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MINIMAL CLONES WITH WEAKLY ABELIAN REPRESENTATIONS

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Dedicated to Béla Csákány on his seventieth birthday

Abstract. We show that a minimal clone has a nontrivial weakly abelian representation iff it has a nontrivial abelian representation, and that in this case all representations are weakly abelian.

1. Introduction

A *concrete clone* is a composition-closed collection of operations on some set containing all the projections. An *abstract clone* is a heterogeneous algebra equipped with operations which mimic the composition operations of concrete clones. (For the formal definition see [11] or [6].)

A *representation* of an abstract clone is a homomorphism into the concrete clone of operations on a given set. Usually one obtains a representation by picking a set of generators of the clone and assigning to each of them an operation of the same arity on a set in such a way that this assignment extends to a clone homomorphism. Thus each representation gives an algebra, and these algebras form a variety. (If we choose another set of generators, then we get another variety which is term-equivalent to the previous one.) Conversely, every variety arises in this way from the clone of term functions of the countably generated free algebra in the variety.

A clone is *minimal* if it has exactly two subclones: the clone itself and the clone which consists of projections only. The latter is called a *trivial clone*, and in this paper we will call an algebra trivial if the clone of its term functions is trivial (even if the algebra has more than one element!). Specially, a groupoid is trivial iff it is a left or right zero semigroup. A nontrivial representation of a minimal clone is also minimal, so if a variety has a minimal clone, then any nontrivial algebra in the variety has a minimal clone.

Let us now recall the definition of four variants of abelianness (cf.[2]). For an algebra $\mathbb{A}$ let $\mathcal{M}(\mathbb{A})$ denote the set of 2×2 matrices of the form $\begin{pmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{pmatrix}$ where t is a polynomial of $\mathbb{A}$ of arity $n+m$ and $\mathbf{a},\mathbf{b}\in A^n,\ \mathbf{c},\mathbf{d}\in A^m$.

Definition 1.1. We say that an algebra $\mathbb{A}$ is

1. *weakly abelian* if $\begin{pmatrix} u & u \\ u & v \end{pmatrix}\in\mathcal{M}(\mathbb{A})$ implies $u=v$;
2. *abelian* if $\begin{pmatrix} u & u \\ v & w \end{pmatrix}\in\mathcal{M}(\mathbb{A})$ implies $v=w$;
3. *rectangular* if $\begin{pmatrix} u & v \\ w & u \end{pmatrix}\in\mathcal{M}(\mathbb{A})$ implies $u=v=w$;
4. *strongly abelian* if it is both abelian and rectangular.

All of these properties are inherited by subalgebras and direct products, but not by homomorphic images. If $\mathbb{A}$ is a groupoid, and we apply (1) to $t(x,y)=xy$ then we get that whenever in the multiplication table of $\mathbb{A}$ we see a configuration like

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this:

$$\begin{array}{c|ccccc}
 & \cdots & c & \cdots & d & \cdots \\ \hline
\vdots & & \vdots & & \vdots & \\
a & \cdots & u & \cdots & u & \cdots \\
\vdots & & \vdots & & \vdots & \\
b & \cdots & u & \cdots & v & \cdots \\
\vdots & & \vdots & & \vdots &
\end{array}$$

then we must have $u = v$. Of course, this is just a necessary condition for $\mathbb{A}$ to be weakly abelian.

Minimal clones with abelian representations have been described by K. Kearnes in [1]. Here we examine the analogous question for the other three concepts. We will show that if a minimal clone has a nontrivial weakly abelian representation, then it also has a nontrivial abelian representation, and all representations are weakly abelian. From this result we will easily deduce that if a minimal clone has a nontrivial rectangular representation, then it also has a nontrivial strongly abelian representation; moreover, all representations are strongly abelian.

2. Preliminary results

Minimal clones are generated by any of their nontrivial elements and it is convenient to choose one of minimum arity. Such a generator must be one of five types according to the following theorem of Rosenberg [9] (see also [10]).

Theorem 2.1. ([9]). *Let f be a nontrivial operation of minimum arity in a minimal clone. Then f satisfies one of the following conditions:*

(I) *f is unary, and $f^2(x) = f(x)$, or $f^p(x) = x$ for some prime p;*
(II) *f is a binary idempotent operation, i.e. $f(x, x) = x$;*
(III) *f is a ternary majority operation, i.e. $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$;*
(IV) *$f(x, y, z) = x + y + z$ for an elementary abelian 2-group with addition $+$;*
(V) *f is a semiprojection, i.e. there exists an i $(1 \le i \le n)$ such that $f(x_1, x_2, \ldots, x_n) = x_i$ whenever the arguments are not pairwise distinct.*

A minimal clone cannot contain operations of two different types, therefore we can speak about five types of minimal clones. Any representation of a clone of type (I) is strongly abelian; any nontrivial representation of a clone of type (IV) is abelian, but not rectangular (hence not strongly abelian). A minimal clone of type (III) or (V) cannot have a nontrivial weakly abelian representation. This is shown in Theorem 3.1 in [1]. (This theorem is about abelian representations, but the proof actually shows that there is no weakly abelian representation either.) Thus we have to consider clones of type (II) only.

To recall the results of [1], we have to define several clones. By the clone of an *affine space* we mean the clone of all idempotent term functions of a vector space over some field. This clone is minimal iff the field is a p-element field for some prime number p. If $p > 2$, then this clone is of type (II): any nontrivial operation of the form $\lambda x + (1 - \lambda)y$ generates the clone. If $p = 2$ then the clone is of type (IV): the minority operation $x + y + z$ is a generator of minimum arity.

For any prime p, let us define the variety of *p-cyclic groupoids* by the identities $xx = x, x(yz) = xy, (xy)z = (xz)y, (\cdots((xy)y)\cdots)y = xy^p = x$. These groupoids have been introduced by Płonka [8]; he also proved that they have minimal clones [7]. *Rectangular bands* are idempotent semigroups satisfying $xyz = xz$, and they have minimal clones, too.

Now we can describe all minimal clones with a nontrivial abelian representation (Theorem 3.11 in [1]).

Theorem 2.2. ([1]). *The minimal clones which have a nontrivial abelian representation are the following:*

(i) *the unary clone generated by an operation f satisfying $f(x) = f(y)$, but not satisfying $f(x) = x$;*
(ii) *the unary clone generated by an operation f satisfying $f^2(x) = f(x)$, but not satisfying $f(x) = f(y)$ or $f(x) = x$;*
(iii) *the unary clone generated by an operation f satisfying $f^p(x) = x$ for some prime p, but not satisfying $f(x) = x$;*
(iv) *the clone of any nontrivial rectangular band;*
(v) *the clone of an affine space over a prime field;*
(vi) *the clone of any nontrivial p-cyclic groupoid (or its dual) for some prime p.*

The following interesting property of abelian representations has also been proved in [1] with the help of absorption identities (see also [6]).

Theorem 2.3. ([1]). *If a minimal clone has a nontrivial abelian representation, then this representation is faithful.*

As a special case of this theorem we have that if a variety $\mathcal{V}$ has a minimal clone and it contains a nontrivial rectangular band or affine space, then $\mathcal{V}$ must be the variety of rectangular bands or a variety of affine spaces. From the proof it is clear that the same is true for p-cyclic groupoids too, although not all of them are abelian, as we will see in the last section.

3. Weak abelianness and distributivity

In the theory of groupoids and quasigroups a different notion of 'weak abelianness' is defined by the identities

$$(*) \qquad (xx)(yz) = (xy)(xz), \qquad (yz)(xx) = (yx)(zx),$$

and a groupoid is called 'abelian' (or medial, or entropic) if $(xy)(zu) = (xz)(yu)$ holds (see [4]). To avoid confusion with the universal algebraic definitions, we will use the word *entropic* in the latter case. Minimal clones are always idempotent, and in this case the identities $(*)$ are equivalent to the *distributive identities*:

Left distributivity: $x(yz) = (xy)(xz)$,
Right distributivity: $(yz)x = (yx)(zx)$.

Any idempotent abelian groupoid is entropic ([1], Theorem 3.2), and one might expect that idempotent weakly abelian groupoids are distributive. We do not know if this is true or not, but for our present purposes the weaker properties stated in the next two lemmas are sufficient.

Lemma 3.1. *If $\mathbb{A}$ is an idempotent weakly abelian groupoid, then $uv_1 = uv_2 = w$ implies $u(v_1v_2) = w$, i.e. $\{v \mid uv = w\}$ is a subuniverse for any given $u, w \in \mathbb{A}$.*

Proof. Applying the definition of weak abelianness with $\mathbf{a} = (u, v_1, u), \mathbf{b} = (u, u, v_1)$, $\mathbf{c} = v_1, \mathbf{d} = v_2$ for $t(x_1, x_2, x_3, x_4) = (x_1x_2)(x_3x_4)$ we get

$$\begin{pmatrix} (uv_1)(uv_1) & (uv_1)(uv_2) \\ (uu)(v_1v_1) & (uu)(v_1v_2) \end{pmatrix} = \begin{pmatrix} ww & ww \\ uv_1 & u(v_1v_2) \end{pmatrix} = \begin{pmatrix} w & w \\ w & u(v_1v_2) \end{pmatrix} \in \mathcal{M}(\mathbb{A}),$$

hence $u(v_1v_2) = w$. □

Lemma 3.2. *Any idempotent weakly abelian groupoid satisfies the following identities:*

(i) $(xy)(xz) = (x(yz))((xy)(xz))$;
(ii) $(yx)(zx) = ((yx)(zx))((yz)x)$;
(iii) $(xy)x = x(yx)$.

Proof. Let $\mathbb{A}$ be an idempotent weakly abelian groupoid. To prove (i), we will use the 8-ary term $((\cdot\cdot)(\cdot\cdot))((\cdot\cdot)(\cdot\cdot))$; the underlined letters show the entries occupied by $\mathbf{c}$ and $\mathbf{d}$ in the definition. We have

$$\begin{pmatrix} ((xy)(x\underline{y}))((x\underline{x})(\underline{z}z)) & ((xy)(x\underline{z}))((x\underline{y})(\underline{x}z)) \\ ((xx)(y\underline{y}))((x\underline{x})(\underline{z}z)) & ((xx)(y\underline{z}))((x\underline{y})(\underline{x}z)) \end{pmatrix}$$
$$= \begin{pmatrix} (xy)(xz) & (xy)(xz) \\ (xy)(xz) & (x(yz))(xy)(xz) \end{pmatrix} \in \mathcal{M}(\mathbb{A}),$$

therefore the equality in (i) holds. Doing the same with the dual $\langle A, yx\rangle$ of $\mathbb{A} = \langle A, xy\rangle$, which is of course also weakly abelian, we obtain the second identity. We could derive the third identity in a similar manner, but it is easier to deduce it from the previous ones. If we put $z = x$ in (i) we get $(xy)x = (x(yx))((xy)x)$; writing $y = x$ and $z = y$ in (ii) yields $x(yx) = (x(yx))((xy)x)$; comparing them gives (iii). □

In light of the last identity we will sometimes omit the parentheses in a product of the form xyx. To make the connection between distributivity and weak abelianness more explicit, we will define a relation $\sim$ on our groupoid by $a \sim b$ iff $ab = a$. Identity (ii) says that $\mathbb{A}$ is right distributive 'modulo $\sim$'. This does not make perfect sense yet, since $\sim$ may not be an equivalence relation. Our strategy will be to reduce the problem to the case when $\sim$ is a congruence relation. As a preparation, we first show that assuming that the clone of $\mathbb{A}$ is minimal, we can conclude that $\mathbb{A}$ satisfies at least one-sided distributivity.

Lemma 3.3. *A weakly abelian groupoid with a minimal clone must satisfy at least one of the distributive laws.*

Proof. Suppose that $\mathbb{A}$ is a weakly abelian groupoid with a minimal clone, and $\mathbb{A}$ is neither left nor right distributive. First we will show that there is a two-element left zero semigroup in $\mathcal{V}(\mathbb{A})$. Since $\mathbb{A}$ is not right distributive, we can find elements x, y, z such that $b = (yz)x \neq (yx)(zx) = a$. The second identity of Lemma 3.2 shows that $ab = a$. If $ba = b$, then $\{a, b\}$ is a two-element left zero subsemigroup of $\mathbb{A}$. If $ba \neq b$, then let c denote the product ba, which is different from a by the weak abelian property (see the figure after Definition 1.1). We have $ab = aa = a$, so Lemma 3.1 yields that $a = a(ba) = ac$. With the help of identity (iii) of Lemma 3.2 we can compute $cb = (ba)b = b(ab) = ba = c$. Thus we have the following part in the multiplication table of $\mathbb{A}$.

	a	b	c
a	a	a	a
b	c	b	
c		c	c

If $bc = b$, then again we have a two-element left zero subsemigroup, $\{b, c\}$. Suppose therefore that $bc \neq b$. Then $x(xy)$ is a nontrivial operation, since $a(ab) = aa = a \neq b$ and $b(ba) = bc \neq b$. However, the operation $x(xy)$ is trivial on the set $\{a, c\}$. The only entry which we need to verify is $c(ca) = c$. We can get this equality by simply applying the definition of weak abelianness on the following matrix:

$$\begin{pmatrix} c(\underline{b}b) & c(\underline{c}b) \\ c(\underline{b}a) & c(\underline{c}a) \end{pmatrix} = \begin{pmatrix} c & c \\ c & c(ca) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Therefore any operation in the clone generated by $x(xy)$ is a first projection on $\{a, c\}$, and the original multiplication must be in this clone since it was supposed to generate a minimal clone. Thus we have $ca = c$, that is, $\{a, c\}$ is a two-element left zero subsemigroup.

Passing from $\mathbb{A}$ to its dual, which is not left or right distributive (since $\mathbb{A}$ itself is not right or left distributive) we see from the fact proved in the preceding paragraph that $\mathbb{A}$ also has a two-element right zero subsemigroup. The product of these two is a nontrivial rectangular band in $\mathcal{V}(\mathbb{A})$, therefore Theorem 2.3 implies that $\mathbb{A}$

itself is a rectangular band. This is a contradiction, since rectangular bands are distributive. □

With the help of Lemma 3.3 we will be able to handle all cases where $\sim$ is not a congruence relation, and finally we will arrive at the quotient groupoid $\mathbb{A}/\sim$, which will turn out to be distributive. This will be a rather lengthy argument, so we postpone it to the next section. Here we give the characterization of distributive groupoids with a minimal clone, which we will need to analyse $\mathbb{A}/\sim$. We will use the classification of entropic groupoids with a minimal clone (cf.[3]). To state this result, we need to define the following varieties.

An idempotent semigroup is called a *left normal band* if it satisfies the identity $xyz = xzy$; similarly *right normal bands* are those satisfying the identity $xyz = yxz$. The variety of *normal bands* is the join of these two varieties. A groupoid is called a *right semilattice* if it satisfies the identities $xx = x, x(yz) = xy, (xy)z = (xz)y$ and $(xy)y = xy$. The dual of a right semilattice is a *left semilattice*.

Now we can describe the entropic groupoids which have a minimal clone. (Note that the statement is slightly different from Theorem 3.20 in [3], because here we formulate the description in terms of concrete clones instead of abstract clones.)

Theorem 3.4. ([3]). *Let $\mathbb{A}$ be an entropic groupoid with a minimal clone. Then $\mathbb{A}$ or its dual is an affine space, a rectangular band, a left normal band, a right semilattice or a p-cyclic groupoid.*

Let us turn to the investigation of distributive groupoids with a minimal clone. It was shown in [5] that every distributive groupoid is trimedial, i.e. any subgroupoid generated by at most three elements is entropic. The next theorem shows that the distributive and entropic properties are equivalent for groupoids with a minimal clone.

Theorem 3.5. *If $\mathbb{A}$ is a distributive groupoid with a minimal clone, then the entropic law holds in $\mathbb{A}$.*

Proof. We know that all three-generated subgroupoids of $\mathbb{A}$ are entropic. If they are all trivial, then there must be a left and a right zero semigroup among them (since the clone of $\mathbb{A}$ is not trivial), and the product of these gives a nontrivial rectangular band in $\mathcal{V}(\mathbb{A})$. Applying Theorem 2.3, we get that $\mathbb{A}$ is a rectangular band. If there is a nontrivial 3-generated subalgebra which is an affine space, a rectangular band, or (the dual of) a p-cyclic groupoid, then again by Theorem 2.3 we have that $\mathbb{A}$ (or its dual) belongs to one of these varieties. Hence in all these cases $\mathbb{A}$ is entropic.

So we can assume that every three-generated subgroupoid of $\mathbb{A}$ is a left or right semilattice or a normal band. If there is a nontrivial right semilattice among them, then the term $x(xy)$ is the first projection on this subalgebra, hence by the minimality of the clone we have that $\mathbb{A} \models x(xy) = x$. This equation does not hold in a left semilattice or in a normal band, except for a left zero semigroup (which is a right semilattice). Thus we have that every 3-generated subalgebra is a right semilattice. This means that all identities involving at most three variables which hold in the variety of right semilattices also hold in $\mathbb{A}$. Since right semilattices are axiomatizable by three-variable identities, we conclude that $\mathbb{A}$ itself is a right semilattice.

The case of left semilattices is similar, so finally we can suppose that we have only normal bands as 3-generated subalgebras, i.e. that $\mathbb{A}$ satisfies all 3-variable identities that hold for normal bands. Associativity is such an identity, so our groupoid is a distributive semigroup, hence entropic (cf.[5], Proposition 2.3). □

Finally let us see which of the varieties mentioned in Theorem 3.4 contain nontrivial weakly abelian algebras.

Theorem 3.6. *If $\mathbb{A}$ is a weakly abelian entropic groupoid with a minimal clone, then $\mathbb{A}$ or its dual is a rectangular band, an affine space or a p-cyclic groupoid.*

Proof. By Theorem 3.4, we only need to show that $\mathbb{A}$ cannot be a left or right normal band, or left or right semilattice. A nontrivial semilattice is clearly not weakly abelian. In a nontrivial left normal band one can find elements a, b such that $a \neq ab$. It is easy to check that $\{a, ab\}$ is a two-element subsemilattice, contradicting weak abelianness. Similarly, a nontrivial right normal band cannot be weakly abelian either.

Finally, let us suppose that $\mathbb{A}$ is a right semilattice (the case of a left semilattice is similar). Considering the matrix

$$\begin{pmatrix} (x\underline{y})(yy) & (x\underline{x})(yy) \\ (x\underline{y})(xy) & (x\underline{x})(xy) \end{pmatrix} = \begin{pmatrix} xy & xy \\ xy & x \end{pmatrix} \in \mathcal{M}(\mathbb{A})$$

we see that $xy = x$ holds for all $x, y \in A$, and this contradicts the assumption that $\mathbb{A}$ has a minimal clone. □

4. Left distributive weakly abelian groupoids with minimal clones

Throughout this section $\mathbb{A}$ will denote a weakly abelian groupoid with a minimal clone. Lemma 3.3 shows that such a groupoid satisfies at least one of the distributive laws, so we will suppose that $\mathbb{A}$ is left distributive. We define a binary relation $\sim$ on $\mathbb{A}$ by $a \sim b$ iff $ab = a$. Clearly, this relation is reflexive. In a sequence of lemmas we will prove that if $\sim$ is not a congruence, then $\mathbb{A}$ is a p-cyclic groupoid for some prime p.

Lemma 4.1. *If $\sim$ is not symmetric, then $\mathbb{A} \models x(xy) = x$.*

Proof. Suppose that there are elements $a, b \in A$ such that $a \sim b$ but $b \not\sim a$, that is, $ab = a$ and $ba = c \neq b$. This situation is the same as in Lemma 3.3, and we will proceed similarly, but this time we go farther. Again, we have $c \neq a$ by the weak abelian property. Let $\mathbb{S}$ be the subgroupoid of $\mathbb{A}$ generated by a and b. According to Lemma 3.1 $\{x \mid ax = a\}$ is a subuniverse of $\mathbb{A}$, and it contains a and b. Therefore it contains $\mathbb{S}$, which implies that a is a left zero element in this subgroupoid. Moreover, $xy = a$ implies $x = a$ for $x, y \in S$. This can be seen in the multiplication table of $\mathbb{S}$ by weak abelianness.

$$\begin{array}{c|ccccc} & a & \cdots & x & \cdots & y \\ \hline a & a & \cdots & a & \cdots & a \\ \vdots & \vdots & & \vdots & & \vdots \\ x & * & \cdots & x & \cdots & a \end{array}$$

(Note that we have $xx = x$ by idempotence, and $*$ indicates xa, its value is irrelevant.)

Next we show that c is almost a left zero element in $\mathbb{S}$; more precisely, $cz = c$ for all $z \in S \setminus \{a\}$. Since z is in the subgroupoid generated by a and b, there is a binary term t such that $t(a, b) = z$. We prove $cz = c$ by induction on the length of t. If this length is zero, then either $t(x, y) = x$ or $t(x, y) = y$. The former is impossible because $z \neq a$. In the latter case we have $cb = (ba)b = b(ab) = ba = c$. Now for the induction step suppose that $z = t(a, b) = uv$ with $u = t_1(a, b), v = t_2(a, b)$. Again, $u \neq a$ follows from $z \neq a$ and therefore $cu = c$ by the induction hypothesis. If v is also different from a, then $cv = c$, so $cz = c(uv) = c$ by Lemma 3.1. If $v = a$, then we have to prove $c(ua) = c$. Let us consider the matrix

$$\begin{pmatrix} c(b\underline{b}) & c(b\underline{a}) \\ c(u\underline{b}) & c(u\underline{a}) \end{pmatrix} = \begin{pmatrix} cb & cc \\ c(ub) & c(ua) \end{pmatrix} = \begin{pmatrix} c & c \\ c(ub) & c(ua) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

We know that $cu = cb = c$, therefore $c(ub) = c$ as before. Therefore our matrix is of the form $\left(\begin{smallmatrix} c & c \\ c & c(ua) \end{smallmatrix}\right)$, hence $cz = c(ua) = c$ by weak abelianness.

What we just proved means that in the multiplication table of the subgroupoid $\mathbf{S}$, the row of c is constant c except for ca which may be different. In the same way as we proved that $xy = a$ implies $x = a$, we can show that $xy = c$ implies $x = c$ or $y = a$, that is, c can appear only in its own row and in the column of a.

The knowledge we gathered about the multiplication table is enough to see that the operation $x(xy)$ preserves $S \setminus \{c\}$. Indeed, if $x(xy) = c$ for some $x, y \in S$, then either $x = c$ or $xy = a$. The latter is impossible since it would force $x = a$, but then $x(xy) = a \neq c$. However, the original multiplication does not preserve this set because $ab = c$. Therefore by the minimality of the clone, $x(xy)$ must be a projection. Since $a(ab) = a \neq b$, it can only be the first projection, i.e. the identity $x(xy) = x$ holds in $\mathbb{A}$. □

Lemma 4.2. *If $\sim$ is symmetric but not transitive, then $\mathbb{A} \models x(xy) = x$.*

Proof. Suppose that there are elements $a, b, c \in A$ such that $a \sim b \sim c$ but $a \not\sim c$. Then a, b, c must be pairwise different, because $\sim$ is reflexive by the idempotence of $\mathbb{A}$. A part of the multiplication table looks like this:

	a	b	c
a	a	a	
b	b	b	b
c		c	c

It is easy to check that we have the same in the multiplication table of $x(xy)$. But for this operation we can compute the missing two entries, too, with the help of the left distributive identity:

$$a(ac) = (ab)(ac) = a(bc) = ab = a,$$
$$c(ca) = (cb)(ca) = c(ba) = cb = c.$$

Thus we see that $x(xy)$ is the first projection on the set $\{a, b, c\}$, but the original operation xy is not, because $a \not\sim c$ implies $ac \neq a$. Therefore, by the minimality of the clone of $\mathbb{A}$, $x(xy)$ must be a trivial operation, hence $\mathbb{A}$ satisfies $x(xy) = x$. □

To finish the investigation of the cases where $\sim$ is not an equivalence relation, we will show that a weakly abelian groupoid with a minimal clone satisfying $x(xy) = x$ must be a p-cyclic groupoid. This will be the consequence of the following lemma, where we do not assume weak abelianness.

Lemma 4.3. *Let $\mathbb{A}$ be a groupoid with a minimal clone such that $\mathbb{A}$ satisfies the identity $x(yz) = xy$. Then either $\mathbb{A}$ is a p-cyclic groupoid, or the identity $(xy)y = xy$ holds in $\mathbb{A}$.*

Proof. Suppose that t_1, t_2 are two terms, and the leftmost variable of t_2 is x. Then it can be shown easily by induction on the length of t_2, that the identity $t_1t_2 = t_1x$ holds in $\mathbb{A}$. This means that any term t of $\mathbb{A}$ can be reduced to a left-associated product: $t = (\cdots((xy_1)y_2)\cdots)y_n$. Let us now compute what happens if we multiply a term with its leftmost variable: $tx = t\underline{t} = t$ because the leftmost variable of the underlined t is also x. Thus we have the same situation as in Claim 3.9 of [3], except that the order of the variables $y_1, \ldots, y_n$ is not irrelevant. However, when we compute binary terms, we do not have to permute them, so every binary term is of the form xy^k, and we can proceed as in [3] to show that either $(xy)y = xy$ or $xy^p = x$ holds for some prime number p. In the first case we are done, so let us suppose that the latter holds. One can check that the term $t(x, y, z) = (((xy^{p-1})z)y)z^{p-1}$ satisfies the identities $t(x, x, z) = t(x, y, x) = t(x, y, y) = x$, i.e., it is a first semiprojection. Therefore t does not generate any nontrivial binary operation, so it must be trivial: $t(x, y, z) = x$. Substituting xy for x in this equality and multiplying both sides on the right with z we get the identity $t(xy, y, z)z = (xy)z$. Computing the left hand side we obtain the identity $(xz)y = (xy)z$. Thus all the defining identities of the variety of p-cyclic groupoids hold in $\mathbb{A}$. □

Remark. One might think that in the case $\mathbb{A} \models (xy)y = xy$ we can conclude that $\mathbb{A}$ is a right semilattice, but this is not true. The variety defined by the identities

$xx = x, x(yz) = xy, (xy)y = xy$ has a minimal clone. Indeed, any nontrivial term can be written in the form $t = (\cdots((xy_1)y_2)\cdots)y_n$, and identifying all the y_is we get $xy^n = xy$. However, these identities do not imply $(xy)z = (xz)y$, so the variety of right semilattices is a proper subvariety of the above variety.

Lemma 4.4. *If $\mathbb{A}$ is a weakly abelian groupoid with a minimal clone that satisfies the identity $x(xy) = x$, then $\mathbb{A}$ is a p-cyclic groupoid.*

Proof. We show that weak abelianness and the identity $x(xy) = x$ imply the stronger identity $x(yz) = xy$. Let $t = t(x, y, z) = x(yz)$, and compute the following matrix:

$$\begin{pmatrix} t(t\underline{z}) & t(t\underline{y}) \\ x(y\underline{z}) & x(y\underline{y}) \end{pmatrix} = \begin{pmatrix} t & t \\ t & xy \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Thus we have $x(yz) = xy$ and we can apply the preceding lemma. The only thing we need to show is that the identity $(xy)y = xy$ cannot hold. We can proceed the same way as we did at the end of the proof of Theorem 3.6 to see that $(xy)y = xy$ would imply $xy = x$. $\square$

So far we have proved that if $\sim$ is not an equivalence relation, then $\mathbb{A}$ is a p-cyclic groupoid. From now on we will assume that $\sim$ is an equivalence relation, and we will force it to be a congruence of $\mathbb{A}$. Using the left distributive identity we can show that $\sim$ is not very far from being a congruence.

Lemma 4.5. *For any $a, b, c \in \mathbb{A}$, if $a \sim b$ then the following relations are true:*

(i) $ca \sim cb$,
(ii) $(ac)(bc) \sim ac$.

Proof. To prove (i) we simply apply the left distributive law: $(ca)(cb) = c(ab) = ca$. For (ii) we substitute $x = c, y = a, z = b$ in the identity $(yx)(zx) = ((yx)(zx))((yz)x)$, which holds in $\mathbb{A}$ by Lemma 3.2. We get $(ac)(bc) = ((ac)(bc))((ab)c) = ((ac)(bc))(ac)$ which is just what we had to prove. $\square$

It would be nice if we had $ac \sim bc$ in (ii), because then $\sim$ would be a congruence. With the next lemma we finish the investigation of the case where $\sim$ is not a congruence.

Lemma 4.6. *If $\sim$ is not a congruence relation, then $\mathbb{A}$ is a p-cyclic groupoid.*

Proof. We prove first that for any $a, b, c \in \mathbb{A}$, if $a \sim b$ then the subalgebra generated by ac and bc satisfies the identity $x(xy) = x$. The second part of the previous lemma shows that $uv \sim u$ holds for $u, v \in S = \{ac, bc\}$. Next we show that this property is inherited when we pass from S to the subgroupoid generated by S. This can be done using the following two rules:

$$(uw \sim u, uv \sim u) \Rightarrow (uv)w \sim uv,$$
$$(wu \sim w, wv \sim w) \Rightarrow w(uv) \sim w.$$

To check the first one, we calculate $u((uv)w) = (u(uv))(uw) = u(uw) = u$, which shows that $u \sim (uv)w$. We have assumed $u \sim uv$ therefore by transitivity and symmetry $(uv)w \sim uv$ follows. The second one is easier: $w(w(uv)) = w((wu)(wv)) = (w(wu))(w(wv)) = ww = w$. With these rules one can show by induction on the length of terms that $uv \sim u$ for all u, v in the subgroupoid generated by S. Hence this subgroupoid satisfies the identity $x(xy) = x$.

If $\sim$ is not a congruence, then we can find elements a, b, c such that $a \sim b$ but $ac \not\sim bc$, that is, $(ac)(bc) \neq (ac)$. If $(ac)(bc) = bc$, then by the second part of Lemma 4.5 we would have $bc \sim ac$, which is impossible since $ac \not\sim bc$. Thus the subalgebra generated by $\{ac, bc\}$ is not trivial. Then it has a minimal clone; it is weakly abelian, and satisfies $x(xy) = x$, therefore by Lemma 4.4 it is a nontrivial p-cyclic groupoid in $\mathcal{V}(\mathbb{A})$. With the help of Theorem 2.3 we conclude that $\mathcal{V}(\mathbb{A})$ is the variety of p-cyclic groupoids. $\square$

Let us summarize what we have proved so far in this section.

Theorem 4.7. *If $\mathbb{A}$ is a weakly abelian left distributive groupoid with a minimal clone such that the relation $\sim$ defined by $a \sim b \Leftrightarrow ab = a$ is not a congruence, then $\mathbb{A}$ is a p-cyclic groupoid for some prime p.*

So finally we can suppose that $\mathbb{A}$ is a left distributive weakly abelian groupoid with a minimal clone, and $\sim$ is a congruence of $\mathbb{A}$. The corresponding factor groupoid $\mathbb{A}/\sim$ is distributive; right distributivity follows, because $\mathbb{A}$ satisfies identity (ii) from Lemma 3.2. Furthermore, $\mathbb{A}/\sim$ has a minimal or trivial clone. Therefore it is entropic by Theorem 3.5, and it must have at least two elements, since $\mathbb{A}$ is not trivial. Using the list of entropic groupoids with a minimal clone, we will prove that $\mathbb{A}$ is also entropic. The key observation is that by the definition of $\sim$ we have

$$\mathbb{A}/\sim \models t_1 = t_2 \Leftrightarrow \mathbb{A} \models t_1 t_2 = t_1.$$

Lemma 4.8. *If $\mathbb{A}/\sim$ has a two-element left or right zero subsemigroup then $\mathbb{A}$ is entropic. It is impossible to have a two-element semilattice among the subgroupoids of $\mathbb{A}/\sim$.*

Proof. First let us suppose that $X, Y \in \mathbb{A}/\sim$ form a left zero semigroup. Then for any $x, y \in X \cup Y$ we have $xy \sim x$. Therefore $x(xy) = x$ holds in $X \cup Y$, which is a nontrivial subgroupoid of $\mathbb{A}$, since X and Y are two different congruence classes. By Lemma 4.4 this subgroupoid must be p-cyclic, and by the minimality of the clone of $\mathcal{V}(\mathbb{A})$, Theorem 2.3 implies that $\mathbb{A}$ itself must also be a p-cyclic groupoid.

Now suppose that $X, Y \in \mathbb{A}/\sim$ form a right zero semigroup. Again, $X \cup Y$ is a subgroupoid of $\mathbb{A}$, and $t_1 t_2 = t_1$ holds in this subalgebra whenever the rightmost variables of t_1 and t_2 are the same (i.e., when $t_1 = t_2$ holds in right zero semigroups). Using this fact and the weak abelian property, we can compute $x(yz)$ for $x, y, z \in X \cup Y$ as follows:

$$\begin{pmatrix} ((xy)\underline{y})z & ((xy)\underline{z})z \\ ((xx)\underline{y})z & ((xx)\underline{z})z \end{pmatrix} = \begin{pmatrix} (xy)z & (xy)z \\ (xy)z & xz \end{pmatrix} \in \mathcal{M}(\mathbb{A}),$$

therefore the identity $(xy)z = xz$ holds in $X \cup Y$. Similarly, $X \cup Y \models x(yz) = xz$ can be shown by considering the following matrix:

$$\begin{pmatrix} (xz)(\underline{z}z) & (xz)(\underline{y}z) \\ (xx)(\underline{z}z) & (xx)(\underline{y}z) \end{pmatrix} = \begin{pmatrix} xz & xz \\ xz & x(yz) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Thus $X \cup Y$ is a rectangular band, and if it is nontrivial, then $\mathbb{A}$ is also a rectangular band by Theorem 2.3, so we are done. If $X \cup Y$ is trivial, then X and Y must be singletons, because X and Y are left zero subsemigroups. Therefore $X \cup Y$ is a right zero subsemigroup in $\mathbb{A}$. Forming the direct product of this with any non-singleton congruence class we get a nontrivial rectangular band in $\mathcal{V}(\mathbb{A})$, so $\mathbb{A}$ is also a rectangular band by Theorem 2.3. If all the $\sim$-blocks of $\mathbb{A}$ are singletons, then $\mathbb{A} = \mathbb{A}/\sim$ is distributive, hence entropic by Theorem 3.5.

Finally, let us suppose that $X, Y \in \mathbb{A}/\sim$ form a semilattice. Then $X \cup Y$ satisfies every equation of the form $t_1 t_2 = t_1$ where $t_1 = t_2$ is valid in every semilattice. Combining this with identity (iii) from Lemma 3.2 allows us to conclude that the identities

$$(xy)y = ((xy)y)(xy) = (xy)(y(xy)) = xy,$$
$$(xy)x = ((xy)x)(xy) = (xy)(x(xy)) = xy$$

hold in $X \cup Y$. Using these identities we can compute the following matrix:

$$\begin{pmatrix} (xy)\underline{y} & (xy)\underline{x} \\ (xx)\underline{y} & (xx)\underline{x} \end{pmatrix} = \begin{pmatrix} xy & xy \\ xy & x \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

Thus $X \cup Y$ is a left zero semigroup, contradicting the fact that X and Y are two different congruence classes. □

Theorem 4.9. *If $\sim$ is a congruence relation of $\mathbb{A}$, then $\mathbb{A}$ is entropic.*

Proof. There are at least two $\sim$-classes, since otherwise $\mathbb{A}$ would be a left zero semigroup. So $\mathbb{A}/\sim$ has at least two elements, and if it is trivial, then we can apply the previous lemma. If this is not the case, then $\mathbb{A}/\sim$ must belong to one of the varieties which have entropic minimal clones. In the case of affine spaces, rectangular bands and p-cyclic groupoids Theorem 2.3 shows that $\mathbb{A}$ also belongs to one of these varieties. As we have seen in the proof of Theorem 3.6, a nontrivial left or right normal band always contains a two-element subsemilattice, but Lemma 4.8 shows that this is impossible for $\mathbb{A}/\sim$. Finally, let us assume that $\mathbb{A}/\sim$ is a nontrivial right semilattice. Then it contains elements a, b such that $a \neq ab$. Using the defining identities of the variety of right semilattices, one can check that a and ab form a two-element left zero subsemigroup in $\mathbb{A}/\sim$, so we can apply Lemma 4.8 again. Similarly, a nontrivial left semilattice must contain a two-element right zero subsemigroup, so Lemma 4.8 applies in this case, too. □

Putting together Theorems 4.7 and 4.9 with Theorem 3.6 we get the main result of this section.

Theorem 4.10. *A left distributive weakly abelian groupoid with a minimal clone is either a rectangular band, an affine space or (the dual of) a p-cyclic groupoid for some prime p.*

5. Summary

We have seen that only minimal clones of types (I), (II) and (IV) can have nontrivial weakly abelian representations, and in case of types (I) and (IV) all representations are abelian. A weakly abelian groupoid with a minimal clone is left or right distributive by Lemma 3.3, thus we can apply Theorem 4.10 (after dualizing if necessary) to see that such a groupoid must be a rectangular band, an affine space or (the dual of) a p-cyclic groupoid. This list does not contain any new items compared to Theorem 2.2.

Theorem 5.1. *If a minimal clone has a nontrivial weakly abelian representation, then it also has a nontrivial abelian representation. Therefore such a clone must be a unary clone, the clone of an affine space, a rectangular band or (the dual of) a p-cyclic groupoid for some prime p.*

Unary algebras, rectangular bands and affine spaces are abelian. A p-cyclic groupoid must be weakly abelian, as we shall see in the following lemma.

Lemma 5.2. *Every p-cyclic groupoid is weakly abelian.*

Proof. Suppose that $\mathbb{A}$ is a p-cyclic groupoid for some prime number p. (Actually, we will not need the fact that p is prime.) Let t be a term of $\mathbb{A}$, with arity $n+m$, and let $\mathbf{a}, \mathbf{b} \in A^n$, $\mathbf{c}, \mathbf{d} \in A^m$ be such that the matrix $\begin{pmatrix} t(\mathbf{a},\mathbf{c}) & t(\mathbf{a},\mathbf{d}) \\ t(\mathbf{b},\mathbf{c}) & t(\mathbf{b},\mathbf{d}) \end{pmatrix}$ is of the form $\begin{pmatrix} u & u \\ u & v \end{pmatrix}$. As we have seen in the proof of Lemma 4.3, every term of $\mathbb{A}$ can be reduced to a left-associated product, so we may assume that t is of the form $t = (\cdots((x_1x_2)x_3)\cdots)x_{n+m}$. Transposing our matrix if necessary, we can suppose that the leftmost variable is occupied by entries belonging to $\mathbf{a}$ and $\mathbf{b}$, say a_1 and b_1. Using the identity $(xy)z = (xz)y$ we can permute the other variables, so that the entries in the first column of the matrix are: $t(\mathbf{a}, \mathbf{c}) = a_1a_2\cdots a_nc_1c_2\cdots c_m$, and $t(\mathbf{b}, \mathbf{c}) = b_1b_2\cdots b_nc_1c_2\cdots c_m$. (Both products are left-associated, we have omitted the parentheses.) Our groupoid is right cancellative, since multiplication by any element on the right is a permutation of order p. Therefore the equation $t(\mathbf{a}, \mathbf{c}) = t(\mathbf{b}, \mathbf{c})$ implies that $a_1a_2\cdots a_n = b_1b_2\cdots b_n$. Multiplying both sides on the right with $d_1, d_2, \cdots, d_m$, we conclude that $t(\mathbf{a}, \mathbf{d}) = t(\mathbf{b}, \mathbf{d})$, that is $u = v$, so $\mathbb{A}$ is weakly abelian. □

Theorem 5.3. *If a minimal clone has a nontrivial weakly abelian representation, then all representations are weakly abelian.*

As the following example shows, there exist nonabelian p-cyclic groupoids. Therefore the two abelianness concepts differ already for groupoids with minimal clones.

Example. For any prime number p let us define the following binary operation on the set $\mathbb{Z}_p \times \{0,1\}$:

$$(a,b) \circ (c,d) = \begin{cases} (a+1,b) & \text{if } b=0 \text{ and } d=1; \\ (a,b) & \text{otherwise.} \end{cases}$$

The algebra $\mathbb{A} = (\mathbb{Z}_p \times \{0,1\}, \circ)$ is a p-cyclic groupoid, therefore it is weakly abelian and has a minimal clone. It is not abelian, as we can see from the following matrix.

$$\begin{pmatrix} (0,1)\circ(0,0) & (0,1)\circ(0,1) \\ (0,0)\circ(0,0) & (0,0)\circ(0,1) \end{pmatrix} = \begin{pmatrix} (0,1) & (0,1) \\ (0,0) & (1,0) \end{pmatrix} \in \mathcal{M}(\mathbb{A}).$$

We conclude with a remark on rectangularity and strong abelianness. A nontrivial affine space or p-cyclic groupoid cannot be rectangular, but unary algebras and rectangular bands are all strongly abelian. Thus these two concepts coincide for concrete minimal clones.

Theorem 5.4. *If a minimal clone has a nontrivial rectangular representation, then it also has a nontrivial strongly abelian representation; moreover, all representations are strongly abelian. Such a clone must be unary, or the clone of rectangular bands.*

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