n$ is simple, as claimed.

Let $\mathcal{C}$ be a majority minimal clone. To simplify the notation we will just write 1, 2 and 3 for the first, second and third ternary projections respectively, and numbers greater than 3 will denote nontrivial elements of $\mathcal{C}^{(3)}$. Our next goal is to prove that if all majority functions in $\mathcal{C}$ are cyclically symmetric, then there is only one majority operation in the clone, i.e. $\mathcal{C}^{(3)} \cong [m_1]^{(3)}$. In preparation, we introduce three binary operations on the ternary part of $\mathcal{C}$.

$$f * g = f(g(1,2,3), g(2,3,1), g(3,1,2))$$
$$f \bullet g = f(g(1,2,3), 2, 3)$$
$$f \odot g = f(1, g(1,2,3), g(1,3,2))$$

Theorem 3.2. *The operations $*$, $\bullet$ and $\odot$ are associative, and if $\mathcal{C}$ is a majority clone, then $\mathcal{C}^{(3)} \setminus \mathcal{I}$ is closed under them. Therefore if $\mathcal{C}^{(3)}$ is finite, then it contains a nontrivial idempotent element for each of these operations.*

Proof. It is easy to check that if f and g are majority operations, then so are $f * g$, $f \bullet g$ and $f \odot g$, hence $\mathcal{C}^{(3)} \setminus \mathcal{I}$ is closed under these three operations. Associativity can be checked by a routine calculation using the three defining axioms of abstract clones. We work out the details for $\odot$, the other two cases are similar. Let us compute $(f \odot g) \odot h$ first:

$$\begin{aligned}(f \odot g) \odot h &= (f \odot g)(1, h(1,2,3), h(1,3,2)) \\ &= f(1, g(1, h(1,2,3), h(1,3,2)), g(1, h(1,3,2), h(1,2,3))).\end{aligned}$$

For $f \odot (g \odot h)$ we have

$$\begin{aligned}f \odot (g \odot h) &= f(1, (g \odot h)(1,2,3), (g \odot h)(1,3,2)) \\ &= f(1, g(1, h(1,2,3), h(1,3,2))(1,2,3), g(1, h(1,2,3), h(1,3,2))(1,3,2)) \\ &= f(1, g(1, h(1,2,3), h(1,3,2)), g(1, h(1,3,2), h(1,2,3))).\end{aligned}$$

The last statement of the theorem follows since every finite semigroup contains an idempotent element. Let us note that this fact is proved for the operation $\bullet$ in Lemma 4.4 of [HM] and for $*$ in Theorem 2.2 of [Wa]. $\qquad\square$

Now we are ready to prove the main result of this section. This theorem is an analogue of a theorem of J. Dudek and J. Gałuszka which states that if a binary minimal clone contains finitely many nontrivial binary operations all of which are commutative, then there is just one nontrivial binary operation in the clone [DG].

Theorem 3.3. *Let $\mathcal{C}$ be a majority minimal clone with finitely many ternary operations. If every nontrivial ternary operation in $\mathcal{C}$ is cyclically symmetric, then $\mathcal{C}$ contains only one nontrivial ternary operation, hence $\mathcal{C}^{(3)} \cong [m_1]^{(3)}$.*

Proof. Let $\mathcal{C}^{(3)} = \{1, 2, \ldots, n\}$, where $1, 2, 3$ are the ternary projections as before. First let us assume that there is no totally symmetric majority function in $\mathcal{C}$, i.e. $\sigma(f) = A_3$ for all $f \geq 4$. By Theorem 3.2 there is a nontrivial $\odot$-idempotent, say $4 \odot 4 = 4$. Since 4 is not invariant under the transposition (23), the element $4(1,3,2)$

is different from 4, thus we may suppose without loss of generality that $4(1,3,2) = 5$. We have $4(1,4,5) = 4 \odot 4 = 4$, hence $4(1,4,5) = 4(4,5,1) = 4(5,1,4) = 4$ because 4 is cyclically symmetric. We can compute $4(1,5,4)$ as well, using the associativity of composition:

$$4(1,5,4) = 4(1(1,3,2), 4(1,3,2), 5(1,3,2)) = 4(1,4,5)(1,3,2) = 4(1,3,2) = 5.$$

Thus we have $4(1,5,4) = 4(5,4,1) = 4(4,1,5) = 5$, therefore 4 preserves $\{1,4,5\}$, and its restriction to this set is isomorphic to m_3. However, m_3 generates majority operations that are not cyclically symmetric (see Table 1), and this contradicts our assumption that every nontrivial ternary operation of $\mathcal{C}$ is cyclically symmetric.

This contradiction shows that $\mathcal{C}$ must contain at least one totally symmetric majority function. If f and g are totally symmetric, then $f \bullet g$ is invariant under the transposition (23):

$$(f \bullet g)(1,3,2) = f(g(1,2,3),2,3)(1,3,2)$$
$$= f(g(1,3,2),3,2) = f(g(1,2,3),2,3) = f \bullet g.$$

Since $f \bullet g$ is nontrivial, it is also cyclically symmetric, hence $\sigma(f \bullet g) = S_3$. Thus totally symmetric majority functions form a finite semigroup under $\bullet$, so there is a totally symmetric $f \in \mathcal{C}^{(3)} \setminus \mathcal{I}$ with $f \bullet f = f$. Then f satisfies the identities in (3.2), hence $[f]^{(3)} \cong [m_1]^{(3)}$. By the minimality of $\mathcal{C}$ we have $[f] = \mathcal{C}$, and this proves the theorem. $\qquad \square$

Corollary 3.4. *If $\mathcal{C}$ is a majority minimal clone with $2 \leq |\mathcal{C}^{(3)} \setminus \mathcal{I}| < \aleph_0$, then the action of S_3 on $\mathcal{C}^{(3)} \setminus \mathcal{I}$ has an orbit with at least 3 elements.*

Proof. By the previous theorem there is a nontrivial operation $f \in \mathcal{C}^{(3)}$ which is not cyclically symmetric. Thus $\sigma(f)$ has at most 2 elements, and therefore the size of the orbit of f is $6/|\sigma(f)| \geq 3$. $\qquad \square$

4. Minimal clones with at most seven ternary operations

In this section we are going to prove the following characterization of majority minimal clones with at most seven ternary operations.

Theorem 4.1. *If $\mathcal{C}$ is a majority minimal clone with at most seven ternary operations, then $\mathcal{C}^{(3)}$ is isomorphic to either $[m_1]^{(3)}$ or $[m_2]^{(3)}$.*

Since there are three ternary projections, the clones under consideration contain $1, 2, 3$ or 4 majority operations. Theorem 3.1 describes the minimal clones with one majority operation, and from Corollary 3.4 we see immediately that there is no minimal clone with exactly two majority operations. We will deal with the cases of three and four majority operations in two separate lemmas.

Lemma 4.2. *If $\mathcal{C}$ is a minimal clone with three majority operations, then $\mathcal{C}^{(3)}$ is isomorphic to $[m_2]^{(3)}$.*

Proof. Let $\mathcal{C}$ be a minimal clone with three majority functions, and let $\mathcal{C}^{(3)} = \{1,2,3,4,5,6\}$, where $1,2,3$ are the ternary projections. Considering the orbits of the action of S_3 on $\{4,5,6\}$ we see by Corollary 3.4 that the only possibility is that there is just one orbit, i.e. any two nontrivial ternary operations can be obtained form each other by cyclic permutations of variables. We can suppose that $4(2,3,1) = 5$ and $5(2,3,1) = 6$ (and then $6(2,3,1) = 4$).

Any composition of majority operations is again a majority operation, therefore the set $\mathcal{C}^{(3)} \setminus \mathcal{I} = \{4,5,6\}$ is preserved by 4. This implies that every operation in $\mathcal{C}$ preserves $\{4,5,6\}$, since $\mathcal{C} = [4]$. Thus we have a clone homomorphism

$$\varphi : \mathcal{C} \to \mathcal{O}_{\{4,5,6\}}, \ f \mapsto f|_{\{4,5,6\}}.$$

We claim that φ is injective on $\{1,2,3,4,5,6\}$. Clearly it suffices to show that $\varphi(4) \neq \varphi(5) \neq \varphi(6) \neq \varphi(4)$. We prove the first inequality, the other two are

similar. Let us compute $5\,(4,5,6)$ using the associativity of composition:

$$5\,(4,5,6) = 4\,(2,3,1)\,(4,5,6) = 4\,(5,6,4)$$
$$= 4\,(4\,(2,3,1)\,,5\,(2,3,1)\,,6\,(2,3,1)) = 4\,(4,5,6)\,(2,3,1)\,.$$

Since $4\,(4,5,6) \in \{4,5,6\}$ and none of these three elements are invariant under the permutation (231), we have $5\,(4,5,6) = 4\,(4,5,6)\,(2,3,1) \neq 4\,(4,5,6)$. Thus $4|_{\{4,5,6\}} \neq 5|_{\{4,5,6\}}$ as claimed.

Now we see that $\mathcal{C}^{(3)}$ is isomorphic to its image under φ, which is the ternary part of a minimal clone on a three-element set. Therefore $\mathcal{C}^{(3)} \cong [m_i]^{(3)}$ for some $i \in \{1,2,3\}$. The cardinality of $\mathcal{C}^{(3)}$ is 6, so we must have $i = 2$, and the lemma is proved. $\square$

Remark. The previous lemma can be formulated in terms of algebras and varieties as follows. Let $\mathcal{M}_2$ be the variety defined by the three-variable identities satisfied by $(\{1,2,3\}\,;m_2)$. If f is a majority operation on a set A, then $[f]$ is a minimal clone with exactly three majority operations iff $(A;f)$ is term-equivalent to an element of $\mathcal{M}_2 \setminus \mathcal{M}_1$. Note that no two different subvarieties of $\mathcal{M}_2$ are term-equivalent, since for any $\mathbb{A} = (A;f) \in \mathcal{M}_2$ the basic operation f is the only nontrivial ternary function in $\mathrm{Clo}\,\mathbb{A}$ which is invariant under the transposition (23). This means that in order to show that there are infinitely many nonisomorphic minimal clones with three majority operations, it suffices to verify that the variety $\mathcal{M}_2$ has infinitely many subvarieties that are not contained in $\mathcal{M}_1$. If d_A is the dual discriminator function on a set A with at least three elements, then $(A;d_A\,(z,y,x)) \in \mathcal{M}_2 \setminus \mathcal{M}_1$, and by Jónsson's lemma we have $(B;d_B\,(z,y,x)) \notin \mathrm{HSP}\,(A;d_A\,(z,y,x))$ if A is finite and $|A| < |B|$. Thus the algebras $(A;d_A\,(z,y,x))$ with $A = \{1,2,\ldots,n\}$ and $n \geq 3$ generate pairwise different subvarieties of $\mathcal{M}_2$ that are not contained in $\mathcal{M}_1$.

Lemma 4.3. *There is no minimal clone with four majority operations.*

Proof. Let us suppose that $\mathcal{C}$ is a minimal clone with four majority functions, and let $\mathcal{C}^{(3)} = \{1,2,3,4,5,6,7\}$, with $1,2,3$ being the ternary projections. Corollary 3.4 shows that there are two orbits under the action of S_3 on $\{4,5,6,7\}$: a three-element and a one-element orbit. Thus one of the four nontrivial operations is totally symmetric, the other three operations have two-element invariance groups, and the latter three functions can be obtained from each other by cyclic permutations of their variables. We may assume without loss of generality that 7 is totally symmetric, and 4, 5 and 6 are invariant under the transpositions (23), (13) and (12) respectively. Then we must have $4\,(2,3,1) = 5$, $5\,(2,3,1) = 6$ and $6\,(2,3,1) = 4$.

Since any composition of majority operations is nontrivial, every operation in $\mathcal{C}$ preserves $\{4,5,6,7\}$. Restricting to this set, we obtain (the ternary part of) a minimal clone on a four-element set. The operation $7\,(4,5,6)$ is easily seen to be totally symmetric: applying a permutation to $7\,(4,5,6)$ will just permute $4,5$ and 6 in the arguments of 7, and this has no effect on the final value, as 7 is totally symmetric. Since the only totally symmetric operation in $\mathcal{C}^{(3)}$ is 7, we must have $7\,(4,5,6) = 7$. This means that the restriction of 7 to $\{4,5,6,7\}$ is a totally symmetric minimal majority operation that is not conservative. Now Theorem 2.3 implies that $7|_{\{4,5,6,7\}}$ is isomorphic to M_1, so $7\,(a,b,c) = 7$ for any pairwise distinct $a,b,c \in \{4,5,6,7\}$. Moreover, since M_1 does not generate any majority operation but itself, the operations $4,5,6,7$ coincide with each other on $\{4,5,6,7\}$:

(4.1) $f\,(a,b,c) = 7$ if $f,a,b,c \in \{4,5,6,7\}$ and a,b,c are pairwise distinct.

In particular, we have $6\,(6,4,5) = 7$, and taking into account that 4 and 5 are obtained from 6 by cyclic permutations of variables, this can be written as $6 * 6 = 7$.

In what follows, we will compute many more compositions until we get a contradiction by constructing a nontrivial ternary operation in $\mathcal{C}$ which is different from $4,5,6$ and 7.

The operation $7\,(1,2,7)$ is invariant under the transposition (12), hence it is either 6 or 7. The latter is impossible, since $7\,(1,2,7) = 7$ implies that 7 satisfies the identities

in (3.2), and then the clone generated by 7 would contain just one nontrivial ternary operation. Thus we have $7(1,2,7) = 6$, and by the total symmetry of 7 it follows that

$$(4.2) \qquad 7(1,2,7) = 7(7,1,2) = 7(2,7,1) = 6.$$

Let us now consider the values of 6 on $(1,2,7), (2,7,1), (7,1,2)$. We have $6(1,2,7) \in \{6,7\}$ since $6(1,2,7)$ is invariant under (12). Applying this transposition to $6(2,7,1)$ we obtain $6(7,1,2)$:

$$6(2,7,1)(2,1,3) = 6(1,7,2) = 6(7,1,2).$$

Therefore either both $6(2,7,1)$ and $6(7,1,2)$ are equal to 6 or 7, or one of them is 4, the other one is 5. The resulting eight possibilities are summarized in the following table.

(4.3)

$6(1,2,7)$	6	6	6	6	7	7	7	7
$6(2,7,1)$	7	6	4	5	7	6	4	5
$6(7,1,2)$	7	6	5	4	7	6	5	4
		↑				↑		

Let us consider any of the eight columns, and let a, b, c be the elements in this column. Then using the fact that $7 = 6 * 6$, we obtain

$$7(1,2,7) = 6(6(1,2,7), 6(2,7,1), 6(7,1,2)) = 6(a,b,c).$$

For the two columns marked by the arrows this gives $7(1,2,7) = 6$ by the majority rule. Similarly, for the first and the fifth column the majority rule yields $7(1,2,7) = 7$, and in the remaining four cases we get $7(1,2,7) = 7$ again, according to (4.1). However, we already know from (4.2) that $7(1,2,7) = 6$, so one of the two possibilities indicated by the arrows takes place. In both cases we have

$$(4.4) \qquad 6(2,7,1) = 6.$$

Now we go on to collect some information about the function 7. For the reader's convenience, we put the number of the equation being used over the equality sign in the following calculations.

First of all, using (4.2) and (4.4) we obtain

$$7(6,2,7) \overset{(4.2)}{=} 7(7,1,2)(2,7,1) \overset{(4.2)}{=} 6(2,7,1) \overset{(4.4)}{=} 6.$$

Permuting variables we get

$$(4.5a) \qquad 7(4,3,7) = 7(6,2,7)(2,3,1) = 6(2,3,1) = 4;$$
$$(4.5b) \qquad 7(5,3,7) = 7(6,2,7)(1,3,2) = 6(1,3,2) = 5.$$

We already know from (4.1) that $7(4,5,7) = 7$, and let us suppose for a moment that $7(4,5,3) = 7$. Then (4.5) shows that 7 preserves $\{3,4,5,7\}$, and its restriction to this four-element set is a totally symmetric nonconservative minimal majority function. Therefore it is isomorphic to M_1 by Theorem 2.3. However, this is clearly not the case. This contradiction shows that $7(4,5,3) \neq 7$. Let us observe that $7(4,5,3)(2,1,3) = 7(5,4,3) = 7(4,5,3)$, i.e. $7(4,5,3)$ is invariant under the transposition (12). Since 6 and 7 are the only nontrivial functions in our clone which are invariant under (12), we must have

$$(4.6) \qquad 7(4,5,3) = 6.$$

Next we calculate the value of $6(4,5,3)$:

$$(4.7) \qquad 6(4,5,3) \overset{(4.2)}{=} 7(1,2,7)(4,5,3) \overset{(4.6)}{=} 7(4,5,6) \overset{(4.1)}{=} 7.$$

Note that $6(3,4,5)(2,1,3) = 6(3,5,4) = 6(5,3,4)$, hence similarly to the previous table, we can list the possible behaviours of 6 on $\{(4,5,3), (5,3,4), (3,4,5)\}$.

(4.8)

$6(4,5,3)$	7	7	7	7
$6(5,3,4)$	7	6	5	4
$6(3,4,5)$	7	6	4	5
	↑			

We can read $7\,(4,5,3)$ from this table in the same way as we read $7\,(1,2,7)$ from (4.3). We see that $7\,(4,5,3) = 7$ in three of the four cases. However, we already know that $7\,(4,5,3) \stackrel{(4.6)}{=} 6$, so the only possibility is the one marked by the arrow.

Finally, to reach the desired contradiction, let us consider $6\,(2,3,6)$. Denoting this composition by f, we show that $f\,(4,5,3) = 5$:

$$f\,(4,5,3) = 6\,(2,3,6)\,(4,5,3) \stackrel{(4.7)}{=} 6\,(5,3,7) \stackrel{(4.2)}{=} 7\,(1,2,7)\,(5,3,7) \stackrel{(4.5b)}{=} 7\,(5,3,5) = 5.$$

The operation f is nontrivial, but it does not coincide with any of 4, 5, 6 or 7, because the value of these functions on $(4,5,3)$ is different from 5. Indeed, we have

$$
\begin{aligned}
4\,(4,5,3) &= 6\,(5,3,4) \stackrel{(4.8)}{=} 6; \\
5\,(4,5,3) &= 6\,(3,4,5) \stackrel{(4.8)}{=} 6; \\
6\,(4,5,3) &\stackrel{(4.7)}{=} 7; \\
7\,(4,5,3) &\stackrel{(4.6)}{=} 6.
\end{aligned}
$$

Thus we have more than four majority operations in our clone, and this contradiction completes the proof. $\square$

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