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ASSOCIATIVE SPECTRA OF BINARY OPERATIONS

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To Ivo Rosenberg on his 65th birthday

Abstract. The distance of a binary operation from being associative can be "measured" by its *associative spectrum*, an appropriate sequence of positive integers. Particular instances and general properties of associative spectra are studied.

1. Introduction

Let n be a positive integer. We call a string consisting of symbols x, (, and) a *bracketing of size* n if it contains n symbols "x", and $n-1$ symbols "(" (left parentheses) as well as ")" (right parentheses) so that they are properly placed to determine a product of n factors x (see, *e.g.* [1,15]). More formally,

1. x is the unique bracketing of size 1,
2. the bracketings of size n are exactly the strings of form (PQ) where P and Q are bracketings of size k resp. l with $k+l=n$.

E.g. (xx) is the only bracketing of size 2, and $((x(xx))(xx))$ is a bracketing of size 5. Note that we always use an outermost pair of parentheses whenever $n>1$, in contrary to the everyday usage of parentheses. We shall denote bracketings by capital letters, and $|B|$ stands for the size of B.

Bracketings are, in fact, the elements of the free groupoid[1] with one free generator x (*cf.* [1], p. 133), or, equivalently, they are the unary groupoid terms. The corresponding unary term operations on special groupoids were investigated by several authors (see, *e.g.* [5,7]). In any bracketing of size n we can indicate the position of symbols x by subscripts $1,\dots,n$, *e.g.* $(x_1x_2), ((x_1(x_2x_3))(x_4x_5))$. Thus, a bracketing of size n provides also an element of the free groupoid with free generators $x_1,\dots,x_n$, *i.e.*, an n-ary groupoid term (although, of course, not all n-ary groupoid terms originate from bracketings in such a way). Here we always study bracketings considered as n-ary groupoid terms, even if in some cases we omit the subscripts $1,\dots,n$. On every groupoid G, these terms give rise to n-ary term operations. We call them *regular n-ary operations of G* (or, *regular over the operation of G*), and, in concrete cases, *operations induced by given bracketings.* For notation of the regular operation induced by the bracketing B,P,Q, *etc.* we use the corresponding lowercase letters b,p,q, *etc.*

If G is associative, then by the generalized associative law there is exactly one regular n-ary operation for each n. In the general case, we have a sequence

$$s_G(1), s_G(2), \dots, s_G(n), \dots$$

of positive integers with $s_G(n)$ denoting the number of distinct n-ary regular operations of G. *E.g.*, $s_G(1)=s_G(2)=1$ for every groupoid G, and $s_G(3)=2$ if and only if G is nonassociative, as then the two possible bracketings of size 3, $(x_1(x_2x_3))$ and $((x_1x_2)x_3)$ induce different ternary term operations.

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[1]Here by a groupoid we mean a nonempty set with one binary operation.

The sequence

$$\{s_G(n)\} = (s_G(1), s_G(2), \ldots, s_G(n), \ldots)$$

measures, in some sense, the distance of G from associativity: the smaller its entries are, the closer the operation of G is to being associative. Hence we call this sequence *the associative spectrum of* G (or, *of the operation of* G). Instead of $s_G(n)$ we write $s(n)$ if this cannot cause misunderstanding. Usually we also omit $s(1)$ and $s(2)$, bearing no information about G.

In this paper we study the introduced notion from several points of view. The next section contains some well-known facts, simple observations, and auxiliary results on bracketings and associative spectra; there and later, the routine inductive proofs will often be omitted. Most frequently we use induction on size; we leave out the words "on size" in these cases. The third section contains samples of determining associative spectra of some familiar nonassociative operations. The problem of characterizing all associative spectra of operations on a set with a given power seems to be hard. However, the case of the two-element set is, as it might be expected, easy (Section 4), and a lot of three-element groupoids are accessible (Section 5). In the final section we present some facts on the general behavior of associative spectra, and formulate several problems.

Further on, we write simply *spectrum* for associative spectrum.

2. Properties of bracketings and spectra

For any bracketing B of size $n(>1)$, we can *pair* its left and right parentheses in a natural way ([9,15]). Induction shows that we can always choose a consecutive quadruple (xx) in B; its left and right parentheses will be associated to form a pair. Replacing then (xx) with x we obtain a bracketing B' of size $n-1$, for which the preceding process can be repeated until no unpaired parentheses remain. This way of forming pairs involves that any pair together with the symbols between them is also a bracketing. It is called a *subbracketing of* B; *e.g.*, if $B = (PQ)$, then P and Q are subbracketings of B, as outermost parentheses of any bracketing are paired. We call P and Q the (*left* resp. *right*) *factors* of B. The symbols x are considered as subbracketings of size 1, too. Observe that pairing is unique, and if a parenthesis lies between a pair then its associate also lies between them. Hence the representation of bracketings of size >1 in form (PQ) is unique, too.

Substituting x for one or several disjoint subbracketings in B we obtain *a quotient bracketing* of B. *E.g.* $(x(xx))$ and $((xx)(xx))$ are (disjoint) subbracketings of $B = (((x(xx))x)((xx)(xx)))$, and replacing them with x provides the quotient bracketing $((xx)x)$ of B. A bracketing is a *nest* if it is either of size 1 (*a trivial nest*) or one out of its factors is x, and the other one is a nest ([5,7]). *E.g.*, all bracketings of size 4 save $(xx)(xx)$ are nests. Given a bracketing B, there are subbracketings of B which are nests; in particular, each x_i is contained in a unique maximal nest. We call these maximal nests simply *the nests of* B. A nontrivial nest has a unique subbracketing of form $(x_i x_{i+1})$; we say that x_i, x_{i+1} are *the eggs of the nest*.

The *Catalan numbers* C_n are defined recursively by

(1) $C_0 = 1$,

(2) $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-2}C_1 + C_{n-1}C_0 \quad (n > 0)$,

or, equivalently, by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Compare (1) and (2) with the formal definition of bracketings in the introduction, and take into account the unicity of the representation of bracketings in form (PQ). Then the following standard result follows:

2.1. *The number of bracketings of size n equals C_{n-1}* (see, *e.g.* [8]).

Hence we infer:

2.2. *For any spectrum* $\{s(n)\}$,

$$1 \le s(n) \le C_{n-1}$$

holds for every n.

If $s_G(n) = C_{n-1}$ for every n, then the groupoid G and its operation are said to be *Catalan*. *E.g.*, free groupoids are Catalan. Another inequality also follows from the definition of bracketings:

2.3. *For any spectrum* $\{s(n)\}$,

$$s(n) \le s(1)s(n-1) + s(2)s(n-2) + \cdots + s(n-2)s(2) + s(n-1)s(1)$$

holds for every $n(\ge 2)$.

Hence if $s_G(n_0) < C_{n_0-1}$ then $s_G(n) < C_{n-1}$ for every $n > n_0$. The following trivial observations are useful, too:

2.4. *If the groupoids* G *and* H *are isomorphic or antiisomorphic, then their spectra coincide.*

2.5. *If the groupoid* H *is a subgroupoid or a factorgroupoid of* G, *then*

$$s_H(n) \le s_G(n)$$

holds for every n.

By 2.5, in order to show that G is Catalan it is sufficient to find a Catalan subgroupoid or factorgroupoid of G. The next fact goes back to Łukasiewicz (for a proof, see [3], Ch. 3.2, or [11], Exercise 1.38):

2.6. *Bracketings are uniquely determined by the places of their right (or left) parentheses between the symbols* $x_1, \dots, x_n$.

Next we introduce sequences of nonnegative integers which arise naturally from bracketings, and also contain full information on them. Consider the free monoid F_2 with unit element e, generated by symbols 0 and 1. A subset M of F_2 is *prefix-free* if no word in M is a prefix (*i.e.*, a left segment) of another word in M. There exist finite maximal prefix-free sets (*FMPF-sets* in short) in F_2, *e.g.*, the set containing the empty word e only, the sets $\{0,1\}, \{00, 010, 011, 10, 11\}$, *etc.* Assign to each bracketing an ordered sequence of words in F_2 inductively by the rule:

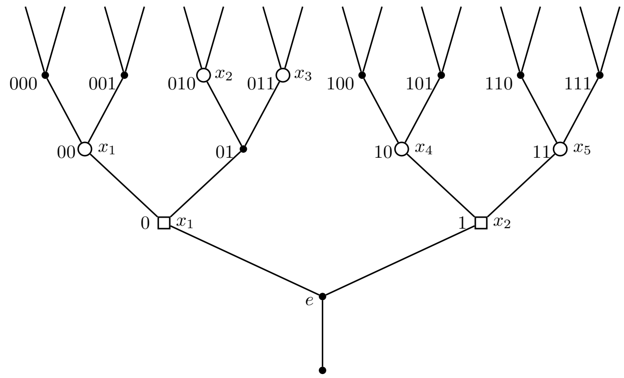
(a) $x \mapsto (e)$,

(b) if $P \mapsto (w_1, \dots, w_k)$ and $Q \mapsto (w_{k+1}, \dots, w_{k+l})$ then $(PQ) \mapsto (0w_1, \dots, 0w_k, 1w_{k+1}, \dots, 1w_{k+l})$.

It is a routine to check that, in this way, a unique, lexicographically listed FMPF-set of n words is assigned to every bracketing of size n. Now we can use the defining properties (1),(2) of Catalan numbers to show that the number of distinct FMPF-sets of n elements equals C_{n-1}. Therefore, (a) and (b) provide a 1-1 correspondence between bracketings and lexicographically ordered FMPF-sets.

Consider a bracketing B of size n viewed with subscripts, *i.e.*, as an n-ary groupoid term. Let $(w_1(B), \dots, w_n(B))$ be the lexicographically ordered FMPF-set corresponding to B. Call the length of $w_i(B)$ *the depth of* x_i *in* B, and the number of 0's (resp. of 1's) in $w_i(B)$ *the left depth* (resp. *the right depth*) *of* x_i *in* B.

Inspecting (a) and (b) we get the intuitive meaning of depth of x_i: the number of pairs of parentheses (or, equivalently, of the subbracketings of size at least 2) containing x_i. Similarly, *e.g.* the right depth of x_i in B is the number of those subbracketings in which x_i is contained in the right factor. The sequence consisting of the depths of $x_1, \dots, x_n$ in B will be called *the depth sequence of* B. *Left* and *right depth sequences of* B are defined analogously. *E.g.*, the depth sequence of $((x(xx))(xx))$ is $(2,3,3,2,2)$, and its right depth sequence is $(0,1,2,1,2)$.



FMPF-sets — and thus also bracketings — can be imagined as such minimal sets of vertices in the infinite binary rooted tree that separate the top of the tree from its bottom. See the figure where the sets of vertices corresponding to (x_1x_2) and $(x_1(x_2x_3))(x_4x_5)$ are marked by squares, resp. circles; correspondence between vertices and binary strings is indicated, too. In this representation, the depth of x_i is the number of edges in the path p connecting e with x_i. Similarly, the left (right) depth is the number of left(right) edges in p.

2.7. *Bracketings are uniquely determined by their depth sequences.*

This is clearly true for bracketings of size ≤ 3. Suppose the bracketings (P_1Q_1) and (P_2Q_2) of size $n(>3)$ have the same depth sequence $(d_1,\dots,d_n)$. From the definition, the equality

$$\sum_{i=1}^{n}\frac{1}{2^{e_i}}=1 \tag{1}$$

follows for every depth sequence $(e_1,\dots,e_n)$. If $|P_1|=j, |P_2|=k$, then, in view of (a) and (b), the depth sequences of P_1 and P_2 are of form $(d_1-1,\dots,d_j-1)$ and $(d_1-1,\dots,d_k-1)$, respectively. Therefore,

$$\sum_{i=1}^{j}\frac{1}{2^{d_i}}=\sum_{i=1}^{k}\frac{1}{2^{d_i}}=1/2.$$

Hence the sizes of P_1 and P_2 are equal. Now the proposition follows by induction.

2.8. *Bracketings are uniquely determined by their right (or left) depth sequences.*

Let $B=(PQ)$ be a bracketing with right depth sequence (in short, *RD-sequence*)

$$(d_1,\dots,d_n). \tag{2}$$

Then there is a k between 1 and n such that the RD-sequence of P is $(d_1,\dots,d_k)$, and that of Q is $(d_{k+1}-1,\dots,d_n-1)$. Induction shows that always

$$d_1=0,\quad d_2=1, \tag{3}$$

and, for $i=1,\dots,n-1$,

$$1\leq d_{i+1}\leq d_i+1. \tag{4}$$

Call a sequence (2) of nonnegative integers a *zag sequence* (*cf.* [6], Ch. 1.2, where *zig* is defined) if it has the properties (3) and (4). We use induction to prove that for any zag sequence (2) there exists at most one bracketing with RD-sequence (2). This is clearly true for $n\leq 2$. As $(d_{k+1}-1,\dots,d_n-1)$ is a zag sequence, we have $d_{k+1}=1$, and $d_j\geq 2$ for $j=k+2,\dots,n$. It follows that if the size of the first factor of B is k, then the last 1 in the RD-sequence of B appears on the $(k+1)$st place. Hence if the RD-sequences of $B=(PQ)$ and $B'=(P'Q')$ are the same, then $|P| = |P'|$. Thus the RD-sequences of P and P' coincide, and, by induction, $P = P'$. Similarly we obtain $Q = Q'$, completing the proof.

An analogous straightforward induction shows that every zag sequence is the RD-sequence of some bracketing. Consequently, the number of zag sequences of length n equals that of the bracketings of size n, *i.e.,* C_{n-1} (*cf.* [14], Ch. 5, Exercise 19(u)).

3. Examples

In this section we determine spectra of several common operations. Given a particular operation, we denote the members of its spectrum by $s(n)$ (without subscript), and we write $s(n) = f(n)$ to indicate that this equality holds for $n \geq 3$.

3.1. *For the subtraction of numbers,* $s(n) = 2^{n-2}$.

Induction shows that any regular operation $b(x_1, x_2, \ldots, x_n)$ over the subtraction is of form $x_1 - x_2 \pm x_3 \pm \cdots \pm x_n$. It is enough to prove that actually every possible sequence of the $+$ and $-$ signs occurs. This is true for $n \leq 3$; suppose $n > 3$, and apply induction. If $b(x_1, x_2, \ldots, x_n) = x_1 - x_2 - \cdots - x_n$, then b is induced by $((\ldots((x_1x_2)x_3)\ldots)x_n)$. Otherwise there exists a first $+$ sign in f, say $b(x_1, x_2, \ldots, x_n) = x_1 - x_2 - \cdots - x_{k+1} + x_{k+2} \pm \cdots \pm x_{k+l}$ $(k + l = n)$. Then $b(x_1, \ldots, x_n) = (x_1 - x_2 - \cdots - x_k) - (x_{k+1} - x_{k+2} \mp \cdots \mp x_{k+l})$, and this is induced by $B = (PQ)$, where $P = ((\ldots((x_1x_2)x_3)\ldots x_k)$, and Q is the bracketing that induces the subtrahend (such a Q exists by induction). In fact, this reasoning is valid for subtraction in arbitrary Abelian groups except those of exponent 2.

3.2. *The arithmetic mean as a binary operation on numbers is Catalan.*

We prove that distinct bracketings induce distinct regular operations over the arithmetic mean. Induction shows that a bracketing B of size n induces $b(x_1, \ldots, x_n) = \sum_{i=1}^{n} 2^{-d_i} x_i$ over the arithmetic mean, where d_i is the depth of x_i in B. Let B' $(\neq B)$ be another bracketing of size n which induces $b'(x_1, \ldots, x_n) = \sum_{i=1}^{n} 2^{-d_i{}'} x_i$. In virtue of 2.7, there exists a j $(1 \leq j \leq n)$ such that $d_j \neq d_j{}'$. Then $b(\delta_1^j, \ldots, \delta_n^j) = 2^{-d_j} \neq 2^{-d_j{}'} = b'(\delta_1^j, \ldots, \delta_n^j)$, *i.e.*, b and b' are distinct operations, as required. This holds for an arbitrary set of numbers closed under arithmetic mean, containing more than one element.

3.3. *The geometric mean and the harmonic mean as binary operations on positive real numbers are Catalan.*

This follows from 3.2 and 2.4, as the groupoids $(\mathbf{R}, (x + y)/2)$ and $(\mathbf{R}_+, \sqrt{xy})$ are isomorphic, as well as $(\mathbf{R}_+, (x + y)/2)$ and $(\mathbf{R}_+, 2xy/(x + y))$.

3.4. *The exponentiation as a binary operation* $(a, b) \mapsto a^b$ *on numbers is Catalan.*

Let $p_1, \ldots, p_n$ be distinct prime numbers. Consider bracketings B, $B'(\neq B)$ and the regular operations b, b' they induce over the exponentiation. We show that $b \neq b'$. Making use of the law $(r^s)^t = r^{st}$, and the usual convention of writing r^{s^t} instead of $r^{(s^t)}$, we can write expressions of form $b(p_1, \ldots, p_n)$ without parentheses, *e.g.*, if $B = ((x_1(x_2x_3))(x_4x_5))$ and p_i are the first primes, we have $b(2, 3, 5, 7, 11) = 2^{3^5 7^{11}}$. Here the exponents are at different levels: say, 2 is at the zeroth, 3 and 7 are at the first level, *etc.* The key observation is that the height of the level of p_i in b always equals the right depth of x_i in B; this can be verified using induction. As $B \neq B'$, by 2.8. there exists a j such that the right depth of x_j in B is different from that of x_j in B'. Then the fundamental theorem of arithmetic implies $b(p_1, \ldots, p_n) \neq b'(p_1, \ldots, p_n)$.

3.5. *The cross product of vectors is Catalan.*

Consider three pairwise perpendicular unit vectors, their additive inverses, and the zero vector. They form a groupoid under cross product, and, if we identify the unit vectors with their negatives, we obtain a four-element factorgroupoid C with Cayley operation table

$\times$	0	u	v	w
0	0	0	0	0
u	0	0	w	v
v	0	w	0	u
w	0	v	u	0

In virtue of 2.5., it is enough to prove that this operation is Catalan. Let B, B', b, b' be as in 3.4. We shall find nonzero elements $c_1, \ldots, c_n \in C$ such that $b(c_1, \ldots, c_n) = 0 \neq b'(c_1, \ldots, c_n)$. The case $n = 3$ is obvious. The general case needs some preparations:

3.5.1. Let F be a nontrivial nest of size k which induces the regular operation f on C. Given i $(1 \leq i \leq k)$, and $c, d \in C$ with $d \notin \{0, c\}$, we can choose elements $c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in C$ so that $f(c_1, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_k) = d$.

This is valid also for any bracketing B and its induced regular operation b instead of F and f. Indeed, apply 3.5.1 to the nest of B containing x_i, if this nest is nontrivial, and replace this nest by x; while if x_i is a trivial nest, replace the eggs of another nest by x. Then, in both cases, use induction for the quotient bracketing. We remark that this generalized form of 3.5.1 implies that any regular operation over the cross product is surjective (*i.e.*, it maps C^n onto C; in fact, this is the case for all surjective binary operations, *cf.* 4.2.1).

3.5.2. If x_j, x_{j+1} are no eggs of any nest of a bracketing B, we can choose $d_1, \ldots, d_{j-1}, d, d_{j+2}, \ldots, d_k$ in C such that $f(d_1, \ldots, d_{j-1}, d, d, d_{j+2}, \ldots, d_k) \neq 0$.

If $B = (PQ)$ with $|P| = k$ and $j + 1 \leq k$, then for suitable elements $d, d_i \in C$ by induction we have $p(d_1, \ldots, d_{j-1}, d, d, d_{j+2}, \ldots, d_k) = e \neq 0$. Now by 3.5.1 there are $d_{k+1}, \ldots, d_n \in C$ such that $q(d_{k+1}, \ldots, d_n) = f \neq 0, e$. Then $b(d_1, \ldots, d, d, \ldots, d_n) = e \times f \neq 0$. The case $k < j$ can be treated in a similar way. Finally, suppose $k = j$. Let us fix $d \neq 0$, and apply 3.5.1 to P and Q with $i = k$ and $i = k + 1$, respectively. Then we have elements $d_1, \ldots, d_{k-1}, d_{k+2}, \ldots, d_n \in C$ such that $p(d_1, \ldots, d_{k-1}, d) = e$ and $q(d, d_{k+2}, \ldots, d_n) = f$, where $C = \{0, d, e, f\}$. Thus $b(d_1, \ldots, d, d, \ldots, d_n) = e \times f = d \neq 0$, completing the proof of 3.5.2.

In order to prove 3.5., first suppose that there is an i $(1 \leq i \leq n)$ such that x_i and x_{i+1} are the eggs of a nest of B as well as of B'. Replacing $(x_i x_{i+1})$ by x in B and B', we obtain quotient bracketings B_1 resp. ${B_1}'$ of size $n - 1$ with induced regular operations b_1 and ${b_1}'$. By induction, there exist nonzero elements $e_1, \ldots, e_{n-1} \in C$ such that $b_1(e_1, \ldots, e_i, \ldots, e_{n-1}) = 0 \neq {b_1}'(e_1, \ldots, e_i, \ldots, e_{n-1})$. Let $e', e'' \in C$ be distinct, and different from 0 and e_i. Then $e' \times e'' = e_i$, and $b(e_1, \ldots, e_{i-1}, e', e'', e_{i+1}, \ldots, e_{n-1}) = b_1(e_1, \ldots, e_i, \ldots, e_{n-1}) = 0 \neq b'(e_1, \ldots, e_{i-1}, e', e'', e_{i+1}, \ldots, e_{n-1})$.

Now suppose that no nests of B and B' have a common pair of eggs. Let x_j and x_{j+1} be the eggs of a nest of B. Then $b(d_1, \ldots, d_{j-1}, d, d, d_{j+1}, \ldots, d_n) = 0$ for any choice of $d_1, \ldots, d_{j-1}, d, d_{j+2}, \ldots, d_n \in C$. However, as x_j and x_{j+1} are eggs of no nest in B', from 3.5.2 it follows that there is a choice of $d_1, \ldots, d_{j-1}, d_{j+2}, \ldots, d_n$ such that $b'(d_1, \ldots, d_{j-1}, d, d, d_{j+2}, \ldots, d_n) \neq 0$.

4. Groupoids on two-element sets

In what follows we consider operations on finite sets. For uniform treatment, we study groupoids of form $(\mathbf{n}, \circ)$, where $\mathbf{n}$ stands for the set $\{0, 1, \ldots, n-1\}$. Each two-element groupoid is isomorphic or antiisomorphic with $(\mathbf{2}, \circ)$, where $x \circ y$ is one of the following seven Boolean functions:

(1) the constant 1 operation; (2) x (the first projection); (3) $x \wedge y$ (*i.e.*, $\min(x, y)$); (4) $x + y \bmod 2$; (5) $x + 1 \bmod 2$; (6) $x|y$ (the Sheffer function: "neither x, nor y"); (7) $x \to y$ (implication).

Here (1) — (4) are associative. We determine the spectra of (5) — (7).

4.1. *For the operation* $x + 1 \bmod 2$, $s(n) = 2$.

Indeed, induction shows that for an arbitrary bracketing B of size n and $c_1, \dots, c_n \in \mathbf{2}$, $b(c_1, \dots, c_n) = c_1 + d \bmod 2$, where d is the left depth of x_1 in B.

4.2. *The Sheffer function is Catalan.*

Recall, that $0|0 = 1$ and $x|y = 0$ otherwise. We shall need some preliminaries.

4.2.1. Regular operations over a surjective operation are surjective (*i.e.*, they take on all elements of their base sets; the inductive proof is trivial).

4.2.2. If the Cayley table of a surjective operation $\circ$ has neither two identical columns nor two identical rows, then each variable of any regular operation over $\circ$ is essential.

This is obvious for at most binary regular operations. Let $B = (PQ)$, $|B| = n \geq 3, |P| = k$. Take a variable x_i of b. We have to prove that there are elements $c_1, \dots, c_{i-1}, u, v, c_{i+1}, \dots, c_n$ in the base set M of $\circ$ such that $b(c_1, \dots, c_{i-1}, u, c_{i+1}, \dots, c_n) \neq b(c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n)$. Without loss of generality, suppose $i \leq k$. Then, by induction there exist $c_1, \dots, c_{i-1}, u, v, c_{i+1}, \dots, c_k \in M$ such that $g = p(c_1, \dots, c_{i-1}, u, c_{i+1}, \dots, c_n) \neq p(c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n) = h$. The rows of g and h in the Cayley table of $\circ$ are not identical, *i.e.*, there is a $d \in M$ such that $g \circ d \neq h \circ d$. Further, by 4.2.1, there are $c_{k+1}, \dots, c_n \in M$ with $q(c_{k+1}, \dots, c_n) = d$. Then $b(c_1, \dots, c_{i-1}, u, c_{i+1}, \dots, c_n) = g \circ d \neq h \circ d = b(c_1, \dots, c_{i-1}, v, c_{i+1}, \dots, c_n)$, which was needed.

4.2.3. If $\circ$ fulfils the conditions of 4.2.2, then regular operations of distinct arities over $\circ$ cannot be identically equal.

Indeed, otherwise the last variable of the regular operation of greater arity could not be essential.

We see that 4.2.1—3 apply to the Sheffer function. Let B_1, B_2 be bracketings of size $n\,(\geq 3), B_1 = (P_1Q_1), B_2 = (P_2Q_2)$, and suppose that their induced operations b_1 and b_2 coincide. We have to prove $B_1 = B_2$. This is true for $n = 3$, as $(0|0)|1 = 0 \neq 1 = 0|(0|1)$. Let $n > 3$, and assume $k = |P_1| \leq |P_2| = l$. First we show that, for arbitrary $c_1, \dots, c_k, \dots, c_l \in \mathbf{2}, p_1(c_1, \dots, c_k) = 0$ if and only if $p_2(c_1, \dots, c_l) = 0$. Let $p_1(c_1, \dots, c_k) = 0$. By 4.2.1, there exist $c_{k+1}, \dots, c_n \in \mathbf{2}$ with $q_1(c_{k+1}, \dots, c_n) = 0$. Hence it follows

$$\begin{aligned} b_1(c_1, \dots, c_k, c_{k+1}, \dots, c_n) &= p_1(c_1, \dots, c_k) \,|\, q_1(c_{k+1}, \dots, c_n) = 1 = \\ = b_2(c_1, \dots, c_l, c_{l+1}, \dots, c_n) &= p_2(c_1, \dots, c_l) \,|\, q_2(c_{l+1}, \dots, c_n), \end{aligned}$$

implying $p_2(c_1, \dots, c_l) = 0$. This reasoning is valid in the opposite direction, too, showing that p_1 identically equals p_2. Now from 4.2.3 we infer $k = l$ and, by induction, $P_1 = P_2$. It remains to establish $Q_1 = Q_2$. Let, once more, $p_1(c_1, \dots, c_k) = 0$. If $Q_1 \neq Q_2$, then, again by induction, there are $c_{k+1}, \dots, c_n \in \mathbf{2}$ such that $q_1(c_{k+1}, \dots, c_n) \neq q_2(c_{k+1}, \dots, c_n)$. Then

$$b_1(c_1, \dots, c_n) = 0 \,|\, q_1(c_{k+1}, \dots, c_n) \neq 0 \,|\, q_2(c_{k+1}, \dots, c_n) = b_2(c_1, \dots, c_n),$$

a contradiction. Thus $Q_1 = Q_2$, as required.

4.3. *Implication is Catalan.*

In view of 2.4., instead of implication we can consider the operation $x * y$, defined by $0 * 1 = 1$ and $x * y = 0$ otherwise, as $(\mathbf{2}, \to)$ and $(\mathbf{2}, *)$ are isomorphic. For $*$, the proof of 4.2. can be literally adapted.

5. Groupoids on three-element sets

There are 3330 essentially distinct three-element groupoids in the sense that each three-element groupoid is isomorphic with exactly one of them (see the Siena Catalog [2], in which code numbers from 1 to 3330 are given to each of these representatives), therefore a plain survey of their spectra such as in the two-element case seems to be impossible. In this section we determine the spectra of all groupoids on **3** with minimal clones of term operations, and give examples for further spectra.

There exist 12 essentially distinct groupoids on **3** with minimal clones, and each of them is idempotent (see [4]). The operations of an *idempotent* groupoid on **3** may be encoded by the numbers $0, 1, \dots, 728$ in the following transparent way: let the code of $\circ$ be

$$(0 \circ 1) \cdot 3^5 + (0 \circ 2) \cdot 3^4 + (1 \circ 0) \cdot 3^3 + (1 \circ 2) \cdot 3^2 + (2 \circ 0) \cdot 3 + (2 \circ 1)$$

(see the examples below). The operations of the groupoids on **3** with minimal clones are (or, more exactly, may be chosen as) $0, 8, 10, 11, 16, 17, 26, 33, 35, 68, 178, 624$ (their codes in the Siena Catalog are $80, 102, 105, 106, 122, 125, 147, 267, 271, 356, 1108, 2346$ respectively). It is easy to check that $0, 8, 10, 11$ and 26 are associative. Here we display the Cayley tables of the remaining seven operations:

$$\begin{array}{|ccc}\hline 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{array} \qquad \begin{array}{|ccc}\hline 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \end{array} \qquad \begin{array}{|ccc}\hline 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{array} \qquad \begin{array}{|ccc}\hline 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 2 \end{array}$$

$$16 \qquad\qquad 17 \qquad\qquad 33 \qquad\qquad 35$$

$$\begin{array}{|ccc}\hline 0 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{array} \qquad \begin{array}{|ccc}\hline 0 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{array} \qquad \begin{array}{|ccc}\hline 0 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{array}$$

$$68 \qquad\qquad 178 \qquad\qquad 624$$

As we apply three different approaches, we parcel our task into three parts.

5.1. *The operations* 16, 17 *and* 178 *are Catalan.*

Observe that **3** with each of the operations 16, 17 and 178 is a groupoid in which $\{0, 1\}$ is a subgroupoid with two-sided zero element 0, while $\{1, 2\}$ and $\{2, 0\}$ are subgroupoids with left zero elements 1 and 2, respectively. Here and in what follows, the just considered operations will be denoted by circle.

Let $B_i = (P_i Q_i)$ $(i = 1, 2)$ be distinct bracketings of size $n\,(\geq 3)$. For $n = 3$, $1 \circ (2 \circ 0) = 1 \circ 2 = 1 \neq 0 = 1 \circ 0 = (1 \circ 2) \circ 0$, *i.e.*, $b_1 \neq b_2$. To prove the same for $n > 3$, first suppose $|P_1| = k < l = |P_2|$. Then

$$b_1(1, \dots, 1, 2, \dots, 2, 0, \dots, 0) = p_1(1, \dots, 1) \circ q_1(2, \dots, 2, 0, \dots, 0) = 1 \circ 2 = 1,$$
$$b_2(1, \dots, 1, 2, \dots, 2, 0, \dots, 0) = p_2(1, \dots, 1, 2, \dots, 2) \circ q_2(0, \dots, 0) = 1 \circ 0 = 0.$$

Thus, we can assume $|P_1| = |P_2| = k$. If $P_1 \neq P_2$, by induction there exist elements $c_1, \dots, c_k \in \mathbf{3}$ with $g_1 = p_1(c_1, \dots, c_k) \neq p_2(c_1, \dots, c_k) = g_2$. Let d be the element of **3** that is different from g_1 and g_2. Then $g_1 \circ d \neq g_2 \circ d$ (see the Cayley tables), and hence $b_1(c_1, \dots, c_k, d, \dots, d) = g_1 \circ d$ differs from $b_2(c_1, \dots, c_k, d, \dots, d) = g_2 \circ d$. It remains to settle the case $Q_1 \neq Q_2$. Again, we can choose elements $c_{k+1}, \dots, c_n \in \mathbf{3}$ with $h_1 = q_1(c_{k+1}, \dots, c_n) \neq q_2(c_{k+1}, \dots, c_n) = h_2$.
Case 17*:* 0 and 2 are left zero elements, whence $c_{k+1} = 1$, and we can assume $h_1 = 0, h_2 = 1$. Now $b_1(1, \dots, 1, c_{k+1}, \dots, c_n) = 1 \circ 0 = 0 \neq 1 = 1 \circ 1 = b_2(1, \dots, 1, c_{k+1}, \dots, c_n)$.

Cases 16 *and* 178*:* for distinct elements $h_1, h_2 \in \mathbf{3}$ there exists $e \in \mathbf{3}$ with $e \circ h_1 \neq e \circ h_2$. Hence it follows $b_1(e, \ldots, e, c_{k+1}, \ldots, c_n) \neq b_2(e, \ldots, e, c_{k+1}, \ldots, c_n)$, concluding the proof.

5.2. *The operation* 33 *is Catalan. For* 35 *and* 68*,* $s(n) = 2^{n-2}$*.*

Consider a groupoid $(G, \circ)$ with idempotent elements $d, e(\neq d), f$ such that

(α) in the Cayley table of $\circ$, d occurs only in its own row;
(β) in the row of e, $e \circ d$ occurs only once;
(γ) f is a right unit element.

First check that $\mathbf{3}$ with 33, 35, or 68 satisfies these conditions. Now let $B_1 = (P_1 Q_1)$ and $B_2 = (P_2 Q_2)$ be bracketings of size n such that their induced operations over $\circ$ coincide. We prove $p_1 = p_2$. Suppose $k = |P_1| < |P_2| = l$. Then

$$b_2(e, \ldots, e, d, \ldots, d) = p_2(e, \ldots, e) \circ q_2(d, \ldots, d) = e \circ d,$$
$$b_1(e, \ldots, e, d, \ldots, d) = p_1(e, \ldots, e) \circ q_1(e, \ldots, e, d, \ldots, d).$$

As by (α) we have $q_1(e, \ldots, e, d, \ldots, d) \neq d$, from ($\beta$) it follows that $b_1(e, \ldots, e, d, \ldots, d) \neq b_2(e, \ldots, e, d, \ldots, d)$. Thus $|P_1| = |P_2|$, and , by (γ), for arbitrary $c_1, \ldots, c_k \in G$ it holds $p_1(c_1, \ldots, c_k) = b_1(c_1, \ldots, c_k, f, \ldots, f) = b_2(c_1, \ldots, c_k, f, \ldots, f) = p_2(c_1, \ldots, c_k)$, which was needed.

Take into account that 33 is surjective, and its Cayley table has no two identical columns. We show that in the case of 33 if $b_1 = b_2$, then $q_1 = q_2$, which together with $p_1 = p_2$ implies *via* induction that 33 is Catalan. Indeed, suppose that, although $b_1 = b_2$, there exist $c_{k+1}, \ldots, c_n \in \mathbf{3}$ such that $q_1(c_{k+1}, \ldots, c_n) \neq q_2(c_{k+1}, \ldots, c_n)$. Then the columns of these two elements are also distinct, *i.e.* $c \circ q_1(c_{k+1}, \ldots, c_n) \neq c \circ q_2(c_{k+1}, \ldots, c_n)$ for some $c \in \mathbf{3}$. In virtue of 4.2.1 we can choose $c_1, \ldots, c_k \in \mathbf{3}$ so that $p_1(c_1, \ldots, c_k) = c$. Now $b_1(c_1, \ldots, c_n) = p_1(c_1, \ldots, c_k) \circ q_1(c_{k+1}, \ldots, c_n) \neq p_1(c_1, \ldots, c_k) \circ q_2(c_{k+1}, \ldots, c_n) = b_2(c_1, \ldots, c_n)$, a contradiction.

Concerning 35 and 68, observe that in these cases if $u \circ v \neq u \circ w$ then at least one of v and w is a left zero which satisfies (α). We have seen that $b_1 = b_2$ implies $p_1 = p_2$; now we prove that the converse implication also holds. Suppose not, *i.e.*, there are $c_1, \ldots, c_n \in \mathbf{3}$ such that $b_1(c_1, \ldots, c_n) = p_1(c_1, \ldots, c_k) \circ q_1(c_{k+1}, \ldots, c_n) \neq p_1(c_1, \ldots, c_k) \circ q_2(c_{k+1}, \ldots, c_n) = b_2(c_1, \ldots, c_n)$. Hence, without loss of generality, the element $d = q_1(c_{k+1}, \ldots, c_n)$ is a left zero, and d does not occur in other rows. We infer that $c_{k+1} = d$, and, as a consequence, $q_2(c_{k+1}, \ldots, c_n) = d = q_1(c_{k+1}, \ldots, c_n)$, whence $b_1(c_1, \ldots, c_n) = b_2(c_1, \ldots, c_n)$, a contradiction.

This shows that, for 35 and 68, $s(n) = s(n-1) + \cdots + s(2) + s(1)$, and this means $s(n) = 2^{n-2}$, as stated.

5.3. *For the operation* 624*,* $s(n) = \lfloor 2^n/3 \rfloor$*.*

624 is actually $2x + 2y \bmod 3$ on $\mathbf{3}$. We shall write it in form $-x-y$; our considerations are valid for this operation on numbers, too. An n-ary regular operation (over $-x-y$) is always of form $t(x_1, \ldots, x_n) = \pm x_1 \pm \cdots \pm x_n$. We call such operations *complete linear*. As $x_1 - x_2 + x_3$ shows, not every complete linear operation is regular. Denote by $\pi(t)$ the number of + signs in a complete linear operation $t = t(x_1, \ldots, x_n)$, and call a complete linear t *subregular*, if $\pi(t) \equiv 2n - 1 \pmod 3$. The following assertion can be checked immediately:

5.3.1. If t, t_1, t_2 are complete linear operations such that the equality $t(x_1, \ldots, x_n) = -t_1(x_1, \ldots, x_k) - t_2(x_{k+1}, \ldots, x_n)$ holds, then every one of t, t_1, t_2 is subregular provided the other two of them are subregular.

Next we characterize the regular operations over $-x-y$.

5.3.2. A complete linear operation $t(x_1, \ldots, x_n)$ is regular over $-x-y$ if and only if it is subregular but not of form

$$x_1 - x_2 + x_3 - \cdots + x_n \tag{5}$$

(*i.e.*, not of odd arity with alternating signs and beginning with a $+$ sign).

Clearly, this is true for $n \leq 3$. Suppose that t is regular. Then $t(x_1, \ldots, x_n) = -t_1(x_1, \ldots, x_k) - t_2(x_{k+1}, \ldots, x_n)$ with t_1 and t_2 regular. By induction, t_1 and t_2 are subregular, and 5.3.1 implies that t is subregular. If t is regular and it is of form (5), then one of t_1 and t_2 — say, t_1 — must be of even arity with alternating signs. However, a complete linear operation t of arity $2m$ with alternating signs cannot be subregular, as $\pi(t) = m \not\equiv 2 \cdot 2m - 1 \pmod 3$. Hence t_1 is not subregular, a contradiction.

Conversely, assume that t is subregular but not regular. We have to prove that t is of form (5). We show that the first sign in t is $+$. If not, then $t(x_1, \ldots, x_n) = -x_1 \pm x_2 \pm \cdots \pm x_n = -x_1 - (\mp x_2 \mp \cdots \mp x_n) = -x_1 - t_2(x_2, \ldots, x_n)$, and from 5.3.1 it follows that t_2 is subregular. If, in addition, t_2 is not of form (5), then by induction t_2 is regular, hence t is regular, in contrary to the assumption. However, if t_2 is of form (5), then

$$\begin{aligned} t(x_1, \ldots, x_n) &= -x_1 - x_2 + x_3 - \cdots + x_{n-1} - x_n = \\ &= -(x_1 + x_2 - x_3 + \cdots - x_{n-1}) - x_n = \\ &= -t_1(x_1, \ldots, x_{n-1}) - x_n, \end{aligned}$$

and here t_1 is regular, implying again the regularity of t.

Thus, t starts with a $+$ sign, and it is enough to prove that the signs alternate in t. If not, consider the first two consecutive identical signs in t. Suppose they are $+$; the other case can be treated analogously. Then

$$\begin{aligned} t(x_1, \ldots, x_n) &= x_1 - x_2 + \cdots - x_{2k-2} + x_{2k-1} + x_{2k} \pm \\ &\quad \pm x_{2k+1} \pm \cdots \pm x_n = \\ &= -\,(-x_1 + x_2 - \cdots + x_{2k-2} - x_{2k-1} - x_{2k}) - \\ &\quad -\,(\mp x_{2k+1} \mp \cdots \mp x_n) = \\ &= -\,t_1(x_1, \ldots, x_{2k}) - t_2(x_{2k+1}, \ldots, x_n). \end{aligned}$$

We can check that t_1 is subregular and not of form (5), hence regular; further, t_2 is subregular by 5.3.1. As above, supposing that t_2 is not of form (5) leads to a contradiction. Hence $t_2(x_{2k+1}, \ldots, x_n) = x_{2k+1} - x_{2k+2} + \cdots - x_{n-1} + x_n$, and

$$\begin{aligned} t(x_1, \ldots, x_n) &= x_1 - x_2 + x_3 - \cdots + x_{2k-1} + x_{2k} - x_{2k+1} + \\ &\quad + x_{2k+2} - \cdots + x_{n-1} - x_n = \\ &= -\,(-x_1 + x_2 - x_3 + \cdots - x_{2k-1} - x_{2k} + x_{2k+1} - \\ &\quad - x_{2k+2} + \cdots - x_{n-1}) - x_n = \\ &= -\,{t_1}'(x_1, \ldots, x_{n-1}) - x_n. \end{aligned}$$

Here ${t_1}'$ is subregular and not of form (5), so it is regular by induction, whence we obtain that t is regular, and this final contradiction proves that a subregular but not regular complete linear operation is of form (5).

From 5.3.2 it follows that the number $s(n)$ of the n-ary regular operations over $-x - y$ equals $\sum_k \binom{n}{3k+i} - (n \bmod 2)$, if $n \equiv 2 - i \pmod 3$ $(i = 0, 1, 2)$. It is known that each of these numbers is equal to $\lfloor 2^n/3 \rfloor$ (see [6], Ch. 5, Exercise 75). This completes the description of spectra of three-element groupoids with minimal clones.

The next seven operations are of some interest from various reasons. The first two pairs have the same spectra but with different coincidences of induced regular operations. Fibonacci numbers appear at the fifth one. Nest structure is exploited in the next example, and the last one is related to the Sheffer operation on **2**. These operations are numbered by their codes in the Siena Catalog [2]:

$$\begin{array}{cccc}
\begin{array}{|ccc}\hline 0&0&2\\0&0&2\\2&2&1\end{array} &
\begin{array}{|ccc}\hline 0&0&0\\0&0&0\\1&0&0\end{array} &
\begin{array}{|ccc}\hline 0&0&1\\0&0&1\\1&1&0\end{array} &
\begin{array}{|ccc}\hline 1&1&1\\2&2&2\\0&0&0\end{array} \\
1066 & 10 & 405 & 3242
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{|ccc}\hline 0&0&0\\0&1&0\\0&0&1\end{array} &
\begin{array}{|ccc}\hline 0&0&0\\0&1&0\\0&1&2\end{array} &
\begin{array}{|ccc}\hline 1&0&0\\0&2&0\\0&0&0\end{array} \\
79 & 82 & 2407
\end{array}$$

5.4. *For the operations* 1066 *and* 10, $s(n) = n - 1$.

Denote by $t(c_1, \dots, c_n)$ the number of occurrences of 2 among $c_1, \dots, c_n$. Concerning 1066, induction shows that, for arbitrary bracketing $B = (PQ)$ with $|B| = n$, $|P| = k$, and $c_1, \dots, c_n \in \mathbf{3}$,

$$b(c_1, \dots, c_n) = 2 \text{ if and only if } t(c_1, \dots, c_n) \text{ is odd,}$$

and

$$b(c_1, \dots, c_n) = 1 \text{ if and only if both } t(c_1, \dots, c_k) \text{ and } t(c_{k+1}, \dots, c_n) \text{ are odd.}$$

As a consequence, $b(c_1, \dots, c_n) = 0$ if and only if both $t(c_1, \dots, c_k)$ and $t(c_{k+1}, \dots, c_n)$ are even. Hence it follows that two bracketings of equal size induce the same operation if and only if the sizes of their left factors are equal.

In order to manage 10 (which, for this once, will be written as multiplication), we introduce the *priority of a bracketing* B ($\mathrm{pr}(B)$ in sign) for $|B| > 2$ as follows: If $B = (PQ)$ and $|P| > 1$, then $\mathrm{pr}(B) = 0$; if $B = (x_1(x_2(\dots(x_k(R))\dots)))$, and $\mathrm{pr}(R) = 0$ or $|R| = 2$, then $\mathrm{pr}(B) = k$. We call the bracketing R *the core of* B. Clearly, if $n > 2$, for every $k = 0, 1, \dots, n-2$ there exist bracketings of size n with priority k. Hence it is sufficient to prove that two bracketings of size n induce the same regular operation over 10 if and only if they are of the same priority.

"If": $\mathrm{pr}(B) = 0$ implies that b is the constant 0 operation. If $k = n-2$ or $k = n-3$, then there is only one bracketing B with $\mathrm{pr}(B) = k$. Suppose B_1 and B_2 are of size n with cores R_1, resp. R_2, and $\mathrm{pr}(B_1) = \mathrm{pr}(B_2) = k < n-3$. Then

$$\begin{aligned}
b_1(c_1, \dots, c_n) &= (c_1(\dots(c_k \cdot r_1(c_{k+1}, \dots, c_n))\dots)) = (c_1(\dots(c_k \cdot 0)\dots)) = \\
&= (c_1(\dots(c_k \cdot r_2(c_{k+1}, \dots, c_n))\dots)) = b_2(c_1, \dots, c_n)
\end{aligned}$$

for arbitrary $c_1, \dots, c_n \in \mathbf{3}$.

"Only if": Let again B_1 and B_2 be bracketings with cores as above, and let $\mathrm{pr}(B_1) = k < l = \mathrm{pr}(B_2)$. Induction on priority shows that bracketings with positive priority induce nonconstant operations over 10. Hence there are $c_{k+1}, \dots, c_l$, $c_{l+1}, \dots, c_n \in \mathbf{3}$ such that $(c_{k+1}(\dots(c_l \cdot r_2(c_{l+1}, \dots, c_n))\dots)) = 1$. For $i = 0, 1$, check the equality $(2(2(\dots(2 \cdot i)\dots))) = (k - i) \bmod 2$, where k is the number of occurrences of 2 in the left side, and choose $c_1 = \dots = c_k = 2$. It follows

$$\begin{aligned}
b_1(c_1, \dots, c_n) &= (c_1(\dots(c_k \cdot r_1(c_{k+1}, \dots, c_n))\dots)) = (c_1(\dots(c_k \cdot 0)\dots)) = \\
&= k \bmod 2 \neq (k-1) \bmod 2 = (c_1(\dots(c_k \cdot 1)\dots)) = \\
&= (c_1(\dots(c_k(c_{k+1}(\dots(c_l \cdot r_2(c_{l+1}, \dots, c_n))\dots)))\dots)) = \\
&= b_2(c_1, \dots, c_n).
\end{aligned}$$

5.5. *For the operations* 405 *and* 3242, $s(n) = 3$ *if* $n > 3$.

Let B_1, B_2 be bracketings of size n, $B_i = (P_i Q_i)$. We show that the induced regular operations b_1, b_2 over 405 coincide if and only if one of the following conditions is satisfied:

(1) $|P_1| = |P_2| = 1$;
(2) $1 < |P_1|, |P_2| < n - 1$;
(3) $|P_1| = |P_2| = n - 1$.

Indeed, in the case (1) the first variable, and in the case (3) the last variable determines the value of b_i. In the case (2) b_i is the constant zero operation. Finally, if $B_1 = (x_1 Q_1)$, $B_2 = (P_2 x_n)$, then $b_1(0, \ldots, 2) = 0 \neq 1 = b_2(0, \ldots, 2)$.

3242 is $x + 1 \bmod 3$. Similarly to 4.1, for any bracketing B and its induced operation b over 3242 we have $b_1(c_1, \ldots, c_n) = c_1 + d \bmod 3$, where d is the left depth of x_1 in B.

5.6. *For the operation* 79*,* $s(n) = F_{n+1} - 1$*, where* F_k *is the* k*th Fibonacci number.*

First we show that, for bracketings B_1, B_2 of equal size, b_1 coincides with b_2 if and only if the eggs of nests of B_1 are the same as the eggs of nests of B_2. Suppose that x_i, x_{i+1} are the eggs of a nest of B_1 but of no nest of B_2. Put $c_j = 2$, if $j = i$ or $j = i + 1$, and $c_j = 1$ otherwise. Then $b_1(c_1, \ldots, c_n) = 1 \neq 0 = b_2(c_1, \ldots, c_n)$. On the other hand, if the eggs of nests of B_1 and B_2 are the same, induction on the number of nests proves $b_1 = b_2$. Note that this number is 1 exactly when B_1 and B_2 are nests, and for nests we can apply the usual induction on size.

Choose several non-overlapping pairs $(i, i + 1)$ in the sequence $1, \ldots, n$. The number of such choices (including the empty choice) is F_{n+1}. Induction shows that for every such nonempty choice C there exists a bracketing B such that x_i, x_{i+1} are the eggs of a nest of B if and only if $(i, i + 1)$ occurs in the choice C. This proves our proposition.

5.7. *The operation* 82 *is Catalan.*

Induction shows that the first (*i.e.*, leftmost) right parenthesis in B together with its left pair encloses just the eggs of the leftmost nontrivial (maximal) nest of B. Let $|B_1| = |B_2| = n$, $b_1 = b_2$, and let the eggs in question of B_1 and B_2 consist of x_k, x_{k+1} and x_l, x_{l+1} $(k < l)$, respectively. For $c_1 = \cdots = c_k = c_{k+2} = \cdots = c_n = 1$, $c_{k+1} = 2$ we get $b_1(c_1, \ldots, c_n) = 0 \neq b_2(c_1, \ldots, c_n)$. Thus, the first right parentheses in B_1 and B_2 cannot be in different positions. Collapsing x_k and x_{k+1} we obtain quotient bracketings B_1' and B_2' of size $n - 1$. Remark that, for arbitrary $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n \in \mathbf{3}$, $b_i'(c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n) = b_i(c_1, \ldots, c_{k-1}, 2, c_{k+1}, \ldots, c_n)$ holds, as 2 is a left unit for 82. In such a way, b_i determines b_i', and the latter determines the place of the first right parenthesis in B_i', which is the second right parenthesis in B_i; *etc.* We see that the induced operation determines the positions of all right parentheses in its parent bracketing. Now 5.7 follows from 2.6.

5.8. *The operation* 2407 *is Catalan.*

The proof consists of a suitable adaptation of 4.2. The observations 4.2.1, 4.2.2, and 4.2.3 apply to 2407. Now, from $B_1 = (P_1 Q_1)$, $B_2 = (P_2 Q_2)$, and $b_1 = b_2$ we can deduce not only the equivalence of $p_1(c_1, \ldots, c_k) = 0$ and $p_2(c_1, \ldots, c_l) = 0$ but also that of $p_1(c_1, \ldots, c_k) = 1$ and $p_2(c_1, \ldots, c_l) = 1$. Thus, again we have $p_1 = p_2$, and, by induction, $P_1 = P_2$. In order to refute $Q_1 \neq Q_2$, assume that there exist $c_{k+1}, \ldots, c_n \in \mathbf{3}$ with $q_1(c_{k+1}, \ldots, c_n) = i \neq j = q_2(c_{k+1}, \ldots, c_n)$; here we can suppose $i \neq 2$. There are $c_1, \ldots, c_k \in \mathbf{3}$ with $p_1(c_1, \ldots, c_k) = i$. Then $b_1(c_1, \ldots, c_n) = i \circ i = i + 1 \bmod 3 \neq i \circ j = b_2(c_1, \ldots, c_n)$.

The Sheffer function on **2** and 2407 on **3** are the smallest instances of groupoids $(\mathbf{n}, \circ)$ with operations

$$i \circ j = \begin{cases} i+1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

All these groupoids are *primal*; *i.e.*, all possible operations on $\mathbf{n}$ are term operations of such a groupoid. The proof of 5.8 can be generalized for them without trouble. Hence we could (in fact, we did) conjecture for a minute that primality implies a Catalan spectrum; however, operation 3233 testifies that this is not the case. Its Cayley table comes from that of 3242 by writing $1 \circ 2 = 0$ instead of $1 \circ 2 = 2$. For 3233 we have $s_6 = 41 < C_5(= 42)$. Actually,

$$x_1 \circ ((x_2 \circ (x_3 \circ (x_4 \circ x_5))) \circ x_6) = x_1 \circ ((x_2 \circ ((x_3 \circ x_4) \circ x_5)) \circ x_6)$$

identically holds for 3233 on **3** (but no other regular operations over 3233 induced by distinct bracketings of size ≤ 6 are equal). On the other hand, the primality of **3** with 3233 as well as of $\mathbf{n}$ with operation (6) follows, *e.g.*, from Rousseau's criterion: a finite algebra with a single operation is primal if and only if it has neither proper subalgebras, nor congruences, nor automorphisms [12].

We have checked all the 3330 entries of the Siena Catalog by computer for the five initial elements of their spectra, *i.e.* $(s(3), s(4), s(5), s(6), s(7))$. It is known that there are 24 nonisomorphic three-element semigroups. The table below shows the number of essentially distinct three-element nonassociative groupoids with a given initial segment of spectrum:

2	2	2	2	2	16	2	5	10	21	42	5
2	3	3	3	3	4	2	5	11	23	47	2
2	3	4	5	6	15	2	5	11	24	53	4
2	4	4	4	4	2	2	5	12	28	65	12
2	4	5	6	7	6	2	5	13	34	87	12
2	4	6	8	10	4	2	5	13	34	89	2
2	4	7	12	20	4	2	5	13	34	90	4
2	4	7	12	21	12	2	5	13	34	91	24
2	4	8	15	27	12	2	5	13	35	96	2
2	4	8	16	32	62	2	5	13	35	97	32
2	5	8	12	16	2	2	5	14	41	123	6
2	5	10	18	31	4	2	5	14	41	124	16
2	5	10	20	40	4	2	5	14	42	132	3038

Several sequences beginning with some quintuples above, *e.g.* $(2, 5, 10, 21, 42)$ (*cf.* 5.3) and $(2, 5, 14, 41, 123)$, are recently missing in the Encyclopedia [13].

6. General remarks and problems

All the spectra considered up to now are monotonic. Groups with the commutator operation provide examples of nonmonotonic spectra: if a group G is nilpotent then there exists an n such that all n-ary regular term operations over the commutator of G are equal (to the constant unit operation), hence $s(n) = 1$, and if G is not nilpotent of class 2 then the commutator is not associative (see, *e.g.* [10]). The spectrum always stabilizes in these examples: $s(n) = 1$ implies $s(m) = 1$ for every $m > n$. In fact, this is a common property of all spectra, which generalizes the generalized associative law:

6.1. *For an arbitrary spectrum s, $s(n) = 1$ for some $n\,(\geq 3)$ implies $s(m) = 1$ for every $m > n$.*

Call two bracketings of size m *adjacent* if there exists a j such that x_j, x_{j+1} are eggs of nests for each of these bracketings. It is easy to see that the transitive closure of the adjacency relation is the trivial equivalence if $m \geq 5$.

Let $n\,(\geq 3)$ be a number such that $s(n) = 1$ for an operation $\circ$ on a set M. Consider bracketings B, B^* of size $n+1$. We have to prove $b = b^*$. For $n = 3$ this is the generalized associative law. Assume $n > 3$. Then $n + 1 \geq 5$, hence there exist bracketings $B_0 = B, B_1, \ldots, B_k = B^*$ such that, for $i = 0, 1, \ldots, k-1$, B_i is adjacent to B_{i+1}. Let x_j, x_{j+1} be common eggs of a nest of B_i and a nest of B_{i+1}. Replacing $(x_j x_{j+1})$ by x_j in both of them, we obtain quotient bracketings B'_i, B'_{i+1} of size n. As $s(n) = 1$, we have $b'_i = b'_{i+1}$, and thus

$$\begin{aligned} b_i(c_1, \ldots, c_{n+1}) &= b'_i(c_1, \ldots, c_{j-1}, c_j \circ c_{j+1}, c_{j+2}, \ldots, c_{n+1}) = \\ &= b'_{i+1}(c_1, \ldots, c_{j-1}, c_j \circ c_{j+1}, c_{j+2}, \ldots, c_{n+1}) = \\ &= b_{i+1}(c_1, \ldots, c_{n+1}) \end{aligned}$$

for arbitrary $c_1, \ldots, c_{n+1} \in M$.

Groups provide also examples showing that the difference $s(n) - s(n-1)$ of consecutive entries of a spectrum can be arbitrarily large:

6.2. *The spectrum of the commutator operation on the dihedral group of degree* 2^t $(t \geq 3)$ *is*

$$s(n) = \begin{cases} 2, & \text{if } n = 3 \\ n, & \text{if } 3 < n \leq t \\ 1, & \text{if } n > t. \end{cases}$$

D_m, the dihedral group of degree m is generated by a rotation α of order m and a reflection ρ. We write i for α^i and j' for $\alpha^j \rho$. Here is the concise Cayley table of the commutator on D_m:

	j	j'
i	0	$-2i \bmod m$
i'	$2j \bmod m$	$2\,(i-j) \bmod m$

The following observations are immediate: If a bracketing B over the commutator on D_n has at least two nests, then it induces the constant zero operation. Further, if B is a nest with eggs x_k, x_{k+1}, then $b(c_1, \ldots, c_n) \neq 0$ only if all $c_i\,(\in D_m)$ but at most one of c_k, c_{k+1} are of form i' (*i.e.*, $\alpha^i \rho$). From the Cayley table we learn that for such a nest B and such elements $c_1, \ldots, c_n$

$$b(c_1, \ldots, c_n) = [c_k, c_{k+1}]\, 2^{k-1}(-2)^{n-k-1} \bmod m \tag{7}$$

holds. From (7) we infer that the position of eggs of B determines the induced operation b. As all commutators are of form $2u \bmod m$, (7) shows also that always $b(c_1, \ldots, c_n) = 2^{n-1} \cdot v \bmod m$ with suitable integers v. This means that b is the zero operation if $m = 2^t$ and $n > t$.

It remains to show that nests of equal size $n\,(\leq t)$ but with distinct eggs induce distinct operations. In fact, besides B consider another nest B' with eggs x_l, x_{l+1} $(l > k)$. Let $c_k = 1$, $c_{k+1} = 2'$, and choose elements c_i $(i \neq k, k+1)$ of form i' arbitrarily. Then $[1, 2'] = -2 \bmod 2^t$, and, by (7), $b(c_1, \ldots, c_n) = (-1)^{n-k} 2^{n-1} \bmod 2^t \neq 0$. On the other hand, $l > k$ implies $b'(c_1, \ldots, c_n) = 0$ because $c_k = 1$, and x_k is out of the egg of B'.

The same reasoning shows that the commutator on D_1, D_2 and D_4 is associative, and if m is not a power of 2 (*e.g.*, in the case of $D_3 = S_3$) the spectrum of the commutator on D_m is $s(n) = n$ for $n > 3$.

The next example leads to groupoids whose spectra begin with arbitrarily many Catalan numbers and still reach 1.

6.3. *The following operation on the nonnegative integers is Catalan:*

$$a \circ b = \begin{cases} \min(a, b) - 1, & \text{if } a, b > 0 \\ 0, & \text{otherwise.} \end{cases}$$

For the proof, denote by $d_B(x_i)$ the depth of x_i in the bracketing B. Consider an arbitrary bracketing $B = (PQ)$ with $|B| = n,\ |P| = k$. First we show that

$$b(d_B(x_1) + 1, \ldots, d_B(x_n) + 1) = 1.$$

Note that, for any B, $b(c_1, \ldots, c_n) > 0$ implies $b(c_1{+}1, \ldots, c_n{+}1) = b(c_1, \ldots, c_n){+}1$. By induction we have $p(d_B(x_1), \ldots, d_B(x_k)) = p(d_P(x_1) + 1, \ldots, d_P(x_k) + 1) = 1$, and similarly $q(d_B(x_{k+1}), \ldots, d_B(x_n)) = 1$, whence it follows

$$\begin{aligned} b(d_B(x_1) + 1, \ldots, d_B(x_n) + 1) &= p(d_B(x_1) + 1, \ldots, d_B(x_k) + 1)\circ \\ &\qquad \circ\, q(d_B(x_{k+1}) + 1, \ldots, d_B(x_n) + 1) = \\ &= (1 + 1) \circ (1 + 1) = 1, \end{aligned}$$

as needed.

Next we show that for any other B' of size n, $b'(d_B(x_1){+}1, \ldots, d_B(x_n){+}1) = 0$. Again, induction shows that for arbitrary B, nonnegative integers $c_1, \ldots, c_n$, and i $(1 \le i \le n)$

$$b(c_1, \ldots, c_n) \le \max(c_i - d_B(x_i), 0) \tag{8}$$

holds; we omit the details. As $B' \ne B$, 2.7 implies that there exists an i such that $d_{B'}(x_i) \ne d_B(x_i)$, and in view of (1) we can suppose even $d_{B'}(x_i) > d_B(x_i)$. Then applying (8) to B' we obtain

$$b'(d_B(x_1) + 1, \ldots, d_B(x_n) + 1) \le \max(d_B(x_i) + 1 - d_{B'}(x_i), 0) = 0,$$

concluding the proof.

For any bracketing B with $|B| = k < n$, and for every $i\,(= 1, \ldots, k)$, we have $d_B(x_i) < k$, hence $d_B(x_i) + 1 \in \mathbf{n}$. Therefore the above reasoning shows that in $(\mathbf{n}, \circ)$, which is a subgroupoid of $(\mathcal{N}_0, \circ)$, distinct bracketings of size $k\,(< n)$ induce different regular operations. On the other hand, every bracketing B whose size exceeds 2^{n-2} has a symbol x_j with $d_B(x_j) \ge n - 1$. Applying (8) to the regular operation b of $(\mathbf{n}, \circ)$ we obtain

$$b(c_1, \ldots, c_n) \le \max(c_j - d_B(x_j), 0) = 0,$$

as $c_j \le n - 1$. Hence any bracketing of size $2^{n-2} + 1$ induces the constant zero operation of $(\mathbf{n}, \circ)$. Thus, for the spectrum of $(\mathbf{n}, \circ)$, $s(k) = C_{k-1}$ if $k < n$, and $s(k) = 1$ if $k > 2^{n-2}$.

The study of spectra of linear operations $px{+}qy$ (and $px{+}qy{+}r$) on numbers (or, more generally, on modules over rings) also offers remarkable facts. As a specimen, we prove the following generalization of 3.2.

6.4. *The linear operations $px + py$ and $x + py$ on the complex numbers are not Catalan if and only if p is a root of unity.*

Concerning $px + py$, induction shows that for any bracketing B of size n, the induced operation over $px + py$ is

$$b(x_1, \ldots, x_n) = \sum_{i=1}^{n} p^{d_i} x_i, \tag{9}$$

where d_i is the depth of x_i in B. From 2.7 it follows that if p is not a root of unity then $px + py$ is Catalan. Suppose $p^k = 1$. Define the bracketings B_i by $B_1 = (xx)$, and $B_{n+1} = (B_n B_n)$ for $n > 0$. The depth sequences of $B' = (xB_k)$ and $B'' = (B_k x)$ are $(1, k{+}1, \ldots, k{+}1)$ and $(k{+}1, \ldots, k{+}1, 1)$, respectively. Now (9) implies $b' = b''$. Hence, for $m = 2^k + 1$, $s(m) < C_{m-1}$.

Analogous considerations apply to $x + py$: (9) remains valid for this case with *right* depths instead of depths. If p is not a root of unity, 2.8 guarantees that

$x + py$ is Catalan. Suppose again $p^k = 1$, and redefine B_i by $B_1 = (xx)$, and $B_{n+1} = (xB_n)$ for $n > 0$. The RD-sequences of $B' = (B_k x)$ and $B'' = B_{k+1}$ are $(0, 1, 2, \ldots, k, 1)$ and $(0, 1, 2, \ldots, k, k+1)$, respectively, implying $b' = b''$, and, for $m = k + 2$, $s(m) < C_{m-1}$.

In conclusion, we formulate a few problems:

1. For every positive integer n there exists a minimal $f(n)$ with the property that, if for two spectra s_1, s_2 of n-element groupoids $s_1(i) = s_2(i)$ holds whenever $i \le f(n)$, then these spectra coincide. Propositions 4.1—3 imply $f(2) = 4$, and the table at the end of Section 5 shows that $f(3) \ge 7$. What is the actual value of $f(3)$ (and that of $f(4)$, *etc.*)?

2. We gave a rough estimation for the subsequent entries of a spectrum with a given initial segment in 2.3 which *e.g.,* for $s(3) = 2$ and $s(4) = 4$ provides $s(5) \le 12$. However, a case-by-case analysis shows that $s(3) = 2$ and $s(4) = 4$ actually imply $s(5) \le 8$. Do they imply $s(n) \le 2^{n-2}$ for all $n\,(>1)$? If so, call $s(n) = 2^{n-2}$ a *maximal extension* of the initial segment $(2, 4)$. Prove or disprove that the maximal extension of $(2, 3)$ is $s(n) = n - 1$, and that of $(2, 2)$ is $s(n) = 2$.

3. All nonconstant spectra we exhibited above are ultimately constant or monotonic. In the latter case their growth rates are either linear or exponential. Is there any other possibility? More concretely: find, *e.g.*, a spectrum with quadratic growth rate.

4. The statistics of the three-element groupoids and the abundance of appropriate examples leave such an impression that a huge majority of binary operations is Catalan. Is it true that, in some sense, almost all operations are Catalan (or almost Catalan)?

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