ON POLYMORPHISM-HOMOGENEITY OF THREE-ELEMENT GROUPOIDS

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Abstract. The study of this paper is focused on polymorphism-homogeneity of three-element groupoids. We group these groupoids by clone-equivalence. We show for some special groupoids that they are polymorphism-homogeneous. For the non-polymorphism-homogeneous groupoids a computer program was used, the results of the program were also mathematically proved.

A PREPRINT

1. Introduction

Various notions of homogeneity appear in several areas of mathematics, such as model theory, group theory, combinatorics, etc. Roughly speaking, a structure $\mathcal{A}$ is said to be homogeneous if certain kinds of local morphisms (i.e., morphisms defined on "small" substructures of $\mathcal{A}$) extend to endomorphisms of $\mathcal{A}$. Specifying the kind of morphisms that are expected to be extendable, one can define many different versions of homogeneity. We consider a variant called polymorphism-homogeneity introduced by C. Pech and M. Pech [9] that involves "multivariable" homomorphisms: we require extendibility of homomorphisms defined on finitely generated substructures of direct powers of $\mathcal{A}$ (see Section 2.4 for the precise definition). We study polymorphism-homogeneity of finite algebraic structures. Since homomorphisms depend on the term operations, not on the particular choice of basic operations, we work mainly with the clone $C = \mathrm{Clo}(\mathbb{A})$ of term operations of the algebraic structure $\mathbb{A} = (A; F)$ (i.e., C is the clone generated by F; see Section 2.1).

Together with Tamás Waldhauser we studied two-element algebras, and – using that clones on the two-element set are described – proved that all of them have a property we later named Property (SDC) [13, 14]. This property states that the sets of n-tuples that are closed under the centralizer clone of the algebra coincide with the sets that are solution sets of some system of equations over the algebra. Later we proved that (together with 2 other properties) Property (SDC) is equivalent to polymorphism-homogeneity [15]. This way we showed that every algebra on the two-element set is polymorphism-homogeneous.

On the three-element set much less is known about the clones themselves, thus a full description like on the two-element set is unlikely (for now). Therefore, as a starting point we investigate polymorphism-homogeneity of three-element groupoids. A study on three-element groupoids was written by Joel Berman and Stanley Burris in 1996 [1]. In [1] several properties of the groupoids are investigated with the help of computers. The goal of the author is to extend their results. This paper provides the first steps in this direction; we investigate polymorphism-homogeneity of 303 out of the 411 groupoids (up to clone equivalence, see Subsection 2.5). In Section 3 we investigate special classes of groupoids. A computer program was also used to find non-extendable local polymorphisms of groupoids, the results of this program are collected

Key words and phrases. Polymorphism-homogeneity, algebraic set, universal algebraic geometry, solution set of a system of equations, quantifier elimination.

in Table 1. The results of the program are proved mathematically in sections 4 and 5 with the help of several lemmas. In this paper we use the same notation for the groupoids as in [1], but we also give the operation table (or structure) of all of the investigated groupoids, either in the lemma or theorem about the groupoid, or in case of the results of the computer program in Appendix B.

2. Preliminaries

2.1. **Clones and relational clones.** Let $\mathcal{O}_A^{(n)}$ denote the set of all *n-ary operations* on a set A (i.e., maps $f\colon A^n \to A$), and let $\mathcal{O}_A$ be the set of all operations of arbitrary finite arities on A. In this paper we will always assume that the set A on which we consider operations and relations is finite. The *composition* of $f \in \mathcal{O}_A^{(n)}$ by $g_1, \ldots, g_n \in \mathcal{O}_A^{(k)}$ is the k-ary operation $f(g_1, \ldots, g_n)$ defined by

$$f(g_1, \ldots, g_n)(\mathbf{a}) = f(g_1(\mathbf{a}), \ldots, g_n(\mathbf{a})) \qquad (\mathbf{a} \in A^k).$$

A set $C \subseteq \mathcal{O}_A$ of operations is a *clone* if C is closed under composition and contains the projections $(x_1, \ldots, x_n) \mapsto x_i$ for $1 \leq i \leq n$. We use the symbol $C^{(n)}$ for the n-ary part of C, i.e., $C^{(n)} = C \cap \mathcal{O}_A^{(n)}$. The clone *generated* by $F \subseteq \mathcal{O}_A$ is the least clone $\mathrm{Clo}(F)$ containing F. By the definition of a term operation, $\mathrm{Clo}(F)$ is the clone of term operations of the algebra $\mathbb{A} = (A; F)$, hence we will also use the notation $\mathrm{Clo}(\mathbb{A})$ for this clone.

A *k-ary partial operation* on A is a map $h\colon \mathrm{dom}\, h \to A$, where the *domain* of h can be any set $\mathrm{dom}\, h \subseteq A^k$. The set of all partial operations on A is denoted by $\mathcal{P}_A$, and the set of all k-ary partial operations on A is denoted by $\mathcal{P}_A^{(k)}$. A *strong partial clone* is a set of partial operations that is closed under composition, contains the projections, and contains all restrictions of its members to arbitrary subsets of their domains. Note that if $C \subseteq \mathcal{O}_A$ is a clone, then the least strong partial clone $\mathrm{Str}(C)$ containing C consists of all restrictions of elements of C, i.e., $h \in \mathcal{P}_A$ belongs to $\mathrm{Str}(C)$ if and only if h can be extended to a total operation $\widehat{h} \in C$.

An n-ary *relation* on A is a subset of A^n; the set of all relations (of arbitrary arities) on A is denoted by $\mathcal{R}_A$. Given a set of relations $R \subseteq \mathcal{R}_A$, a *primitive positive formula* $\Phi(x_1, \ldots, x_n)$ over R is an existentially quantified conjunction:

$$\Phi(x_1, \ldots, x_n) = \exists y_1 \cdots \exists y_m \mathop{\&}_{i=1}^{t} \rho_i\big(z_1^{(i)}, \ldots, z_{r_i}^{(i)}\big), \tag{2.1}$$

where $\rho_i \in R$ is a relation of arity r_i, and each $z_j^{(i)}$ is a variable from the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ for $i = 1, \ldots, t,\ j = 1, \ldots, r_i$. The relation $\rho = \{(a_1, \ldots, a_n) : \Phi(a_1, \ldots, a_n) \text{ is true}\} \subseteq A^n$ is then said to be *defined by the primitive positive formula* Φ. The set of all primitive positive definable relations over R is denoted by $\langle R \rangle_\exists$, and such sets of relations are called *relational clones*. If we allow only quantifier-free primitive positive formulas, then we obtain the *weak relational clone* $\langle R \rangle_\nexists$.

2.2. **Galois connections between operations and relations.** If $M = (m_{ij}) \in A^{n \times k}$ is an $n \times k$ matrix over the set A, then we denote the i-th row and the j-th column of M by M_{i*} and M_{*j}, respectively:

$$M_{i*} = (m_{i1}, \ldots, m_{ik}) \qquad (i = 1, \ldots, n),$$
$$M_{*j} = (m_{1j}, \ldots, m_{nj}) \qquad (j = 1, \ldots, k).$$

If $h \in \mathcal{P}_A^{(k)}$ is a partial operation of arity k such that the rows of M are in the domain of h, then we can apply h to each row of M, and then we obtain the n-tuple $\big(h(M_{1*}), \ldots, h(M_{n*})\big)$. We may also denote this tuple by $h(M_{*1}, \ldots, M_{*k})$, as it is nothing but the componentwise application of h to the k columns of M. Therefore, we have

$$\big(h(M_{1*}), \dots, h(M_{n*})\big) = h(M_{*1}, \dots, M_{*k}).$$

We will often use the above equality without further mention.

We say that a k-ary (partial) operation h *preserves* the relation $\rho \subseteq A^n$, denoted as $h \rhd \rho$, if for every matrix $M \in A^{n \times k}$ such that each column of M belongs to ρ (and each row of M is in the domain of h), we have $h(M_{*1}, \dots, M_{*k}) \in \rho$. If R is a set of relations, then we write $h \rhd R$ to indicate that h preserves all elements of R. In other words, $h \rhd R$ holds if and only if h is a (partial) polymorphism of the relational structure $\mathcal{A} = (A; R)$, i.e., h is a homomorphism from (the substructure $\operatorname{dom} h$ of) $\mathcal{A}^k$ to $\mathcal{A}$. The set of all (partial) operations preserving each relation of R is denoted by $\operatorname{Pol} R$ ($\operatorname{pPol} R$), and the set of all relations preserved by each member of a set F of (partial) operations is denoted by $\operatorname{Inv} F$:

$$\operatorname{Pol} R = \big\{ h \in \mathcal{O}_A : h \rhd \rho \text{ for every } \rho \in R \big\};$$
$$\operatorname{pPol} R = \big\{ h \in \mathcal{P}_A : h \rhd \rho \text{ for every } \rho \in R \big\};$$
$$\operatorname{Inv} F = \big\{ \rho \in \mathcal{R}_A : h \rhd \rho \text{ for every } h \in F \big\}.$$

Note that $\operatorname{Pol} R = \operatorname{pPol} R \cap \mathcal{O}_A$.

The closed sets under the Galois connection $\operatorname{Pol} - \operatorname{Inv}$ ($\operatorname{pPol} - \operatorname{Inv}$) between (partial) operations and relations are exactly the (strong partial) clones and the (weak) relational clones; this makes these Galois connections fundamental tools in clone theory.

Theorem 2.1 ([2, 8, 11])**.** *For any set of operations $F \subseteq \mathcal{O}_A$ and any set of relations $R \subseteq \mathcal{R}_A$, we have $\operatorname{Clo}(F) = \operatorname{Pol} \operatorname{Inv} F$ and $\langle R \rangle_{\exists} = \operatorname{Inv} \operatorname{Pol} R$. For any set of partial operations $F \subseteq \mathcal{P}_A$ and any set of relations $R \subseteq \mathcal{R}_A$, we have $\operatorname{Str}(F) = \operatorname{pPol} \operatorname{Inv} F$ and $\langle R \rangle_{\nexists} = \operatorname{Inv} \operatorname{pPol} R$.*

2.3. **Universal algebraic geometry and centralizers.** Let $\mathbb{A}$ be a finite algebra and let $C = \operatorname{Clo}(\mathbb{A})$ be the clone of term operations of $\mathbb{A}$. If f and g are n-ary term operations of $\mathbb{A}$, then $f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$ is an *equation* in n variables over $\mathbb{A}$, which we may simply write as a pair (f, g). The *solution set* of (f, g) is then the set $\operatorname{Sol}(f, g) = \{(a_1, \dots, a_n) \in A^n : f(a_1, \dots, a_n) = g(a_1, \dots, a_n)\}$. Of special interest are the equations of the form $f(x_1, \dots, x_n) = x_{n+1}$; the solution set of this equation is the $(n+1)$-ary relation $f^\bullet = \{(a_1, \dots, a_n, a_{n+1}) \in A^{n+1} : f(a_1, \dots, a_n) = a_{n+1})\}$, which is called the *graph* of f. We use the symbols $C^\bullet$ and C° for the set of graphs and for the set of all solution sets of equations over C:

$$C^\bullet = \big\{ f^\bullet : f \in C \big\};$$
$$C^\circ = \big\{ \operatorname{Sol}(f, g) : f, g \in C^{(n)}, n \in \mathbb{N} \big\}.$$

Note that $C^\bullet \subseteq C^\circ$, and it is easy to verify that $\langle C^\bullet \rangle_{\exists} = \langle C^\circ \rangle_{\exists}$ (see Lemma 3.2 of [14]), but in general $\langle C^\bullet \rangle_{\nexists}$ and $\langle C^\circ \rangle_{\nexists}$ may be different weak relational clones.

The members of $\langle C^\circ \rangle_{\nexists}$ are intersections of solution sets of finitely many equations, i.e., $\langle C^\circ \rangle_{\nexists}$ consists of solution sets of finite systems of equations over $\mathbb{A}$. Allowing infinite systems of equations, we obtain the so-called *algebraic sets*, which are the main objects of study in universal algebraic geometry [10]. Since we deal only with finite algebras, every system of equations is equivalent to a finite system of equations, thus the elements of $\langle C^\circ \rangle_{\nexists}$ are exactly the algebraic sets.

Basic results of linear algebra hint at the possibility that algebraic sets can sometimes be described by closure conditions. It turns out that if there is a clone D such that algebraic sets are exactly those sets of tuples that are closed under D, then D must be the clone $C^* = \operatorname{Pol} C^\bullet$ (see Corollary 3.7 in [14]). This clone is called the *centralizer* of C, since it consists of those operations that commute with every member of C; in other words, a k-ary operation h belongs to C^* if and only if h is a homomorphism from $\mathbb{A}^k$ to $\mathbb{A}$. (Observe that since $\langle C^\bullet\rangle_\exists = \langle C^\circ\rangle_\exists$, the centralizer can equivalently be defined as $C^* = \operatorname{Pol} C^\circ$, by Theorem 2.1.)

If the algebraic sets (i.e., solution sets of systems of equations) of $\mathbb{A}$ coincide with the C^*-closed sets of tuples, then we say that the algebra $\mathbb{A}$ has property (SDC); this abbreviation stands for "Solution sets are Definable by closure under the Centralizer".

Property (SDC). *Let $\mathbb{A} = (A; F)$ be an algebra and $C = \operatorname{Clo}(\mathbb{A})$. The following are equivalent for all $n \in \mathbb{N}$ and $T \subseteq A^n$:*

(a) *there exists a system $\mathcal{E}$ of equations over $\mathbb{A}$ such that $T = \operatorname{Sol}(\mathcal{E})$;*
(b) *the set T is closed under C^*.*

With Tamás Waldhauser we proved in [13] that every two-element algebra has this property, and in [14] finite semilattices and lattices with property (SDC) were characterized. In general, property (SDC) is easily seen to be equivalent to the condition $\langle C^\circ\rangle_\exists = \langle C^\circ\rangle_{\nexists}$, i.e., the algebra $\mathbb{A}$ has property (SDC) if and only if quantifiers can be eliminated from primitive positive formulas over C°. Let us state this fact explicitly for later reference together with a few more equivalent conditions (see Theorem 3.6 of [14]).

Theorem 2.2 ([14])**.** *For every clone C on a finite set A, the following five conditions are equivalent:*

(i) *C has Property* (SDC)*;*
(ii) $\langle C^\circ\rangle_{\nexists} = \operatorname{Inv}(C^*)$*;*
(iii) $\langle C^\circ\rangle_{\nexists} = \langle C^\circ\rangle_\exists$*;*
(iv) *every primitive positive formula over C° is equivalent to a quantifier-free primitive positive formula over C°;*
(v) *$\langle C^\circ\rangle_{\nexists}$ is a relational clone.*

2.4. **Polymorphism-homogeneity.** Let $\mathcal{A}$ be a first-order structure (i.e., a set A equipped with relations and/or operations). If $h\colon \mathcal{B} \to \mathcal{A}$ is a homomorphism defined on a finitely generated substructure $\mathcal{B} \leq \mathcal{A}^k$, then we say that h is a *local polymorphism* of $\mathcal{A}$. (Considering only finite structures, the assumption that $\mathcal{B}$ is finitely generated can be omitted from the definition.) And if $f\colon \mathcal{A}^k \to \mathcal{A}$ is a homomorphism, then we say that f is a *total polymorphism* of $\mathcal{A}$.

A first-order structure $\mathcal{A}$ is said to be *k-polymorphism-homogeneous*, if every local polymorphism $h\colon \mathcal{B} \to \mathcal{A}$ defined on a finitely generated substructure $\mathcal{B} \leq \mathcal{A}^k$ extends to a total polymorphism $\widehat{h}\colon \mathcal{A}^k \to \mathcal{A}$. (Again, in our case the assumption that $\mathcal{B}$ is finitely generated can be omitted from the definition.) The case $k = 1$ gives the notion of *homomorphism-homogeneity* introduced by P. J. Cameron and J. Nešetřil [4]. If $\mathcal{A}$ is k-polymorphism-homogeneous for every natural number k, then we say that $\mathcal{A}$ is *polymorphism-homogeneous* [9].

Together with Tamás Waldhauser we proved that property (SDC) of an algebra is equivalent to polymorphism-homogeneity of the algebra and also of the relational structure (A, C°).

Theorem 2.3 ([15])**.** *If $\mathbb{A}$ is a finite algebra and $C = \operatorname{Clo}(\mathbb{A})$, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*
(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*
(iii) $(A; C^\circ)$ *is polymorphism-homogeneous;*
(iv) $\mathbb{A}$ *is injective in* $\operatorname{SP_{fin}}(\mathbb{A})$.

As mentioned in the introduction, we always consider the clone generated by the basic operations of an algebra, and not the basic operations themselves. We can do this, because by the following lemma polymorphism-homogeneity of an algebra $(A;F)$ is equivalent to the polymorphism-homogeneity of the algebra $\mathrm{Clo}(A;F)$.

Lemma 2.4. *Let* $\mathbb{A} = (A;F)$ *be an arbitrary algebra and* $C = \mathrm{Clo}(\mathbb{A})$. *Then* $(A;F)$ *is polymorphism-homogeneous* $\iff$ $(A;C)$ *is polymorphism-homogeneous.*

Proof. By the previous lemma an algebra is polymorphism-homogeneous if and only if it has property (SDC). Note that property (SDC) only depends on the clone generated by the basic operations and not on the basic operations themselves, since any operation appearing in an equation can be obtained by compositions of the basic operations. Using that $F^* = \mathrm{Clo}(A;F)^* = C^*$ and that $\mathrm{Clo}(A;C) = C$ all together we can see that polymorphism-homogeneity of the two algebras in the lemma are indeed equivalent. □

2.5. **Three-element groupoids.** There are 19683 groupoids on the three-element set, but their number up to isomorphism is only 3330. In the paper [1] Joel Berman and Stanley Burris investigate 12 properties of three-element groupoids with the help of computers. They further reduced the number of groupoids that need to be investigated: they research the properties up to *clone-equivalence*, which is an equivalence relation on the groupoids induced by the pre-order given by

$$\mathbb{A} \leq \mathbb{B} \iff \mathrm{Clo}(\mathbb{A}) \text{ is a subset of the clone of some isomorphic copy of } \mathbb{B}.$$

This equivalence relation has only 411 equivalence classes. By Lemma 2.4, polymorphism-homogeneity of an algebra $\mathbb{A} = (A;F)$ only depends on the clone $\mathrm{Clo}(\mathbb{A})$ generated by F, thus we only need to investigate representatives of these 411 cases.

3. Polymorphism-homogeneity of special groupoids

In this section we investigate polymorphism-homogeneity of special three-element groupoids. We consider Abelian group(s), affine algebras, semilattices, and monounary algebras regarded as groupoids. We also give a conjecture about groupoids generating a congruence-distributive variety.

3.1. **Abelian group(s).** We investigate three-element Abelian group(s) with the help of the following theorem.

Theorem 3.1. [15] *If* $\mathbb{A}$ *is a finite Abelian group and* $C = \mathrm{Clo}(\mathbb{A})$, *then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*
(ii) $\mathbb{A}$ *is homomorphism-homogeneous;*
(iii) $\mathbb{A}$ *is polymorphism-homogeneous;*
(iv) $(A;C^\circ)$ *is polymorphism-homogeneous;*
(v) $(A;C^\bullet)$ *is polymorphism-homogeneous;*
(vi) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*;*
(vii) $\mathbb{A}$ *is injective in* $\mathrm{HSP}(\mathbb{A})$*;*
(viii) *each Sylow-subgroup of* $\mathbb{A}$ *is homocyclic, i.e.,* $\mathbb{A} \cong \mathbb{Z}_{q_1}^{m_1} \times \cdots \times \mathbb{Z}_{q_k}^{m_k}$, *where* $q_1, \ldots, q_k$ *are powers of different primes and* $m_1, \ldots, m_k \in \mathbb{N}$.

The only Abelian group on the three-element set is $\mathbb{Z}_3$ (up to isomorphism). The corresponding groupoid, groupoid (2124) is polymorphism-homogeneous by the previous theorem.

3.2. **Affine algebras.** Affine algebras are a special kind of Mal'tsev algebras, and they are well known in universal algebra. One way of defining them is the following. An n-ary operation f is called *affine with respect to* $\mathbb{G}$, if $\mathbb{G} = (A; +, -, 0)$ is an Abelian group and f commutes with the ternary operation $x - y + z$. An algebra is *affine* if and only if there exists an Abelian group $\mathbb{G} = (A; +, -, 0)$ such that $x - y + z$ is a term operation of $\mathbb{G}$ and every basic operation of the algebra is affine with respect to $\mathbb{G}$. (If we have an algebra $\mathbb{A}$ and a group $\mathbb{G}$ showing that $\mathbb{A}$ is affine, then we will also say that $\mathbb{A}$ is affine with respect to $\mathbb{G}$.) If we choose the Abelian group $\mathbb{G}$ in the definition to be a finite field (regarded as a group), then we can say the following.

Lemma 3.2. *Let $A = \{0, 1, \ldots, p-1\}$, where p is a prime. If an algebra $\mathbb{A}$ defined on A is affine with respect to $(\mathbb{Z}_p; +, -, 0)$, and $\mathrm{Clo}(\mathbb{A}) = \{a_1x_1 + a_2x_2 + \ldots a_nx_n \mid \sum_{i=1}^{n} a_i = 1, n \in \mathbb{N}\}$, then $\mathbb{A}$ is polymorphism-homogeneous.*

Proof. Let $C = \mathrm{Clo}(\mathbb{A})$. By Theorems 2.2 and 2.3, $\mathbb{A}$ is polymorphism-homogeneous if and only if every primitive-positive formula over C° is equivalent to a quantifier-free primitive-positive formula. Let us consider a primitive-positive formula over C° with only one quantifier. We prove that it can be omitted, and iterating this argument we get the proof. Since C° consists of sets of tuples that are defined by single equations, a primitive-positive formula over C° with one quantifier is of the form

$$\Phi(x_1, x_2, \ldots, x_n, u) = \exists u \in A :$$

$$\mathrm{Sol}(f_1(x_1, \ldots, x_n, u), g_1(x_1, \ldots, x_n, u)) \& \ldots \& \mathrm{Sol}(f_k(x_1, \ldots, x_n, u), g_k(x_1, \ldots, x_n, u)),$$

where $k \in \mathbb{N}$ and $f_i, g_i \in \mathrm{Clo}(\mathbb{A})$. We reformulate every equation $f_i(x_1, \ldots, x_n, u) = g_i(x_1, \ldots, x_n, u)$ containing u using the usual properties of the basic operations of $\mathbb{Z}_p$.

Let us consider an equation over C containing u. This equation can be written in the form

$$\sum_{i=1}^{n+1} a_i x_i = \sum_{i=1}^{n+1} b_i x_i,$$

where $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i = 1$ with $u = x_{n+1}$. We of course permit $a_i = 0$ or $b_i = 0$ (for any $i \in \{0, 1, \ldots, n+1\}$). Then we have

$$\sum_{i=1}^{n} (a_i - b_i) x_i = (b_{n+1} - a_{n+1}) x_{n+1}.$$

Since Z_p is a field, the element $(b_{n+1} - a_{n+1})$ has a multiplicative inverse, hence the following equation also holds:

$$(b_{n+1} - a_{n+1})^{-1} \sum_{i=1}^{n} (a_i - b_i) x_i = \sum_{i=1}^{n} (b_{n+1} - a_{n+1})^{-1} (a_i - b_i) x_i = x_{n+1} = u. \tag{3.1}$$

We need to check if the left-hand side of the equation above belongs to C. Since

$$\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n+1} b_i = 1, \text{ we have}$$

$$a_{n+1} = 1 - \sum_{i=1}^{n} a_i \text{ and } b_{n+1} = 1 - \sum_{i=1}^{n} b_i.$$

But then

$$b_{n+1} - a_{n+1} = (1 - \sum_{i=1}^{n} b_i) - (1 - \sum_{i=1}^{n} a_i) = \sum_{i=1}^{n} (a_i - b_i),$$

therefore the sum of the coefficients in the left-hand side of (3.1) is

$$(b_{n+1} - a_{n+1})^{-1} \sum_{i=1}^{n} (a_i - b_i) = \big(\sum_{i=1}^{n} (a_i - b_i)\big)^{-1} \sum_{i=1}^{n} (a_i - b_i) = 1.$$

Let us suppose then that every equation containing u is of the form (3.1). There are two cases. Suppose first that there is only one equation that contains u, for example the first one. Then it can be replaced with $x_1 = x_1$: for any n-tuple $(x_1, \dots x_n)$ it holds that there exists $u \in A$ such that (3.1) holds.

If there are more that one equations containing u, then we replace u everywhere with the other side of the equation first containing u (that is, if the first equation containing u is of the form (3.1), then we replace u in every equation with $(b_{n+1} - a_{n+1})^{-1} \sum_{i=1}^{n} (a_i - b_i)$). We can also delete the equation where u first appeared. And then we get a system of equations equivalent to the one defining Φ. □

Corollary 3.3. *Let* $A = \{0, 1, \dots, p-1\}$*, where* p *is a prime. If an algebra* $\mathbb{A}$ *defined on* A *is affine with respect to* $(\mathbb{Z}_p; +, -, 0)$*, and it satisfies* $\mathrm{Clo}(\mathbb{A}) = \{a_1x_1 + a_2x_2 + \dots a_nx_n + c \mid \sum_{i=1}^{n} a_i = 1, c \in \mathbb{Z}_p, n \in \mathbb{N}\}$*, then* $\mathbb{A}$ *is polymorphism-homogeneous.*

Proof. The same argument as in the proof of Lemma 3.2 works with a slight variation. Having constants only changes that an additional constant can be on the sides of an equation, thus using the same argument we can change each equation into the form (3.1) with some constant appearing on the left-hand side. (And the constants do not interfere with the sum of the coefficients.) □

We can use the previous lemma and its corollary to investigate affine three-element groupoids. Groupoid (2124) has already been considered, since it is (essentially) $\mathbb{Z}_3$. The other two groupoids (up to clone equivalence), groupoid (2346) and (2934) are polymorphism-homogeneous.

Corollary 3.4. *The three-element affine groupoid given by*

(2346)	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

is polymorphism-homogeneous.

Proof. Notice that the basic operation of groupoid (2346) is $x * y = -x - y$, where "$-$" is the inverse of the additive operation of $\mathbb{Z}_3$. It is easy to see that every operation we get from compositions of the basic operation is of the form $\pm x_1 \pm x_2 \cdots \pm x_n$ $(n \in \mathbb{N})$. It is also not hard to prove that an operation $a_1x_1 + a_2x_2 + \dots + a_nx_n$ belongs to $C = \mathrm{Clo}(*)$ if and only if $\sum_{i=1}^{n} a_i = 1$ (modulo 3). This can be done by induction. The basic operation $-x - y$ satisfies this condition since $-1 - 1 = -2 = 1$. And if we have two operations f and g such that the sum of the coefficients is 1 in both of them, then the sum of the coefficients of $f * g = -f - g$ is $-1 - 1 = -2 = 1$. Therefore we have that $C = \{a_1x_1 + a_2x_2 + \dots a_nx_n \mid \sum_{i=1}^{n} a_i = 1, n \in \mathbb{N}\}$, and thus Lemma 3.2 can be used. □

Corollary 3.5. *The three-element affine groupoid given by*

(2934)	0	1	2
0	1	0	2
1	0	2	1
2	2	1	0

is polymorphism-homogeneous.

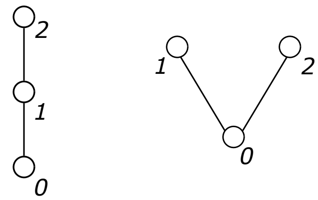

FIGURE 1. A schematic view on groupoids (80) and (105).

Proof. Notice that the basic operation of groupoid (2346) is $x * y = -x - y + 1$, where "$-$" is the inverse of the additive operation of $\mathbb{Z}_3$. As in the previous corollary, it is easy to see that every operation we get from compositions of the basic operation is of the form $\pm x_1 \pm x_2 \cdots \pm x_n + c$ ($n \in \mathbb{N}, c \in \mathbb{Z}_3$). Let C denote the clone generated by $*$. By the following two equalities we can see that $x + 1 \in C$ and $x + 2 \in C$.

$$x*x = -x-x+1 = -2x+1 = x+1, \quad (x+1)*(x+1) = -(x+1)-(x+1)+1 = -2x-1 = x+2$$

Using the same argument as in the previous corollary and that $x + 1, x + 2 \in C$ we can see that an operation $a_1x_1 + a_2x_2 + \cdots + a_nx_n + c$ belongs to C if and only if $\sum_{i=1}^{n} a_i = 1$ (modulo 3). Therefore we have that $C = \{a_1x_1 + a_2x_2 + \dots a_nx_n + c \mid \sum_{i=1}^{n} a_i = 1, c \in \mathbb{Z}_3, n \in \mathbb{N}\}$, and thus Corollary 3.3 can be used. $\square$

3.3. **Semilattices.** We investigate three-element semilattices with the help of the following theorem.

Theorem 3.6. [15] *If* $\mathbb{A}$ *is a finite semilattice and* $C = \mathrm{Clo}(\mathbb{A})$*, then the following conditions are equivalent:*

(i) $\mathbb{A}$ *has property* (SDC)*;*
(ii) $\mathbb{A}$ *is polymorphism-homogeneous;*
(iii) $(A; C^{\circ})$ *is polymorphism-homogeneous;*
(iv) $\mathbb{A}$ *is injective in* $\mathrm{SP}_{\mathrm{fin}}(\mathbb{A})$*;*
(v) $\mathbb{A}$ *is injective in* $\mathrm{HSP}(\mathbb{A})$*;*
(vi) $\mathbb{A}$ *is the semilattice reduct of a finite distributive lattice.*

There are only two (meet) semilattices on the three-element set, namely the three-element chain and the "V-shaped" semilattice (see Figure 1).

The groupoid that is essentially the second semilattice (groupoid (80)) is proven to be non-polymorphism homogeneous by the computer program; but also from the previous theorem we can see that it is not a semilattice reduct of a distributive lattice. The three-element chain on the other hand is polymorphism-homogeneous by the previous theorem. The corresponding groupoid is groupoid (105).

3.4. **Monounary algebras.** A *monounary algebra* is an algebra $\mathbb{A} = (A; f)$ with a single unary operation $f \in \mathcal{O}_A^{(1)}$. An element $a \in A$ is *cyclic* if there is a natural number k such that $f^k(a) = a$. (Here $f^k(a)$ stands for $f(\cdots f(a) \cdots)$ with a k-fold repetition of f, and we also use the convention $f^0(a) = a$.) If A is finite, then for every element $a \in A$ there is a least nonnegative integer $\mathrm{ht}(a)$, called the *height* of a, such that $f^{\mathrm{ht}(a)}(a)$ is cyclic. If $a \in A \setminus f(A)$, i.e., a has no preimage, then we say that a is a *source*. (Note that $\mathrm{ht}(a) = 0$ if and only if a is cyclic; in particular, $\mathrm{ht}(a) \geq 1$ for any source a.)

Polymorphism-homogeneous monounary algebras were characterized by Z. Farkasová and D. Jakubíková-Studenovská in [7].

Theorem 3.7 ([7])**.** *If $\mathbb{A} = (A; f)$ is a finite monounary algebra, then the following conditions are equivalent:*

(i) *$\mathbb{A}$ is polymorphism-homogeneous;*
(ii) *Either $\mathbb{A}$ has no sources, or all sources of $\mathbb{A}$ have the same height: $\forall a, b \in A \setminus f(A)\colon \operatorname{ht}(a) = \operatorname{ht}(b)$.*

We consider all the three-element groupoids (up to clone equivalence) that have an essentially unary basic operation. These groupoids are groupoids (1), (14), (27), (275), (366), (2466) and (3242).

Lemma 3.8. *The following groupoids are polymorphism-homogeneous.*

(1)	0	1	2	(14)	0	1	2	(27)	0	1	2	(275)	0	1	2	(366)	0	1	2	(2466)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
1	0	0	0	1	0	0	0	1	0	0	0	1	1	1	1	1	2	2	2	1	1	0	0
2	0	0	0	2	1	1	1	2	2	2	2	2	2	2	2	2	1	1	1	2	1	0	0

(3242)	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

.

Proof. By Theorem 3.7 we only need to check the height of the sources of each groupoid (regarded as a monounary algebra). Groupoid (1) has two sources, 1 and 2, and their heights are $\operatorname{ht}(1) = \operatorname{ht}(2) = 1$. The other groupoids either have only one source, or have no source at all. □

3.5. **Groupoids generating a congruence distributive variety.** Recall some general facts about algebras generating a congruence distributive variety. These results apply to 57 of the 411 groupoids (up to clone equivalence). If a groupoid generates a congruence distributive variety, then it will be called a CD-groupoid.

Definition 3.9 ([3, Definition IV§11.4])**.** Let $\mathbb{A}_1, \ldots, \mathbb{A}_n$ be algebras of the same type and let $\vartheta_i \in \operatorname{Con}(\mathbb{A}_i)$ for all $i \in \{1, \ldots, n\}$. The *product congruence* $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$ on $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is defined by $\big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big) \in \vartheta \Leftrightarrow \forall i \in \{1, \ldots, n\}\colon (a_i, b_i) \in \vartheta_i$.

Lemma 3.10 ([3, Lemma IV§11.10])**.** *Let $\mathcal{V}$ be a congruence distributive variety, $\mathbb{A}_1, \ldots, \mathbb{A}_n \in \mathcal{V}$ and $\vartheta \in \operatorname{Con}(\mathbb{A}_1 \times \cdots \times \mathbb{A}_n)$. Then there exist $\vartheta_i \in \operatorname{Con}(\mathbb{A}_i)$ for all $i \in \{1, \ldots, n\}$ such that $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$.*

The next theorem and its corollary is implicit in [12, Theorem 2]. (Similar ideas also appear in [5, 6].)

Lemma 3.11 ([12, Theorem 2])**.** *Let $\mathcal{V}$ be a congruence distributive variety and $\mathbb{A} \in \mathcal{V}$. Then the following are equivalent:*

(i) *There exists an essentially n-ary operation $f \in \mathcal{O}_A$ that is a homomorphism from $\mathbb{A}^n$ to $\mathbb{A}$.*
(ii) *There exist $\vartheta_i \in \operatorname{Con}(A)$ such that $\vartheta_i \neq A^2$ for all $i \in \{1, \ldots, n\}$, and $\mathbb{A}_1/\vartheta_1 \times \cdots \times \mathbb{A}_n/\vartheta_n$ embeds into $\mathbb{A}$.*

Corollary 3.12 ([12, Theorem 2])**.** *Let $\mathcal{V}$ be a congruence distributive variety, let $(A; F) = \mathbb{A} \in \mathcal{V}$ be a finite algebra and let C denote the clone of term operations of $\mathbb{A}$. If there is an essentially n-ary operation in C^*, then we have $n \leq \log_2 |A|$. In other words, the essential operations in C^* are at most $\log_2 |A|$-ary.*

Therefore, if a three-element groupoid generates a congruence distributive variety, then we can say the following.

Corollary 3.13. *Let G be a three-element groupoid that generates a congruence distributive variety. If G has an essentially k-ary local polymorphism, where $k \geq 2$, then G is not polymorphism-homogeneous.*

Proof. By the previous theorem any total polymorphism of G is at most unary. Therefore, if G has a essentially k-ary local polymorphism, where $k \geq 2$, then this local polymorphism is obviously non-extendable. □

If we take the results of the computer program mentioned in the paper into consideration, we see that 3 of the CD-groupoids, namely groupoids (219), (222) and (239) are not polymorphism-homogeneous. But the other 54 CD-groupoids do not have a unary counterexample, nor a binary local polymorphism. After some emailing with Mike Behrisch, he used a SAT program to see if the CD-groupoids (all 57) have any ternary local polymorphisms. His programs result was that none of the 57 groupoids have an essentially ternary local polymorphism. Therefore, the following conjecture is given by the author:

Conjecture 3.14. The three-element CD-groupoids not including (219), (222) and (239) are polymorphism-homogeneous.

We have one example somewhat strengthening the conjecture. Groupoid (2407) is a special groupoid along the CD-groupoids: it is primal, meaning that the clone generated by its basic operation contains every operation on the three-element set. It is not hard to see that this groupoid is polymorphism-homogeneous.

Lemma 3.15. *The three-element CD-groupoid given by*

(2407)	0	1	2
0	1	0	0
1	0	2	0
2	0	0	0

is polymorphism-homogeneous.

Proof. Let G denote groupoid (2407), and let $C = \mathrm{Clo}(G)$. Note that G is primal. By Theorem 2.3 the groupoid is polymorphism-homogeneous if and only if it has property (SDC); we will prove the latter one. Since G is primal, we have that the centralizer of G only contains the projections; and every set of n-tuples is closed under the projections (for any $n \in \mathbb{N}$). Therefore, we need to prove that any set of n-tuples is a solution set of some system of equations over C.

Let S be a set of n-tuples and let $\overline{S} = A^n \setminus S$. For any $(a_1, \ldots, a_n) \in \overline{S}$ we give an equation over C such that the solution set of this equation is exactly $A^n \setminus \{(a_1, \ldots, a_n)\}$. Then the solution set of the system of these equations have exactly S as its solution set. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \overline{S}$. Let us define $f(x_1, \ldots, x_n)$ at $\mathbf{a}$ as 1, and as 0 everywhere else. Let us define $g(x_1, \ldots, x_n)$ at $\mathbf{a}$ as 2, and as 0 everywhere else. Since G is primal, these operations belong to C. Then the solution set of the equation $f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$ is $A^n \setminus \{(a_1, \ldots, a_n)\}$. □

4. Proving that the partial operations given in Table 1 are local polymorphisms

In this section we prove that the operations given in Table 1 in Appendix A are local polymorphisms of the listed groupoids. The operation tables for the groupoids can be found in Appendix B.

When investigating whether a partial operation is a partial polymorphism we will often use (partially) the following notation for the commutation tables:

$$\begin{array}{ccccl} & a & b & \overset{\varphi}{\longrightarrow} & \qquad x \\ & c & d & \longrightarrow & \qquad y \\ * & \downarrow & \downarrow & & \qquad \downarrow \\ & u & v & \longrightarrow & \varphi(u,v) \stackrel{?}{=} x * y, \end{array}$$

where $x = \varphi(a,b), y = \varphi(c,d), u = a * c$ and $v = b * d$.

Lemma 4.1. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 1 \\ (1,1) \mapsto 1 \end{cases}$$

is a local polymorphism of the groupoids (2), (4), (5), (6), (8), (10), (11), (12), (13), (15), (16), (18), (22), (26), (30), (31), (32), (33), (34), (35), (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46), (48), (49), (51), (52), (53), (59), (61), (63), (65), (66), (67), (69), (73), (376), (377), (378), (379), (380), (381), (382), (384), (385), (387), (388), (390), (391), (405), (407), (410), (417), (434), (436), (437), (439), (1014), (1038), (1040), (1066).

Proof. Notice that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

*	0	1
0	0	0
1	0	0.

Therefore, the set $\{0,1\} \times \{0,1\}$ is a subalgebra of G^2 for each groupoid G. From this observation it also follows that φ is a local polymorphism of all of the groupoids, since for any $a, b, c, d \in \{0,1\}$ we have

$$\begin{array}{ccccl} & a & b & \overset{\varphi}{\longrightarrow} & 0 \text{ or } 1 \\ & c & d & \longrightarrow & 0 \text{ or } 1 \\ * & \downarrow & \downarrow & & \qquad \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}$$

□

Lemma 4.2. *The partial operation*

$$\varphi : \begin{cases} 0 \mapsto 2 \\ 2 \mapsto 0 \end{cases}$$

is a local polymorphism of the groupoids (21), (24), (47), (50), (72), (75), (78), (96), (99), (119), (141), (144), (162), (165), (168), (182), (185), (188), (201), (204), (218), (221), (235), (252) (255), (1227), (1233).

Proof. Notice that all of the listed groupoids have their basic operation defined on $\{0,2\} \times \{0,2\}$ as

*	0	2
0	0	0
2	2	2

or

*	0	2
0	0	2
2	0	2.

Therefore, $\{0,2\}$ is a subalgebra of G for each listed groupoid G. It is easy to see that φ is an automorphism on $(\{0,2\}, *)$ for both cases, hence it is a partial polymorphism of each listed groupoid.

□

Lemma 4.3. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 0 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$$

is a local polymorphism of the groupoids (60), (147).

Proof. Notice that these groupoids have their basic operation defined on $\{0,2\}\times\{0,2\}$ as

$$\begin{array}{c|cc} (60) & 0 & 2 \\ \hline 0 & 0 & 0 \\ 2 & 0 & 2, \end{array} \qquad \text{and} \qquad \begin{array}{c|cc} (147) & 0 & 2 \\ \hline 0 & 0 & 0 \\ 2 & 2 & 2. \end{array}$$

Therefore, $\{0,2\} \times \{0,2\}$ is a subalgebra of G^2 whenever G is either groupoid. Using that $\varphi(z,0) = \varphi(0,z) = 0$ for any $z \in \{0,2\}$, the following tables show that φ is a local polymorphism of groupoid (60) (here $a,b,c,d \in \{0,2\}$, and thus $x,y,u,v \in \{0,2\}$):

$$\begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 2 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & b & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & d & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0=0 \end{array}.$$

For groupoid (147) we also use that $\varphi(z,0) = \varphi(0,z) = 0$ for any $z \in \{0,2\}$:

$$\begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 2 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & b & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 0 & d & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & c & 0 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2=2 \end{array}.$$

(In the last two tables it suffices to only investigate the cases $a, b \neq 0$, i.e., when $a = b = 2$, since we already considered those cases previously.) □

Lemma 4.4. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 0 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \end{cases}$$

is a local polymorphism of the groupoids (79), (81), (82), (83), (85), (87), (88), (89), (90), (91), (93), (97), (101), (111), (112), (113), (115), (117), (120), (121), (122), (123), (124), (125), (130), (132), (134), (136), (137), (138), (142), (143), (406), (454), (455), (456), (457), (458), (459), (460), (462), (463), (465), (469), (483), (484), (485), (487), (493), (494), (495), (496), (512), (513), (514), (515), (517), (519), (522), (1086), (1107), (1108), (1133).

Proof. Notice that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

$$\begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0, \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1. \end{array}$$

Therefore $\{0,1\} \times \{0,1\}$ is a subalgebra of G^2 for each listed groupoid G. For the groupoids that satisfy $1 * 1 = 0$ we can use the same argument as in the proof of Lemma 4.1 to prove that φ is a local polymorphism. And for the groupoids where $1 * 1 = 1$ we can use the same argument as in the previous lemma for (60), but with interchanging the "2"-s with"1"-s (notice that the local polymorphism is also the same if we interchange 2 and 1). $\square$

Lemma 4.5. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 2 \\ (1,0) \mapsto 1 \end{cases}$$

is a local polymorphism of the groupoids (80), (100), (102), (239), (241), (1132).

Proof. We do this case-by-case. The shown tables prove that $\{(0,0),(0,2),(1,0)\}$ is a subalgebra, and also that φ is a partial polymorphism of the given groupoid.

$$(80): \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}1 \end{array},$$

$$(100): \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}1 \end{array},$$

$$(102):\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},\quad \begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},\quad \begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$\begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1 \end{array},$$

$$(239):\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},\quad \begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1 \end{array},\quad \begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1 \end{array},$$

$$\begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$(241):\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 0 & \xrightarrow{\varphi} & 0 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},\quad \begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},\quad \begin{array}{ccccc} & 0 & 2 & \xrightarrow{\varphi} & 2 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}2 \end{array},\quad \begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1 \end{array},\quad \begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1 \end{array},$$

$$\begin{array}{ccccc} & 1 & 0 & \xrightarrow{\varphi} & 1 \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$(1132):\quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 0 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 1 & 0 & \longrightarrow & 1 \\ 1 & 0 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}1 \end{array},$$

□

Lemma 4.6. *The partial operation*

$$\varphi : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases}$$

is a local polymorphism of the groupoids (116), (135), (178), (194), (203), (281), (308), (316), (488), (520), (565), (758), (780), (984), (1793), (1799), (1818), (1962), (2430), (2436), (2539), (2545), (2636).

Proof. Notice that all of the listed groupoids have their basic operation defined on $\{1,2\} \times \{1,2\}$ as one of the following

$$\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 2, \end{array} \qquad \begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 1 & 2, \end{array} \qquad \begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 2 & 1, \end{array} \qquad \begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 2 \\ 2 & 1 & 1. \end{array}$$

Therefore, $\{1,2\}$ is a subalgebra of G for each listed groupoid G. It is easy to see that φ is an automorphism on $(\{1,2\};*)$ for all four cases, hence it is a partial polymorphism of each groupoid. □

Lemma 4.7. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 0 \\ (0,2) \mapsto 0 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$$

is a local polymorphism of the groupoids (149), (175).

Proof. Note that $S = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(2,2)\} = \{0,1,2\} \times \{0,1,2\} \setminus \{(1,2),(2,1)\}$. Thus, to prove that this set is a subalgebra of both groupoids it suffices to show that neither of $(1,2)$ and $(2,1)$ can be obtained from S using the basic operations of (149) and (175). Let us suppose that (1,2) or (2,1) can be obtained by $(a,b)*(c,d)$. For both groupoids the basic operation only gives 2 if we take $1*1$. So either a and c or b and d are 1. This gives us seven possible candidates: $(0,1)*(0,1)$; $(0,1)*(1,1)$; $(1,0)*(1,0)$; $(1,0)*(1,1)$; $(1,1)*(0,1)$; $(1,1)*(1,0)$; $(1,1)*(1,1)$. But for both groupoids we have $0*0 = 0*1 = 1*0 = 0$, hence we can not get $(1,2)$ or $(2,1)$, that is, the set S is a subalgebra for both groupoids. Now for proving that φ is a local polymorphism, note that $\varphi(z,0) = \varphi(0,z) = 0$, and that $\varphi(z,z) = z$ for all $z \in \{0,1,2\}$. Let us consider (149) first: in this case we have $0*z = z*0 = 0$ for all $z \in \{0,1,2\}$, thus the following tables cover all possible cases. (Here $(a,b),(c,d) \in S$.)

$$\begin{array}{ccccc} & & & \varphi & \\ & 0 & b & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & d & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$\begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & 1 & 1 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & 2 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 1 & 1 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 2 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array},$$

Next we deal with (175): in this case we have $0*z = 0$, $2*z = 1$ and $1*0 = 0$ for all $z \in \{0,1,2\}$, thus the following tables cover all possible cases.

$$\begin{array}{ccccc} & & & \varphi & \\ & 0 & b & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & 0 & d & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & v & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & c & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & u & 0 & \longrightarrow & 0{=}0 \end{array},$$

$$\begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 0 & d & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & c & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & 1 & 1 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & 2 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array},$$

$$\begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 1 & 1 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & 2 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array},$$

□

Lemma 4.8. *The partial operation*

$$\varphi : \begin{cases} (1,1) \mapsto 2 \\ (1,2) \mapsto 1 \\ (2,1) \mapsto 1 \\ (2,2) \mapsto 2 \end{cases}$$

is a local polymorphism of the groupoids (176), (198), (306), (320), (563), (756).

Proof. Notice that the listed groupoids have their basic operation defined on $\{1,2\} \times \{1,2\}$ as

$$\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 2, \end{array} \quad \text{or} \quad \begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 2. \\ 2 & 2 & 2 \end{array}$$

Therefore, $\{1,2\} \times \{1,2\}$ is a subalgebra of G^2 for each listed groupoid G. The following tables show that φ is a local polymorphism of the groupoids from the first case:

$$\begin{array}{rccc} & & \varphi & \\ & 1\;1 & \longrightarrow & 2 \\ & 1\;1 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;2 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 1\;1 & \longrightarrow & 2 \\ & 1\;2 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;1 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 1\;1 & \longrightarrow & 2 \\ & 2\;1 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;2 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 1\;1 & \longrightarrow & 2 \\ & 2\;2 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;1 & \longrightarrow & 2=2 \end{array},$$

$$\begin{array}{rccc} & & \varphi & \\ & 1\;2 & \longrightarrow & 1 \\ & 1\;1 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;1 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 1\;2 & \longrightarrow & 1 \\ & 1\;2 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;2 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 1\;2 & \longrightarrow & 1 \\ & 2\;1 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;1 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 1\;2 & \longrightarrow & 1 \\ & 2\;2 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;2 & \longrightarrow & 1=1 \end{array},$$

$$\begin{array}{rccc} & & \varphi & \\ & 2\;1 & \longrightarrow & 1 \\ & 1\;1 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;2 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 2\;1 & \longrightarrow & 1 \\ & 1\;2 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;1 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 2\;1 & \longrightarrow & 1 \\ & 2\;1 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;2 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 2\;1 & \longrightarrow & 1 \\ & 2\;2 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;1 & \longrightarrow & 1=1 \end{array},$$

$$\begin{array}{rccc} & & \varphi & \\ & 2\;2 & \longrightarrow & 2 \\ & 1\;1 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;1 & \longrightarrow & 2=2 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 2\;2 & \longrightarrow & 2 \\ & 1\;2 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 1\;2 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 2\;2 & \longrightarrow & 2 \\ & 2\;1 & \longrightarrow & 1 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;1 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{rccc} & & \varphi & \\ & 2\;2 & \longrightarrow & 2 \\ & 2\;2 & \longrightarrow & 2 \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;2 & \longrightarrow & 2=2 \end{array}.$$

For the second case, since $*$ is the constant 2 operation on $\{1,2\}\times\{1,2\}$ we have

$$\begin{array}{rccc} & & \varphi & \\ & a\;b & \longrightarrow & x \\ & c\;d & \longrightarrow & y \\ * & \downarrow\;\downarrow & & \downarrow \\ & 2\;2 & \longrightarrow & 2=2 \end{array} \qquad \text{for all } a,b,c,d\in\{1,2\} \text{ (and thus } x,y\in\{1,2\}). \qquad \square$$

Lemma 4.9. *The partial operation*

$$\varphi:\begin{cases}(0,0)\mapsto 0\\(1,1)\mapsto 1\\(1,2)\mapsto 2\\(2,1)\mapsto 2\\(2,2)\mapsto 2\end{cases}$$

is a local polymorphism of the groupoids (216), (284), (1708), (1837), (2088), (2472), (2558).

Proof. Notice that the listed groupoids have their basic operation defined on $\{1,2\}\times\{1,2\}$ as

$$\begin{array}{c|cc} * & 1 & 2\\ \hline 1 & 0 & 0\\ 2 & 0 & 0, \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 1 & 2\\ \hline 1 & 1 & 2.\\ 2 & 2 & 2 \end{array}$$

Hence, to prove that $S=\{(0,0),(1,1),(1,2),(2,1),(2,2)\}$ is a subalgebra of G^2 for all listed groupoids G, we only need to consider the products in the form $(0,0)*(z_1,z_2)$ and $(z_1,z_2)*(0,0)$, where $z_1,z_2\in\{1,2\}$ and $z_1\neq z_2$. Then we can prove that S is a subalgebra by proving that we can not obtain $(0,1),(0,2),(1,0)$ or $(2,0)$ from the products $(0,0)*(z_1,z_2)$ and $(z_1,z_2)*(0,0)$. Note that all of these tuples contain exactly one 0. Therefore, we could only get one of $(0,1),(0,2),(1,0)$ or $(2,0)$ with a product $(0,0)*(z_1,z_2)$ or $(z_1,z_2)*(0,0)$, if one of $0*1, 0*2, 1*0$ or $2*0$ is 0 and $0*1\neq 0*2$ or $1*0\neq 2*0$. This is not the case for any of the groupoids, i.e., either $0*1=0*2=0$, or $0*1\neq 0\neq 0*2$ (and similarly neither $1*0=2*0=0$, or $1*0\neq 0\neq 2*0$). Therefore, S is a subalgebra of G^2 for all listed groupoids G.

Now to prove that φ is a local polymorphism of the groupoids, consider the following tables with $a,b,c,d \in \{1,2\}$ (and thus $x,y \in \{1,2\}$) for the given groupoids:

$$(216):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & a & b & \longrightarrow & x{=}x \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array};$$

$$(284):\quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & d & \longrightarrow & y{=}y \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 2 & \longrightarrow & 2 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 1 & \longrightarrow & 2 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & d & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2{=}2 \end{array},$$

$$\begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array};$$

$$(1708):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & a & b & \longrightarrow & x{=}x \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array};$$

$$(1837):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array};$$

$$(2088):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & d & \longrightarrow & y{=}y \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & a & b & \longrightarrow & x{=}x \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array};$$

$$(2472):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & a & b & \longrightarrow & x{=}x \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array};$$

$$(2558):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2{=}2 \end{array}, \quad \begin{array}{ccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}1 \end{array}.$$

□

Lemma 4.10. *The partial operation*

$$\varphi : \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 2 \end{cases}$$

is a local polymorphism of the groupoids (219), (222), (2090) .

Proof. Notice that the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

(219), (222)	0	1
0	0	0
1	1	0,

(2090)	0	1
0	0	1.
1	1	0

Therefore, $\{0,1\}$ is a subalgebra of G for each listed groupoid G. The following tables show that φ is a local polymorphism of the given groupoid(s):

$$\text{(219), (222):}\quad \begin{array}{ccc} 0 & \xrightarrow{\varphi} & 0 \\ 0 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 0{=}0 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\varphi} & 0 \\ 1 & \longrightarrow & 2 \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 0{=}0 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\varphi} & 2 \\ 0 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ 1 & \longrightarrow & 2{=}2 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\varphi} & 2 \\ 1 & \longrightarrow & 2 \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 0{=}0 \end{array}$$

$$\text{(2090):}\quad \begin{array}{ccc} 0 & \xrightarrow{\varphi} & 0 \\ 0 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 0{=}0 \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{\varphi} & 0 \\ 1 & \longrightarrow & 2 \\ *\downarrow & & \downarrow \\ 1 & \longrightarrow & 2{=}2 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\varphi} & 2 \\ 0 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ 1 & \longrightarrow & 2{=}2 \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{\varphi} & 2 \\ 1 & \longrightarrow & 2 \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 0{=}0 \end{array}$$

□

Lemma 4.11. *The partial operation*

$$\varphi : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$$

is a local polymorphism of the groupoids (257), (258), (259), (260), (261), (262), (263), (265), (266), (267), (268), (269), (270), (271), (272), (274), (278), (282), (677), (678), (679), (680), (682), (684), (687), (690), (691), (693), (695), (696), (697), (698), (704), (705), (707), (710), (712), (1271), (1277), (1281), (2460), (2461), (2462), (2463), (2464), (2467), (2476), (2478), (2479), (2480), (2483), (2486), (2487), (2493), (2739).

Proof. Notice that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

*	0	1
0	0	0
1	1	1,

or

*	0	1
0	1	0
1	1	0,

Therefore, $\{0,1\}$ is a subalgebra of G for each listed groupoid G. It is easy to see that φ is an automorphism on $(\{0,1\};*)$ for both cases, hence it is a partial polymorphism of each groupoid. □

Lemma 4.12. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 1 \\ (1,1) \mapsto 0 \end{cases}$$

is a local polymorphism of the groupoids (273), (681), (2116).

Proof. Notice that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

(273), (681)	0	1
0	0	0
1	1	1,

or

(2116)	0	1
0	0	1
1	1	0.

Therefore $\{0,1\} \times \{0,1\}$ is a subalgebra of G^2 for each listed groupoid G. To prove that φ is a local polymorphism of the groupoids of the first case (i.e., groupoids (273) and (681)), consider the following tables: (here $a, b, c, d \in \{0,1\}$ and thus $x, y \in \{0,1\}$)

$$\begin{array}{rccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0 \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 0 & 1 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 1 & \longrightarrow & 1{=}1 \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 1 & 0 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1 \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 0{=}0 \end{array}\quad .$$

Then for the other case (i.e., groupoid (2116)) let $\tau : \{0,1\} \to \{0,1\}$ with $\tau(0) = 1$ and $\tau(1) = 0$. Using that $\varphi(\tau(z_1), z_2) = \varphi(z_1, \tau(z_2)) = \tau(\varphi(z_1, z_2))$ and that $\varphi(\tau(z_1), (\tau(z_2)) = \varphi(z_1, z_2)$ for all $z_1, z_2 \in \{0,1\}$, the following tables prove that φ is a local polymorphism:

$$\begin{array}{rccccc} & & & \varphi & \\ & 0 & 0 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & d & \longrightarrow & y{=}y \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 0 & 1 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & \tau(d) & \longrightarrow & \tau(y){=}\tau(y) \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 1 & 0 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & \tau(c) & d & \longrightarrow & \tau(y){=}\tau(y) \end{array}\quad ,$$

$$\begin{array}{rccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 0 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & \tau(c) & \tau(d) & \longrightarrow & y{=}y \end{array}\quad .$$

□

Lemma 4.13. *The partial operation*

$$\varphi : \begin{cases} (1,1) \mapsto 1 \\ (1,2) \mapsto 2 \\ (2,1) \mapsto 2 \\ (2,2) \mapsto 1 \end{cases}$$

is a local polymorphism of the groupoids (283), (287), (353), (356), (359).

Proof. Notice that all listed groupoids satisfy $1*1, 1*2, 2*1, 2*2 \in \{1,2\}$. Therefore, $\{1,2\} \times \{1,2\}$ is a subalgebra of G^2 for each listed groupoid G. Let $\tau : \{1,2\} \to \{1,2\}$ with $\tau(1) = 2$ and $\tau(2) = 1$. Using that $\varphi(\tau(z_1), z_2) = \varphi(z_1, \tau(z_2)) = \tau(\varphi(z_1, z_2))$ and that $\varphi(\tau(z_1), (\tau(z_2)) = \varphi(z_1, z_2)$ for all $z_1, z_2 \in \{1,2\}$, the following tables show that φ is a local polymorphism of the given groupoids (here $a, b, c, d \in \{1,2\}$, and thus $\{x, y\} \in \{1,2\}$):

(283):

$$\begin{array}{rccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & d & \longrightarrow & y{=}y \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 1 & 2 & \longrightarrow & 2 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & \tau(d) & \longrightarrow & \tau(y){=}\tau(y) \end{array}\quad , \quad \begin{array}{rccccc} & & & \varphi & \\ & 2 & 1 & \longrightarrow & 2 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & \tau(c) & d & \longrightarrow & \tau(y){=}\tau(y) \end{array}\quad ,$$

$$\begin{array}{rccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & \tau(c) & \tau(d) & \longrightarrow & y{=}y \end{array}\quad ;$$

(287), (359):

$$\begin{array}{rccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & d & \longrightarrow & y{=}y \end{array}\quad ;$$

$$(353):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1=1 \end{array} ;$$

$$(356):\quad \begin{array}{ccccc} & & & \varphi & \\ & a & b & \longrightarrow & x \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & a & b & \longrightarrow & x=x \end{array} ;$$

□

Lemma 4.14. *The partial operation*

$$\varphi : \begin{cases} (1,1) \mapsto 1 \\ (1,2) \mapsto 1 \\ (2,1) \mapsto 1 \\ (2,2) \mapsto 2 \end{cases}$$

is a local polymorphism of the groupoid (354).

Proof. The groupoids basic operation defined on $\{1,2\} \times \{1,2\}$ is

*	1	2
1	1	1
2	1	2.

Therefore, $\{1,2\} \times \{1,2\}$ is a subalgebra of G^2 for the given groupoid G. The following tables show that φ is a local polymorphism of G (here $c, d \in \{1,2\}$ and thus $y \in \{1,2\}$):

$$\begin{array}{ccccc} & & & \varphi & \\ & 1 & 1 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1=1 \end{array} , \quad \begin{array}{ccccc} & & & \varphi & \\ & 1 & 2 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & d & \longrightarrow & 1=1 \end{array} , \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 1 & \longrightarrow & 1 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & 1 & \longrightarrow & 1=1 \end{array} , \quad \begin{array}{ccccc} & & & \varphi & \\ & 2 & 2 & \longrightarrow & 2 \\ & c & d & \longrightarrow & y \\ * & \downarrow & \downarrow & & \downarrow \\ & c & d & \longrightarrow & y=y \end{array} .$$

□

Lemma 4.15. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 2 \\ (2,0) \mapsto 2 \\ (2,2) \mapsto 0 \end{cases}$$

is a local polymorphism of the groupoids (1012), (1084), (1151), (1153), (1176), (1200), (1219), (1221), (1242), (1321), (1433), (1437), (1481), (2102), (2104).

Proof. Notice that the listed groupoids have their basic operation defined on $\{0,2\} \times \{0,2\}$ as

*	0	2
0	0	2
2	2	0.

Therefore, $\{0,2\} \times \{0,2\}$ is a subalgebra of G^2 whenever G is either groupoid. Let $\tau : \{0,2\} \to \{0,2\}$ with $\tau(0) = 2$ and $\tau(2) = 0$. Using that $\varphi(\tau(z_1), z_2) = \varphi(z_1, \tau(z_2)) = \tau(\varphi(z_1, z_2))$ and that $\varphi(\tau(z_1), (\tau(z_2)) = \varphi(z_1, z_2)$ for all $z_1, z_2 \in \{0,2\}$, the following tables show that φ is a local polymorphism of each groupoid:

$$\begin{array}{cccc} & & \varphi & \\ 0 & 0 & \longrightarrow & 0 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ c & d & \longrightarrow & y=y \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 0 & 2 & \longrightarrow & 2 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ c & \tau(d) & \longrightarrow & \tau(y)=\tau(y) \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 2 & 0 & \longrightarrow & 2 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ \tau(c) & d & \longrightarrow & \tau(y)=\tau(y) \end{array},$$

$$\begin{array}{cccc} & & \varphi & \\ 2 & 2 & \longrightarrow & 0 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ \tau(c) & \tau(d) & \longrightarrow & y=y \end{array},$$

□

Lemma 4.16. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 0 \\ (1,1) \mapsto 1 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$$

is a local polymorphism of the groupoids (2654), (2686), (2698), (2702).

Proof. Notice that the listed groupoids have their basic operation defined on $\{0,2\} \times \{0,2\}$ as

*	0	2
0	1	1
2	1	1.

Hence, to prove that $S = \{(0,0),(0,2),(1,1),(2,0),(2,2)\}$ is a subalgebra of G^2 for all listed groupoids G, we only need to consider the products $(1,1)*(z_1,z_2)$ and $(z_1,z_2)*(1,1)$ for all $z_1,z_2 \in \{0,2\}$. Then we can prove that S is a subalgebra by proving that we can not obtain $(0,1),(1,0),(1,2)$ or $(2,1)$ from the products $(1,1)*(z_1,z_2)$ and $(z_1,z_2)*(1,1)$. Note that all of these tuples contain exactly one 1. Therefore, we could only get one of $(0,1),(1,0),(1,2)$ or $(2,1)$ with a product $(1,1)*(z_1,z_2)$ or $(z_1,z_2)*(1,1)$, if one of $1*0, 1*2, 0*1$ or $2*1$ is 1 and $1*0 \neq 1*2$ or $0*1 \neq 2*1$. This is not the case for any of the groupoids, since we have $1*0 \neq 1 \neq 1*2$ (and similarly $0*1 \neq 1 \neq 2*1$). Therefore, S is a subalgebra of G^2 for all listed groupoids G. Now to prove that φ is a local polymorphism of the groupoids, consider the following tables with $a,b,c,d \in \{0,2\}$ (and thus $x,y \in \{0,2\}$) for the given groupoids:

(2654): $$\begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ a & b & \longrightarrow & x=x \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0=0 \end{array};$$

(2686): $$\begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ c & d & \longrightarrow & y=y \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ a & b & \longrightarrow & x=x \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0=0 \end{array};$$

(2698): $$\begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 1=1 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0=0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 2 & 2 & \longrightarrow & 2=2 \end{array};$$

$$(2702):\quad \begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 1{=}1 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ c & d & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}0 \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ a & b & \longrightarrow & x \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ a & b & \longrightarrow & x{=}x \end{array}, \quad \begin{array}{cccc} & & \varphi & \\ 1 & 1 & \longrightarrow & 1 \\ 1 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 2 & 2 & \longrightarrow & 2{=}2 \end{array};$$

□

5. Proving that the partial operations given in Table 1 are non-extendable

In this section we prove that the partial operations given in Table 1 in Appendix A are non-extendable local polymorphisms of the listed groupoids. The operation tables for the groupoids can be found in Appendix B.

Lemma 5.1. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 1 \\ (1,1) \mapsto 1 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (2), (4), (5), (6), (8), (10), (11), (12), (13), (15), (16), (18), (22), (26), (30), (31), (32), (33), (34), (35), (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46), (48), (49), (51), (52), (53), (59), (61), (63), (65), (66), (67), (69), (73), (376), (377), (378), (379), (380), (381), (382), (384), (385), (387), (388), (390), (391), (405), (407), (410), (417), (434), (436), (437), (439), (1014), (1038), (1040), (1066).

Proof. As in the proof of Lemma 4.1, it helps to observe that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

*	0	1
0	0	0
1	0	0

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Let us suppose first, that the groupoid satisfies $2*2 = 1$. Then, by the following tables we must have $\hat{\varphi}(2,2) = \hat{\varphi}(0,2) = \hat{\varphi}(2,0) = \hat{\varphi}(1,2) = \hat{\varphi}(2,1) = 2$ respectively:

$$\begin{array}{cccc} & & \hat{\varphi} & \\ 2 & 2 & \longrightarrow & x_1 \\ 2 & 2 & \longrightarrow & x_1 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 1{=}x_1*x_1 \end{array}, \quad \begin{array}{cccc} & & \hat{\varphi} & \\ 0 & 2 & \longrightarrow & x_2 \\ 0 & 2 & \longrightarrow & x_2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 1 & \longrightarrow & 1{=}x_2*x_2 \end{array}, \quad \begin{array}{cccc} & & \hat{\varphi} & \\ 2 & 0 & \longrightarrow & x_3 \\ 2 & 0 & \longrightarrow & x_3 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}x_3*x_3 \end{array}, \quad \begin{array}{cccc} & & \hat{\varphi} & \\ 1 & 2 & \longrightarrow & x_4 \\ 1 & 2 & \longrightarrow & x_4 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 1 & \longrightarrow & 1{=}x_4*x_4 \end{array},$$

$$\begin{array}{cccc} & & \hat{\varphi} & \\ 2 & 1 & \longrightarrow & x_5 \\ 2 & 1 & \longrightarrow & x_5 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}x_5*x_5 \end{array}.$$

Then if a groupoid also satisfies any of the conditions $0*2 = 2,\ 1*2 = 2, 2*0 = 2,\ 2*1 = 2,\ 0*2 = 2*0 = 0,\ 0*2 = 2*1 = 0,\ 1*2 = 2*0 = 0,\ 1*2 = 2*1 = 0$ we get a contradiction: $\begin{array}{cccc} a & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & 0 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 2 & v & \longrightarrow & 2 \neq 1 \end{array}$, $\begin{array}{cccc} 0 & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & d & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ u & 2 & \longrightarrow & 2 \neq 1 \end{array}$, $\begin{array}{cccc} 0 & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & 0 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0 \neq 1 \end{array}$,

$$\begin{array}{cccc} 0 & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & 1 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0 \neq 1 \end{array} , \quad \begin{array}{cccc} 1 & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & 0 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0 \neq 1 \end{array} , \quad \begin{array}{cccc} 1 & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & 1 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0 \neq 1 \end{array} .$$

(Here $a, d \in \{0, 1\}$ and $u, v \in \{0, 1, 2\}$, and the tables are listed respectively to the conditions.)

This proves that for groupoids (2), (5), (8), (11), (26), (31), (34), (37), (43), (46), (49), (52), (59), (65), (377) and (1066) φ is non-extendable.

Now for the next case let us suppose that the investigated groupoid satisfies $2 * 2 = 0$. Let us also suppose that $0 * 2 = 1$. Then we have $\hat{\varphi}(2, 2) = \hat{\varphi}(0, 2) = 2$:

$$\begin{array}{cccc} 0 & 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 2 & 2 & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 1{=}0 * y \end{array} , \quad \begin{array}{cccc} 0 & 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 0 & 2 & \longrightarrow & y \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 1 & \longrightarrow & 1{=}0 * y \end{array} .$$

But now we have a contradiction: $\begin{array}{cccc} 0 & 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2 & 2 & \longrightarrow & 2 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1 \neq 0 \end{array}$.

The case where the groupoid satisfies $2 * 0 = 1$ (and $2 * 2 = 0$) can be checked similarly (we just need to interchange the rows in the tables above). We can also get a contradiction for the groupoids that satisfy $1 * 2 = 1$ (and $2 * 2 = 0$), or $2 * 1 = 1$ (and $2 * 2 = 0$) with the same reasoning as above (just change the 0-s to 1-s). These cases together give us that for groupoids (4), (10), (13), (16), (22), (30), (33), (36), (39), (42), (45), (48), (51), (61), (67), (73), (376), (379), (382), (385), (388), (391), (405), (417), (434), (437), (1014) and (1038) φ is non-extendable.

Now suppose that the investigated groupoid satisfies $2 * 2 = 2$. Then if a groupoid also satisfies any of the conditions

- $0 * 2 = 1$ and $2 * 0 \in \{0, 1\}$;
- $0 * 2 = 1$ and $2 * 1 \in \{0, 1\}$;
- $1 * 2 = 1$ and $2 * 0 \in \{0, 1\}$;
- $1 * 2 = 1$ and $2 * 1 \in \{0, 1\}$;
- $2 * 0 = 1$ and $0 * 2 \in \{0, 1\}$;
- $2 * 0 = 1$ and $1 * 2 \in \{0, 1\}$;
- $2 * 1 = 1$ and $0 * 2 \in \{0, 1\}$;
- $2 * 1 = 1$ and $1 * 2 \in \{0, 1\}$

we must have $\hat{\varphi}(2, 2) = \hat{\varphi}(0, 2) = \hat{\varphi}(2, 0) = \hat{\varphi}(1, 2) = \hat{\varphi}(2, 1) = 2$. We prove this for the first condition, the others can be checked the same way (with interchanging the rows in the tables, or changing all 0-s to 1-s). $\begin{array}{rcl} 0\ 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 2\ 2 & \longrightarrow & y \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ 1 & \longrightarrow & 1{=}0*y \end{array}$, $\begin{array}{rcl} 0\ 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 0\ 2 & \longrightarrow & y \\ *\downarrow\ \downarrow & & \downarrow \\ 0\ 1 & \longrightarrow & 1{=}0*y \end{array}$,

$$\begin{array}{rcl} 0\ 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 2\ 0 & \longrightarrow & y \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ 0 & \longrightarrow & 1{=}0*y \end{array}, \quad \begin{array}{rcl} 0\ 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 1\ 2 & \longrightarrow & y \\ *\downarrow\ \downarrow & & \downarrow \\ 0\ 1 & \longrightarrow & 1{=}0*y \end{array}, \quad \begin{array}{rcl} 0\ 0 & \xrightarrow{\hat{\varphi}} & 0 \\ 2\ 1 & \longrightarrow & y \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ 0 & \longrightarrow & 1{=}0*y \end{array}.$$

But then we have a contradiction (respectively to the 8 cases above):

$$\begin{array}{rcl} 0\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2\ 0 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array}, \quad \begin{array}{rcl} 0\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2\ 1 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array}, \quad \begin{array}{rcl} 1\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2\ 0 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array}, \quad \begin{array}{rcl} 1\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ 2\ 1 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array},$$

$$\begin{array}{rcl} 2\ 0 & \xrightarrow{\hat{\varphi}} & 2 \\ 0\ 2 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array}, \quad \begin{array}{rcl} 2\ 1 & \xrightarrow{\hat{\varphi}} & 2 \\ 0\ 2 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array}, \quad \begin{array}{rcl} 2\ 0 & \xrightarrow{\hat{\varphi}} & 2 \\ 1\ 2 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array}, \quad \begin{array}{rcl} 2\ 1 & \xrightarrow{\hat{\varphi}} & 2 \\ 1\ 2 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & 1\neq 2 \end{array},$$

where $v \in \{0,1\}$. These cases together give us that for groupoids (6), (12), (15), (18), (32), (35), (38), (41), (44), (63), (66), (69), (378), (381), (384), (387), (390), (407), (410), (436), (439) and (1040) φ is non-extendable.

It only remains to prove the lemma for groupoids (40), (53) and (380). Groupoid (40) and (380) satisfy $2*2=1$, and groupoid (53) satisfies $2*2=2$. For groupoid (40) and (380) we can use that $\hat{\varphi}(2,1)=2$ must hold (see the beginning of the proof of this lemma) to get a contradiction: $\begin{array}{rcl} (40) & \hat{\varphi} & \\ 0\ 1 & \longrightarrow & 1 \\ 2\ 1 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 0\ 0 & \longrightarrow & 0\neq 1 \end{array}$, $\begin{array}{rcl} (380) & \hat{\varphi} & \\ 1\ 1 & \longrightarrow & 1 \\ 2\ 1 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 0\ 0 & \longrightarrow & 0\neq 1 \end{array}$.

For groupoid (53), note that $\hat{\varphi}(1,2)=2$ must hold: $\begin{array}{rcl} 1\ 1 & \xrightarrow{\hat{\varphi}} & 1 \\ 1\ 2 & \longrightarrow & y \\ *\downarrow\ \downarrow & & \downarrow \\ 0\ 1 & \longrightarrow & 1{=}1*y \end{array}$, and then by the following table we have our contradiction: $\begin{array}{rcl} 1\ 0 & \xrightarrow{\hat{\varphi}} & 1 \\ 1\ 2 & \longrightarrow & 2 \\ *\downarrow\ \downarrow & & \downarrow \\ 0\ 0 & \longrightarrow & 0\neq 1 \end{array}$. □

Lemma 5.2. *The partial operation*

$$\varphi : \begin{cases} 0 \mapsto 2 \\ 2 \mapsto 0 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (21), (24), (47), (50), (72), (75), (78), (96), (99), (119), (141), (144), (162), (165), (168), (182), (185), (188), (201), (204), (218), (221), (235), (252) (255), (1227), (1233).

Proof. As in the proof of Lemma 4.2, it helps to observe that all of the listed groupoids have their basic operation defined on $\{0,2\}\times\{0,2\}$ as

$$\begin{array}{c|cc} * & 0 & 2\\ \hline 0 & 0 & 0\\ 2 & 2 & 2 \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 0 & 2\\ \hline 0 & 0 & 2\\ 2 & 0 & 2. \end{array}$$

Let us suppose that for each groupoid there exists an extension $\hat\varphi$ that is a total polymorphism of the groupoid. Let us suppose first, that the groupoid belongs to the first case, and that $1*1=0$. Then, by the following table we must have $\hat\varphi(1)=2$:

$$\begin{array}{rcl} & \hat\varphi & \\ 1 & \longrightarrow & \hat\varphi(1)\\ 1 & \longrightarrow & \hat\varphi(1)\\ *\downarrow & & \downarrow\\ 0 & \longrightarrow & 2=\hat\varphi(1)*\hat\varphi(1) \end{array}$$

. And if we have $2*1\neq 2$ or $1*2\notin\{0,1\}$, then we have a contradiction:

$$\begin{array}{rcl} & \hat\varphi & \\ 2 & \longrightarrow & 0\\ 1 & \longrightarrow & 2\\ *\downarrow & & \downarrow\\ u & \longrightarrow & \hat\varphi(u)=0 \end{array} \qquad \begin{array}{rcl} & \hat\varphi & \\ 1 & \longrightarrow & 2\\ 2 & \longrightarrow & 0\\ *\downarrow & & \downarrow\\ v & \longrightarrow & \hat\varphi(v)=2 \end{array}.$$

This proves that for groupoids (21), (24), (47), (50), (72), (75), (78), (218), (221), (235), (252), and (255) φ is non-extendable. Similarly, if the groupoid belongs to the first case and $1*1=2$, then by the following table we must have $\hat\varphi(1)=0$:

$$\begin{array}{rcl} & \hat\varphi & \\ 1 & \longrightarrow & \hat\varphi(1)\\ 1 & \longrightarrow & \hat\varphi(1)\\ *\downarrow & & \downarrow\\ 2 & \longrightarrow & 0=\hat\varphi(1)*\hat\varphi(1) \end{array}$$

, and if we have $2*1\neq 2$ or $1*2\neq 2$, then we have a contradiction:

$$\begin{array}{rcl} & \hat\varphi & \\ 2 & \longrightarrow & 0\\ 1 & \longrightarrow & 0\\ *\downarrow & & \downarrow\\ u & \longrightarrow & \hat\varphi(u)=0 \end{array} \qquad \begin{array}{rcl} & \hat\varphi & \\ 1 & \longrightarrow & 0\\ 2 & \longrightarrow & 0\\ *\downarrow & & \downarrow\\ v & \longrightarrow & \hat\varphi(v)=0 \end{array}.$$

This proves that for groupoids (162), (165), (168), (182), (185), (188), (201), and (204) φ is non-extendable.

For the remaining groupoids in the first case, note that all listed groupoids satisfy $0*0=0*1=0*2=0$. Using this, if $1*1=1$, then by the following table we must have $2*1\neq 0$:

$$\begin{array}{rcl} & \hat\varphi & \\ 2 & \longrightarrow & 0\\ 1 & \longrightarrow & \hat\varphi(1)\\ *\downarrow & & \downarrow\\ u & \longrightarrow & \hat\varphi(u)=0 \end{array}$$

. This condition rules out groupoids (96), (119) and (141). If $2*1=1$, then we must have $\hat\varphi(1)=0$:

$$\begin{array}{rcl} & \hat\varphi & \\ 2 & \longrightarrow & 0\\ 1 & \longrightarrow & \hat\varphi(1)\\ *\downarrow & & \downarrow\\ 1 & \longrightarrow & \hat\varphi(1)=0 \end{array}$$

But then we have a contradiction for groupoids (99) and (144):

$$\begin{array}{rcl} & \hat\varphi & \\ 1 & \longrightarrow & 0\\ 0 & \longrightarrow & 2\\ *\downarrow & & \downarrow\\ 0 & \longrightarrow & 2\neq 0 \end{array}$$

The last two groupoids (1227), (1233) belong to case two, the proof is similar to the beginning; both groupoids satisfy $1 * 1 = 0$. And by the following table we must have $\hat{\varphi}(1) = 2$:

$$\begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1 & \longrightarrow & \hat{\varphi}(1) \\ 1 & \longrightarrow & \hat{\varphi}(1) \\ * \downarrow & & \downarrow \\ 0 & \longrightarrow & 2=\hat{\varphi}(1)*\hat{\varphi}(1) \end{array}$$

. And if we have $1 * 2 \neq 2$, then we have a contradiction:

$$\begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1 & \longrightarrow & 2 \\ 2 & \longrightarrow & 0 \\ * \downarrow & & \downarrow \\ v & \longrightarrow & \hat{\varphi}(v)=0 \end{array}$$

. ☐

Lemma 5.3. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 0 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (60), (147).

Proof. Let us suppose that for both groupoids there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Let us consider groupoid (60) first. By the following two-two tables, we must have $\hat{\varphi}(1,2) = \hat{\varphi}(2,1) = 1$:

$$\begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1\ 2 & \longrightarrow & x \\ 2\ 2 & \longrightarrow & 2 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 2 & \longrightarrow & 2=x*2 \end{array}, \quad \begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1\ 2 & \longrightarrow & x \\ 1\ 2 & \longrightarrow & x \\ * \downarrow\ \downarrow & & \downarrow \\ 0\ 2 & \longrightarrow & 0=x*x \end{array}; \quad \begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 2\ 1 & \longrightarrow & x \\ 2\ 2 & \longrightarrow & 2 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 2 & \longrightarrow & 2=x*2 \end{array}, \quad \begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 2\ 1 & \longrightarrow & x \\ 2\ 1 & \longrightarrow & x \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 0 & \longrightarrow & 0=x*x \end{array}.$$

But then the table

$$\begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1\ 2 & \longrightarrow & 1 \\ 2\ 1 & \longrightarrow & 1 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 2 & \longrightarrow & 2 \neq 0 \end{array}$$

gives us a contradiction.

Now let us consider groupoid (147). By the following two-two tables, we must have $\hat{\varphi}(1,2) = \hat{\varphi}(2,1) = 1$:

$$\begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1\ 2 & \longrightarrow & x \\ 2\ 2 & \longrightarrow & 2 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 2 & \longrightarrow & 2=x*2 \end{array}, \quad \begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1\ 2 & \longrightarrow & x \\ 0\ 0 & \longrightarrow & 0 \\ * \downarrow\ \downarrow & & \downarrow \\ 0\ 2 & \longrightarrow & 0=x*0 \end{array}; \quad \begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 2\ 1 & \longrightarrow & x \\ 2\ 2 & \longrightarrow & 2 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 2 & \longrightarrow & 2=x*2 \end{array}, \quad \begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 2\ 1 & \longrightarrow & x \\ 0\ 0 & \longrightarrow & 0 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 0 & \longrightarrow & 0=x*0 \end{array}.$$

But then the table

$$\begin{array}{rcl} & \overset{\hat{\varphi}}{} & \\ 1\ 2 & \longrightarrow & 1 \\ 2\ 1 & \longrightarrow & 1 \\ * \downarrow\ \downarrow & & \downarrow \\ 2\ 2 & \longrightarrow & 2 \neq 1 \end{array}$$

gives us a contradiction. ☐

Lemma 5.4. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 0 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (79), (81), (82), (83), (85), (87), (88), (89), (90), (91), (93), (97), (101), (111), (112), (113), (115), (117), (120), (121), (122), (123), (124), (125), (130), (132), (134), (136), (137), (138), (142), (143), (406), (454), (455), (456), (457), (458), (459), (460), (462), (463), (465), (469), (483), (484), (485), (487), (493), (494), (495), (496), (512), (513), (514), (515), (517), (519), (522), (1086), (1107), (1108), (1133).

Proof. As in the proof of Lemma 4.4, it helps to observe that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

$$\begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Only groupoid (406) belongs to the first case, we deal with it first. By the following two tables we must have $\hat{\varphi}(2,0), \hat{\varphi}(0,2) \in \{0,1\}$:

$$\begin{array}{rccl} & & \hat\varphi & \\ & 2\ 0 & \longrightarrow & x \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 0{=}x*0 \end{array}, \quad \begin{array}{rccl} & & \hat\varphi & \\ & 0\ 2 & \longrightarrow & x \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 0{=}x*0 \end{array}$$

, but then by the next table we get a contradiction:

$$\begin{array}{rccl} & & \hat\varphi & \\ & 2\ 0 & \longrightarrow & 0/1 \\ & 0\ 2 & \longrightarrow & 0/1 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1 \neq 0 \end{array}.$$

Now we deal with all the other groupoids. Suppose first that we have $2*2=1$. We investigate what the value of $\hat{\varphi}(1,2)$ can be.

Firstly, $\hat{\varphi}(1,2) \neq 0$, since then we have a contradiction:

$$\begin{array}{rccl} & & \hat\varphi & \\ & 1\ 2 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 0 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1 \neq 0 \end{array}.$$

If $\hat{\varphi}(1,2) = 1$, then we must have $0*2 = 2*0 = 0$ and the groupoid can not satisfy the equalities $1*2=0$ and $2*1=0$:

$$\begin{array}{rccl} & & \hat\varphi & \\ & 1\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 1 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat\varphi(1,v){=}0 \end{array}, \quad \begin{array}{rccl} & & \hat\varphi & \\ & 1\ 2 & \longrightarrow & 1 \\ & 1\ 0 & \longrightarrow & 0 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat\varphi(1,v){=}0 \end{array}, \quad \begin{array}{rccl} & & \hat\varphi & \\ & 1\ 1 & \longrightarrow & 1 \\ & 1\ 2 & \longrightarrow & 1 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat\varphi(1,v){=}1 \end{array}, \quad \begin{array}{rccl} & & \hat\varphi & \\ & 1\ 2 & \longrightarrow & 1 \\ & 1\ 1 & \longrightarrow & 1 \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat\varphi(1,v){=}1 \end{array}.$$

If $\hat{\varphi}(1,2) = 2$, then using that $\hat{\varphi}(0,2) = \hat{\varphi}(2,0) = 0$ stands by the tables

$$\begin{array}{rccl} & & \hat\varphi & \\ & 0\ 2 & \longrightarrow & x \\ & 0\ 2 & \longrightarrow & x \\ * & \downarrow\downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 0{=}x*x \end{array}, \quad \begin{array}{rccl} & & \hat\varphi & \\ & 2\ 0 & \longrightarrow & x \\ & 2\ 0 & \longrightarrow & x \\ * & \downarrow\downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 0{=}x*x \end{array}$$

, we must have $0*2 = 2*0 = 0$:

$$\begin{array}{rccl} & & \hat\varphi & \\ & 0\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 2 \\ * & \downarrow\downarrow & & \downarrow \\ & 0\ v & \longrightarrow & 0{=}0*2 \end{array}, \quad \begin{array}{rccl} & & \hat\varphi & \\ & 1\ 2 & \longrightarrow & 2 \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\downarrow & & \downarrow \\ & 0\ v & \longrightarrow & 0{=}2*0 \end{array}.$$

We show that if $\hat{\varphi}(1,2) = 2$ the groupoids can not satisfy the equalities $1*2=0$ and $2*1=0$. According to the following two tables, if $1*2=0$ we must have $\hat{\varphi}(2,2)=2$.

$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 2\ 2 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 0 & \longrightarrow & 0{=}1*x \end{array}\quad , \quad \begin{array}{cccc} & & \hat{\varphi} & \\ & 2\ 2 & \longrightarrow & x \\ & 2\ 2 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1{=}x*x \end{array}\quad .$$

But then by the following table we have a contradiction:
$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 2\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 0 \neq 1 \end{array}\quad .$$
We can show that the groupoid can not satisfy $2*1 = 0$ the same way (just interchange the rows of the previous three tables above).

Sofar we got that if φ is extendable and $2 * 2 = 1$, then the investigated groupoid must satisfy $0 * 2 = 2 * 0 = 0$ and it can not satisfy $1 * 2 = 0$ or $2 * 1 = 0$. This shows that for groupoids (79), (81), (83), (89), (101), (112), (115), (121), (124), (134), (137), (143), (455), (458), (484), (487), (493), (496), (513), (519), (522) and (1133) φ is non-extendable.

For the next case suppose that we have $2 * 2 = 0$. We again investigate what the value of $\hat{\varphi}(1, 2)$ can be. Firstly, $\hat{\varphi}(1, 2) \neq 1$, since then we have a contradiction:
$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 1 \\ & 1\ 2 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 0 \neq 1 \end{array}\quad .$$

If $\hat{\varphi}(1, 2) = 0$, then the groupoid can not satisfy the equalities $0 * 2 = 1$, $2 * 0 = 1$,$1 * 2 = 1$ and $2 * 1 = 1$:
$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v){=}0 \end{array} , \begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 0 \\ & 1\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v){=}0 \end{array} , \begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 1\ 2 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v){=}0 \end{array} , \begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 0 \\ & 1\ 1 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v){=}0 \end{array} .$$

If $\hat{\varphi}(1, 2) = 2$, then the groupoid can not satisfy the equalities $0*2 = 1$ and $2*0 = 1$:
$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 0\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ v & \longrightarrow & \hat{\varphi}(0,v){=}v \end{array} , \begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ v & \longrightarrow & \hat{\varphi}(0,v){=}v \end{array} .$$

We show that if $\hat{\varphi}(1, 2) = 2$ the groupoids can not satisfy the equalities $1 * 2 = 1$ and $2 * 1 = 1$ either. According to the following two tables, if $1 * 2 = 1$ we must have $\hat{\varphi}(2, 2) = 2$.
$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 2\ 2 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1{=}1*x \end{array}\quad , \quad \begin{array}{cccc} & & \hat{\varphi} & \\ & 2\ 2 & \longrightarrow & x \\ & 2\ 2 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 0 & \longrightarrow & 0{=}x*x \end{array}\quad .$$

But then by the following table we have a contradiction:
$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 2\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 0 \neq 1 \end{array}\quad .$$
We can show that the groupoid can not satisfy $2*1 = 1$ the same way (just interchange the rows of the previous three tables above).

In the case $2*2=0$ altogether we got that if φ is extendable then the investigated groupoid can not satisfy the equalities $0*2=1$, $2*0=1$,$1*2=1$ and $2*1=1$. This shows that for groupoids (85), (88), (91), (97), (111), (117), (120), (123), (130), (136), (142), (454), (457), (460), (463), (469), (483), (495), (512), (515), (1086), (1107) φ is non-extendable.

Lastly, suppose that we have $2*2=2$. We again investigate what the value of $\hat{\varphi}(1,2)$ can be. If $\hat{\varphi}(1,2)=0$, then the groupoid can not satisfy the equalities $0*2=1$ and $2*0=1$:

$$\begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v)=0 \end{array} , \quad \begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 0 \\ & 1\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v)=0 \end{array} .$$

If $\hat{\varphi}(1,2)=1$, then the groupoid can not satisfy the equalities $0*2=1$ and $2*0=1$:

$$\begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v)=0 \end{array} , \quad \begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 1 \\ & 1\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ v & \longrightarrow & \hat{\varphi}(1,v)=0 \end{array} .$$

If $\hat{\varphi}(1,2)=2$, then the groupoid can not satisfy the equalities $0*2=1$ and $2*0=1$:

$$\begin{array}{rccc} & & \hat{\varphi} & \\ & 0\ 0 & \longrightarrow & 0 \\ & 1\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ v & \longrightarrow & \hat{\varphi}(0,v)=v \end{array} , \quad \begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ v & \longrightarrow & \hat{\varphi}(0,v)=v \end{array} .$$

In the case $2*2=2$ so far we got that if φ is extendable then the investigated groupoid can not satisfy the equalities $0*2=1$ and $2*0=1$. This shows that for groupoids (87), (90), (93), (113), (132), (138), (456), (459), (462), (465), (485), (494), (514) and (517) φ is non-extendable.

We deal with the four remaining groupoids (82), (122), (125) and (1108). If the investigated groupoid satisfies $2*2=2$, $2*0=2$ and $1*2=1$, then we must have

$$\hat{\varphi}(2,0)=0 \text{ and } \hat{\varphi}(0,2)=0: \quad \begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 2\ 0 & \longrightarrow & y \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 0=1*y \end{array} , \quad \begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 0\ 2 & \longrightarrow & y \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 0=1*y \end{array} .$$

Then by the following two tables we also have $\hat{\varphi}(1,2)=2$:

$$\begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & x \\ & 2\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 2 & \longrightarrow & x=x*0 \end{array} , \quad \begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 1\ 2 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1=1*x \end{array} .$$

But then we have a contradiction:

$$\begin{array}{rccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 2 & \longrightarrow & 0 \neq 2 \end{array} .$$

This proves that for groupoids (122), (125) and (1108) φ is non-extendable.

And finally we deal with groupoid (82). If φ is extendable, then we must have $\hat{\varphi}(1,2)=\hat{\varphi}(2,1)=2$ by the following two-two tables:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & x \\ & 1 & 1 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}x*1 \end{array}, \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 1 & \longrightarrow & 1 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 0{=}1*x \end{array}; \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 1 & \longrightarrow & x \\ & 1 & 1 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}x*1 \end{array}, \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 1 & \longrightarrow & 1 \\ & 2 & 1 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 1 & \longrightarrow & 0{=}1*x \end{array}.$$

But then the next contradiction shows that φ is non-extendable for groupoid (82):

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & 2 \\ & 2 & 1 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 1 & \longrightarrow & 0 \neq 2 \end{array}.$$

□

Lemma 5.5. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 2 \\ (1,0) \mapsto 1 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (80), (100), (102), (239), (241), (1132).

Proof. Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. We will prove that such extension can not exists by showing that $\hat{\varphi}$ can not be defined on $(1,2)$. The following two-two tables immediately give us a contradiction for the given groupoids, because according to them $\hat{\varphi}(1,2)$ should be defined as both 1 and 2 at the same time (respectively):

(80): $$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & x \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}x*1 \end{array} \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & x \\ & 0 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}x*2 \end{array}$$

(100): $$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 0 & \longrightarrow & 1 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1*x \end{array} \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 0 & 2 & \longrightarrow & 2 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}2*x \end{array}$$

(102): $$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 0 & \longrightarrow & 1 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 0 & \longrightarrow & 1{=}1*x \end{array} \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & x \\ & 0 & 0 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}x*0 \end{array}$$

(239): $$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 0 & \longrightarrow & 1 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}1*x \end{array} \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 0 & 2 & \longrightarrow & 2 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}2*x \end{array}$$

(241): $$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 0 & \longrightarrow & 1 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}1*x \end{array} \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & x \\ & 1 & 0 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}x*1 \end{array}$$

$$
(1132):\quad
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 1\ 2 & \longrightarrow & x \\
 & 1\ 2 & \longrightarrow & x \\
* & \downarrow\ \downarrow & & \downarrow \\
 & 1\ 0 & \longrightarrow & 1{=}x*x
\end{array}
\qquad
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 1\ 2 & \longrightarrow & x \\
 & 0\ 2 & \longrightarrow & 2 \\
* & \downarrow\ \downarrow & & \downarrow \\
 & 0\ 0 & \longrightarrow & 0{=}x*2
\end{array}
$$

□

Lemma 5.6. *The partial operation*

$$
\varphi : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 1 \end{cases}
$$

is a non-extendable local polymorphism of the groupoids (116), (135), (178), (194), (203), (281), (308), (316), (488), (520), (565), (758), (780), (984), (1793), (1799), (1818), (1962), (2430), (2436), (2539), (2545), (2636).

Proof. As in the proof of Lemma 4.6, it helps to observe that all of the listed groupoids have their basic operation defined on $\{1,2\} \times \{1,2\}$ as one of the following

$$
\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 1 & 1 \\ 2 & 2 & 2, \end{array}
\qquad
\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 1 & 2 \\ 2 & 1 & 2, \end{array}
\qquad
\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 2 & 1, \end{array}
\qquad
\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 2 \\ 2 & 1 & 1. \end{array}
$$

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Consider first the groupoids in the first case. If the investigated groupoid satisfies $2*0 = 1$, then by the following table
$$
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 2 & \longrightarrow & 1 \\
 & 0 & \longrightarrow & x \\
* & \downarrow & & \downarrow \\
 & 1 & \longrightarrow & 2{=}1*x
\end{array}
$$
we must have $2 = 1*a$ for some $a \in \{0,1,2\}$, which gives a contradiction for groupoids (116) and (488).

Consider next the groupoids in the second case. If the investigated groupoid satisfies $0*2 = 1$, then by the following table
$$
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 0 & \longrightarrow & x \\
 & 2 & \longrightarrow & 1 \\
* & \downarrow & & \downarrow \\
 & 1 & \longrightarrow & 2{=}x*1
\end{array}
$$
we must have $2 = x*1$ for some $x \in \{0,1,2\}$, which gives a contradiction for groupoid (520). And for the other two groupoids in this case, (135) and (281), consider the tables (using that both groupoids satisfy $2*0 = 1$ and $0*2 = 0$)
$$
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 0 & \longrightarrow & x \\
 & 2 & \longrightarrow & 1 \\
* & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & x{=}x*1
\end{array}
\qquad
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 2 & \longrightarrow & 1 \\
 & 0 & \longrightarrow & x \\
* & \downarrow & & \downarrow \\
 & 1 & \longrightarrow & 2{=}1*x
\end{array}
$$
. But such $x \in \{0,1,2\}$ does not exists for these groupoids.

Lastly, consider the groupoids in the third and fourth case. If the investigated groupoid satisfies $0*0 = 0$, then by the following table we must have $\hat{\varphi}(0) = 0$:
$$
\begin{array}{rcccc}
 & & \hat{\varphi} & \\
 & 0 & \longrightarrow & x \\
 & 0 & \longrightarrow & x \\
* & \downarrow & & \downarrow \\
 & 0 & \longrightarrow & x{=}x*x
\end{array}
$$
. Then the investigated groupoid must satisfy either $0*1 = 0*2 = 0$

or $0*1, 0*2 \in \{1,2\}$ and $0*1 \neq 0*2$: $\begin{array}{ccccc} & & \hat{\varphi} & & \\ & 0 & \longrightarrow & 0 \\ & 1 & \longrightarrow & 2 \\ * & \downarrow & & \downarrow \\ & 0 & \longrightarrow & 0{=}0*2 \end{array}$ or $\begin{array}{ccccc} & & \hat{\varphi} & & \\ & 0 & \longrightarrow & 0 \\ & 1 & \longrightarrow & 2 \\ * & \downarrow & & \downarrow \\ & a & \longrightarrow & \tau(a){=}0*2 \end{array}$,
where $a \in \{1,2\}$, and $\tau(1) = 2$ and $\tau(2) = 1$. It can be shown similarly (by interchanging the rows in the tables above) that the groupoid needs to satisfy either $1*0 = 2*0 = 0$ or $1*0, 2*0 \in \{1,2\}$ and $1*0 \neq 2*0$. This proves that for groupoids (178), (194), (203), (308), (316), (565), (758), (780), (984), (1793), (1799), (1818) and (1962) φ is non-extendable. The remaining groupoids in these two cases all satisfy $0*0 = 1$. Then by the following table we must have $\hat{\varphi}(0) = 1$: $\begin{array}{cccc} & & \hat{\varphi} & \\ & 0 & \longrightarrow & x \\ & 0 & \longrightarrow & x \\ * & \downarrow & & \downarrow \\ & 1 & \longrightarrow & 2{=}x*x \end{array}$.
But then the groupoids in the third case should also satisfy $1*0 = 1$ by the table $\begin{array}{cccc} & & \hat{\varphi} & \\ & 1 & \longrightarrow & 2 \\ & 0 & \longrightarrow & 1 \\ * & \downarrow & & \downarrow \\ & u & \longrightarrow & \hat{\varphi}(u){=}2 \end{array}$, and groupoids in the last case should also satisfy $0*1 = 1$ by the table $\begin{array}{cccc} & & \hat{\varphi} & \\ & 0 & \longrightarrow & 1 \\ & 1 & \longrightarrow & 2 \\ * & \downarrow & & \downarrow \\ & u & \longrightarrow & \hat{\varphi}(u){=}2 \end{array}$
which gives us a contradiction for groupoids (2430), (2436), (2539). (2545) and (2636). ☐

Lemma 5.7. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 0 \\ (0,2) \mapsto 0 \\ (1,0) \mapsto 0 \\ (1,1) \mapsto 1 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (149), (175).

Proof. Note that the local polymorphism is not defined only on $(1,2)$ and $(2,1)$. Let us suppose that φ is extendable for both groupoids, let $\hat{\varphi}$ denote the extension. First we consider groupoid (149). By the following tables we must have $\hat{\varphi}(1,2) = 2$:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 2 & \longrightarrow & 2 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}2*x \end{array}, \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 1 & \longrightarrow & 1 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 0 & \longrightarrow & 0{=}1*x \end{array}, \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 0 & 0 & \longrightarrow & 0 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0{=}0*x \end{array}.$$

Similarly, the following tables show that we must have $\hat{\varphi}(2,1) = 2$ as well:

$$\begin{array}{ccccc} & 2\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ & 2\ 1 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1{=}2*x \end{array}, \quad \begin{array}{ccccc} & 1\ 1 & \xrightarrow{\hat{\varphi}} & 1 \\ & 2\ 1 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 2 & \longrightarrow & 0{=}1*x \end{array}, \quad \begin{array}{ccccc} & 0\ 0 & \xrightarrow{\hat{\varphi}} & 0 \\ & 2\ 1 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 0 & \longrightarrow & 0{=}0*x \end{array}$$

. But then the table $\begin{array}{ccccc} & 1\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ & 2\ 1 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 0 \neq 1 \end{array}$ gives us a contradiction.

We can prove that there is no extension of φ for groupoid (175) the same way: By the following tables we must have $\hat{\varphi}(1,2) = 1$:

$$\begin{array}{ccccc} & 1\ 2 & \xrightarrow{\hat{\varphi}} & x \\ & 2\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1{=}x*2 \end{array}, \quad \begin{array}{ccccc} & 1\ 2 & \xrightarrow{\hat{\varphi}} & x \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 0{=}x*0 \end{array}.$$

Similarly, the following tables show that we must have $\hat{\varphi}(2,1) = 1$ as well:

$$\begin{array}{ccccc} & 2\ 1 & \xrightarrow{\hat{\varphi}} & x \\ & 2\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1{=}x*2 \end{array}, \quad \begin{array}{ccccc} & 2\ 1 & \xrightarrow{\hat{\varphi}} & x \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 0{=}x*0 \end{array}$$

. But then the table $\begin{array}{ccccc} & 1\ 2 & \xrightarrow{\hat{\varphi}} & 1 \\ & 2\ 1 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1 \neq 2 \end{array}$ gives us a contradiction.

□

Lemma 5.8. *The partial operation*

$$\varphi : \begin{cases} (1,1) \mapsto 2 \\ (1,2) \mapsto 1 \\ (2,1) \mapsto 1 \\ (2,2) \mapsto 2 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (176), (198), (306), (320), (563), (756).

Proof. As in the proof of Lemma 4.8, it helps to observe that all of the listed groupoids have their basic operation defined on $\{1,2\} \times \{1,2\}$ as

$$\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 1 & 2, \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 2 & 2. \\ 2 & 2 & 2 \end{array}$$

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Note that all groupoids satisfy $0*0 = 0$, $0*1 = 0$ and $2*0 = 1$. Also note that We start with the groupoids in the first case. We must have $\hat{\varphi}(0,0) = 2$ for all of these groupoids: $\begin{array}{ccccc} & 2\ 2 & \xrightarrow{\hat{\varphi}} & 2 \\ & 0\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 2{=}2*x \end{array}$. But then the following two tables show that we can not define $\hat{\varphi}(0,1)$, as we would need to define it as both 0 and 2 (respectively): $\begin{array}{cccc} & & \hat{\varphi} & \\ & 0\ 1 & \longrightarrow & x \\ & 1\ 2 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & x{=}x*1 \end{array}$, $\begin{array}{cccc} & & \hat{\varphi} & \\ & 0\ 0 & \longrightarrow & 2 \\ & 0\ 1 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 0 & \longrightarrow & 2{=}2*x \end{array}$. This proves that for groupoids (176), (306), (563) and (756) φ is non-extendable. Two groupoids remain: (198) and (320). For groupoid (198) we can not define $\hat{\varphi}(0,2)$, there is no such x where the following table holds: $\begin{array}{cccc} & & \hat{\varphi} & \\ & 2\ 1 & \longrightarrow & 1 \\ & 0\ 2 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 2 & \longrightarrow & 1{=}1*x \end{array}$. And for groupoid (320) we can only define $\hat{\varphi}$ on $(0,2)$ as $\hat{\varphi}(0,2)=0$ (using the previous table). And we can only define $\hat{\varphi}$ on $(0,0)$ as $\hat{\varphi}(0,0)=2$, as the following two tables show: $\begin{array}{cccc} & & \hat{\varphi} & \\ & 0\ 0 & \longrightarrow & x \\ & 1\ 1 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 0 & \longrightarrow & x{=}x*2 \end{array}$, $\begin{array}{cccc} & & \hat{\varphi} & \\ & 2\ 2 & \longrightarrow & 2 \\ & 0\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 2{=}2*x \end{array}$. But then the table $\begin{array}{cccc} & & \hat{\varphi} & \\ & 0\ 0 & \longrightarrow & 2 \\ & 2\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 0 & \longrightarrow & 0 \neq 1 \end{array}$ gives a contradiction. □

Lemma 5.9. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (1,1) \mapsto 1 \\ (1,2) \mapsto 2 \\ (2,1) \mapsto 2 \\ (2,2) \mapsto 2 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (216), (284), (1708), (1837), (2088), (2472), (2558).

Proof. As in the proof of Lemma 4.9, it helps to observe that all of the listed groupoids have their basic operation defined on $\{1,2\} \times \{1,2\}$ as

$$\begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 0 & 0 \\ 2 & 0 & 0, \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 1 & 2 \\ \hline 1 & 1 & 2. \\ 2 & 2 & 2 \end{array}$$

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Note that all listed groupoids satisfy $2*0=2$. Suppose first that the investigated groupoid belongs to the first case and that it also satisfies $1*0=1$. Then by the following two tables we must either have $\hat{\varphi}(1,0)=0$ and $\hat{\varphi}(0,1)=1$, or $\hat{\varphi}(1,0)=1$ and $\hat{\varphi}(0,1)=0$. $\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 1\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & y{=}1*x \end{array}$, $\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 0\ 1 & \longrightarrow & y \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & x{=}1*y \end{array}$.

But in both cases we have a contradiction: $\begin{array}{cccc} & & \hat{\varphi} & \\ & 2\ 1 & \longrightarrow & 2 \\ & 1\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & 1 \neq 2 \end{array}$, $\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 0\ 1 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 1 \neq 2 \end{array}$.

This shows that for groupoids (216), (1708), (2088) and (2472) φ is non-extendable.

Next consider groupoid (1837). Since $0*1=1$ and $1*0=2$, by the following two tables we should have that $\hat\varphi(1,0)*\hat\varphi(0,1)=\hat\varphi(0,1)*\hat\varphi(1,0)=2$:
$$\begin{array}{cccc} & & \hat\varphi & \\ & & \hat\varphi & \\ & 1\ 0 & \longrightarrow & x \\ & 0\ 1 & \longrightarrow & y \\ * & \downarrow\ \downarrow & & \downarrow \\ & 2\ 1 & \longrightarrow & 2=x*y \end{array},$$
$$\begin{array}{cccc} & & \hat\varphi & \\ & 0\ 1 & \longrightarrow & y \\ & 1\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 2 & \longrightarrow & 2=y*x \end{array}.$$
But there are no such x and y that satisfies $x*y=y*x=2$.

Next consider groupoid (2558). Since $0*0=1$ and $1*0=2$, by the following two tables we have that $\hat\varphi(1,0),\hat\varphi(0,1)\in\{1,2\}$.
$$\begin{array}{cccc} & & \hat\varphi & \\ & & \hat\varphi & \\ & 1\ 0 & \longrightarrow & x \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 2\ 1 & \longrightarrow & 2=x*0 \end{array}, \quad \begin{array}{cccc} & & \hat\varphi & \\ & 0\ 1 & \longrightarrow & y \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 2 & \longrightarrow & 2=y*0 \end{array}.$$

But then the next table gives us a contradiction:
$$\begin{array}{cccc} & & \hat\varphi & \\ & 1\ 0 & \longrightarrow & x \\ & 1\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ 1 & \longrightarrow & y=x*x \end{array}.$$

The only groupoid belonging to case two is groupoid (284). By the following two tables we must have $\hat\varphi(1,0)=1$:
$$\begin{array}{cccc} & & \hat\varphi & \\ & & \hat\varphi & \\ & 1\ 1 & \longrightarrow & 1 \\ & 1\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 1 & \longrightarrow & 1=1*x \end{array}, \quad \begin{array}{cccc} & & \hat\varphi & \\ & & \hat\varphi & \\ & 2\ 1 & \longrightarrow & 2 \\ & 1\ 0 & \longrightarrow & x \\ * & \downarrow\ \downarrow & & \downarrow \\ & 2\ 1 & \longrightarrow & 2=2*x \end{array}.$$

But then we have a contradiction:
$$\begin{array}{cccc} & & \hat\varphi & \\ & 1\ 0 & \longrightarrow & 1 \\ & 1\ 2 & \longrightarrow & 2 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & 1\neq 2 \end{array}.$$
□

Lemma 5.10. *The partial operation*

$$\varphi:\begin{cases}0\mapsto 0\\ 1\mapsto 2\end{cases}$$

is a non-extendable local polymorphism of the groupoids (219), (222), (2090).

Proof. Let us suppose that for each groupoid there exists an extension $\hat\varphi$ that is a total polymorphism of the groupoid. Fist we consider groupoid (219). By the table
$$\begin{array}{cccc} & & \hat\varphi & \\ & 2 & \longrightarrow & x \\ & 1 & \longrightarrow & 2 \\ * & \downarrow & & \downarrow \\ & 1 & \longrightarrow & 2=x*2 \end{array}$$
we have a contradiction (there is no such x.) Next we

deal with groupoid (222). By the table $\begin{array}{ccc} 1 & \overset{\hat{\varphi}}{\longrightarrow} & 2 \\ 2 & \longrightarrow & x \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 2{=}2*x \end{array}$ we have $\hat{\varphi}(2) = 2$, but then $\begin{array}{ccc} 2 & \overset{\hat{\varphi}}{\longrightarrow} & 2 \\ 1 & \longrightarrow & 2 \\ *\downarrow & & \downarrow \\ 2 & \longrightarrow & 2 \neq 0 \end{array}$ gives us a contradiction. And lastly, for groupoid (2090) by $\begin{array}{ccc} 2 & \overset{\hat{\varphi}}{\longrightarrow} & x \\ 1 & \longrightarrow & 2 \\ *\downarrow & & \downarrow \\ 1 & \longrightarrow & 2{=}x*2 \end{array}$ we must have $\hat{\varphi}(2) = 0$. But then $\begin{array}{ccc} 1 & \overset{\hat{\varphi}}{\longrightarrow} & 2 \\ 2 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \neq 2 \end{array}$ gives us a contradiction. □

Lemma 5.11. *The partial operation*

$$\varphi : \begin{cases} 0 \mapsto 1 \\ 1 \mapsto 0 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (257), (258), (259), (260), (261), (262), (263), (265), (266), (267), (268), (269), (270), (271), (272), (274), (278), (282), (677), (678), (679), (680), (682), (684), (687), (690), (691), (693), (695), (696), (697), (698), (704), (705), (707), (710), (712), (1271), (1277), (1281), (2460), (2461), (2462), (2463), (2464), (2467), (2476), (2478), (2479), (2480), (2483), (2486), (2487), (2493), (2739).

Proof. As in the proof of Lemma, it helps to observe that all of the listed groupoids have their basic operation defined on $\{0,1\} \times \{0,1\}$ as

$$\begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1, \end{array} \qquad \text{or} \qquad \begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0, \end{array}$$

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. Let us consider first the groupoids belonging to the first case. If the investigated groupoid satisfies $2 * 2 = 1$, then we must have $\hat{\varphi}(2) = 0$ by the following table: $\begin{array}{ccc} 2 & \overset{\hat{\varphi}}{\longrightarrow} & x \\ 2 & \longrightarrow & x \\ *\downarrow & & \downarrow \\ 1 & \longrightarrow & 0{=}x*x \end{array}$. From this we get that the investigated groupoid must not satisfy the equalities $2 * 1 = 0$, $1 * 2 = 0$, $2 * 0 = 0$, and that the groupoid must satisfy the equality $0 * 2 = 0$. This is shown by the next four tables (respectively).

$$\begin{array}{ccc} 2 & \overset{\hat{\varphi}}{\longrightarrow} & 0 \\ 1 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array} \quad \begin{array}{ccc} 1 & \overset{\hat{\varphi}}{\longrightarrow} & 0 \\ 2 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array} \quad \begin{array}{ccc} 2 & \overset{\hat{\varphi}}{\longrightarrow} & 0 \\ 0 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array} \quad \begin{array}{ccc} 0 & \overset{\hat{\varphi}}{\longrightarrow} & 1 \\ 2 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array}$$

This proves that for groupoids (258), (261), (266), (268), (270), (678), (695), (698), (704), (707), (712), and (1271) φ is non-extendable.

If the investigated groupoid (belonging to the first case) satisfies $2*2=0$, then we must have $\hat{\varphi}(2)=1$ by the following table:

$$\begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & x \\ 2 & \longrightarrow & x \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 1{=}x*x \end{array}$$

. From this we get that the investigated groupoid must not satisfy the equalities $2*1=1$, $2*0=1$ and $0*2=1$, and that the groupoid must satisfy the equalities $1*2=1$. This is shown by the next four tables (respectively).

$$\begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & 1 \\ 1 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & 1 \\ 0 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 0 & \longrightarrow & 1 \\ 2 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array}$$

This proves that for groupoids (257), (260), (263), (272), (274), (282), (677), (680), (682), (687), (690), (691), (697), (1277) and (1281) φ is non-extendable.

Now for the remaining groupoids in the first case, they all satisfy $2*2=2$. Obviously, if φ is extendable, then we have either $\hat{\varphi}(2)=0$, $\hat{\varphi}(2)=1$ or $\hat{\varphi}(2)=2$. If $\hat{\varphi}(2)$ is 2, then $\hat{\varphi}$ is an automorphism of the investigated groupoid (since it is bijective and a homomorphism). But then it should satisfy all of the implications $0*2=0 \Leftrightarrow 1*2=1$, $0*2=1 \Leftrightarrow 1*2=0$, $2*0=0 \Leftrightarrow 2*1=1$, $2*0=1 \Leftrightarrow 2*1=0$. And none of the groupoids (259), (262), (265), (267), (269), (271), (278), (679), (684), (693), (696), (705), (710) satisfy all of these implications, hence $\hat{\varphi}(2)\neq 2$. Therefore, we either have $\hat{\varphi}(2)=0$ or $\hat{\varphi}(2)=1$. Notice that in the previous part of the lemma (i.e., investigating when $2*2=0$ and $2*2=1$) we did not use what the value of $2*2$ is, only that it implied what the value of $\hat{\varphi}$ is. Therefore, with the same reasoning the investigating groupoid either must not satisfy the equalities $2*1=0$, $1*2=0$, $2*0=0$, and the groupoid must satisfy the equality $0*2=0$, or the investigated groupoid must not satisfy the equalities $2*1=1$, $2*0=1$ and $0*2=1$, and the groupoid must satisfy the equalities $1*2=1$. These conditions also are not satisfied by groupoids (259), (262), (265), (267), (269), (271), (278), (679), (684), (693), (696), (705), (710), hence altogether we have a contradiction for all of them.

For the second case, suppose first that the investigated groupoid satisfies $2*2=0$. Then, by the following table we must have $\hat{\varphi}(2)=0$:

$$\begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & x \\ 2 & \longrightarrow & x \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 1{=}x*x \end{array}$$

From this we get that the investigated groupoid must not satisfy the equality $2*0=0$ and that the groupoid must satisfy the equalities $2*1=0$, $1*2=0$ and $0*2=0$. This is shown by the next four tables (respectively).

$$\begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & 0 \\ 0 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & 0 \\ 1 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 0 & \longrightarrow & 1 \\ 2 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array}$$

This proves that for groupoids (2460), (2462), (2464), (2476), (2478), (2480), (2483), (2486), (2487) and (2493) φ is non-extendable.

If the investigated groupoid (belonging to the second case) satisfies $2*2=1$, then we must have $\hat{\varphi}(2)=1$ by the following table:

$$\begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & x \\ 2 & \longrightarrow & x \\ *\downarrow & & \downarrow \\ 0 & \longrightarrow & 1{=}x*x \end{array}.$$

From this we get that the investigated groupoid must not satisfy the equality $2*1=1$, and that the groupoid must satisfy the equalities $1*2=1$, $2*0=1$ and $0*2=1$. This is shown by the next four tables (respectively).

$$\begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & 1 \\ 1 & \longrightarrow & 0 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}1 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 2 & \longrightarrow & 1 \\ 0 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array} \quad \begin{array}{rcc} & \hat{\varphi} & \\ 0 & \longrightarrow & 1 \\ 2 & \longrightarrow & 1 \\ *\downarrow & & \downarrow \\ u & \longrightarrow & \hat{\varphi}(u){=}0 \end{array}$$

This proves that for groupoids (2461), (2463), (2467), (2479), (2739) φ is non-extendable. □

Lemma 5.12. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,1) \mapsto 1 \\ (1,0) \mapsto 1 \\ (1,1) \mapsto 0 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (273), (681), (2116).

Proof. Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. We prove the lemma case-by-case. For groupoid (273), by the following two tables we have a contradiction, since according to them we can not define $\hat{\varphi}$ on $(2,2)$ (we would need to define $\hat{\varphi}(2,2)$ as both 0 and 1 respectively):

$$\begin{array}{rrcc} & & \hat{\varphi} & \\ 2 & 2 & \longrightarrow & x \\ 0 & 0 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 0{=}x*0 \end{array}, \quad \begin{array}{rrcc} & & \hat{\varphi} & \\ 2 & 2 & \longrightarrow & x \\ 0 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}x*1 \end{array}.$$

Similarly, for groupoid (681), by the following two tables we have a contradiction, since according to them we can not define $\hat{\varphi}$ on $(2,0)$ (we would need to define $\hat{\varphi}(2,0)$ as both 0 and 1 respectively):

$$\begin{array}{rrcc} & & \hat{\varphi} & \\ 2 & 0 & \longrightarrow & x \\ 0 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 0 & 0 & \longrightarrow & 0{=}x*1 \end{array}, \quad \begin{array}{rrcc} & & \hat{\varphi} & \\ 2 & 0 & \longrightarrow & x \\ 1 & 1 & \longrightarrow & 0 \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 0 & \longrightarrow & 1{=}x*0 \end{array}.$$

And lastly, for groupoid (2116), by the following two tables we have a contradiction, since according to them we can not define $\hat{\varphi}$ on $(2,2)$ (we would need to define $\hat{\varphi}(2,2)$ as both $x \in \{0,1\}$ and 2 respectively):

$$\begin{array}{rrcc} & & \hat{\varphi} & \\ 2 & 2 & \longrightarrow & x \\ 2 & 2 & \longrightarrow & x \\ *\downarrow & \downarrow & & \downarrow \\ 1 & 1 & \longrightarrow & 0{=}x*x \end{array}, \quad \begin{array}{rrcc} & & \hat{\varphi} & \\ 2 & 2 & \longrightarrow & x \\ 0 & 1 & \longrightarrow & 1 \\ *\downarrow & \downarrow & & \downarrow \\ 2 & 2 & \longrightarrow & x{=}x*1 \end{array}. \quad \square$$

Lemma 5.13. *The partial operation*

$$\varphi : \begin{cases} (1,1) \mapsto 1 \\ (1,2) \mapsto 2 \\ (2,1) \mapsto 2 \\ (2,2) \mapsto 1 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (283), (287), (353), (356), (359).

Proof. Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. We prove the lemma case-by-case. For groupoid (283), by the following two tables we have a contradiction, since according to them we can not define $\hat{\varphi}$ on $(0,2)$ (we would need to define $\hat{\varphi}(0,2)$ as both 0 and 1 respectively):

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 0 & 2 & \longrightarrow & x \\ & 2 & 1 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & x{=}x*2 \end{array} \quad , \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 1 & \longrightarrow & 2 \\ & 0 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 2 & \longrightarrow & 2{=}2*x \end{array}$$

. Similarly, for groupoid (287) by the following two tables we have a contradiction, since according to them we can not define $\hat{\varphi}$ on $(0,2)$ (we would need to define $\hat{\varphi}(0,2)$ as both 1 and 2 respectively):

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 1 & \longrightarrow & 2 \\ & 0 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 1{=}2*x \end{array} \quad , \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 1 & \longrightarrow & 1 \\ & 0 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 2 & \longrightarrow & 2{=}1*x \end{array}$$

. For groupoid (353) we immediately get a contradiction from the table

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & 2 \\ & 0 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 1 & \longrightarrow & 2{=}2*x \end{array}$$

, since such x does not exists.

For groupoid (356) we get a contradiction from the following two tables:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 1 & \longrightarrow & 1 \\ & 0 & 0 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 1{=}1*x \end{array} \quad , \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & 2 \\ & 0 & 0 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 1 & \longrightarrow & 2{=}2*x \end{array}$$

, since such x does not exists.

And lastly, for groupoid (359) we get a contradiction from the following two tables, since $2*x = 1 \neq 2 = 2*x$:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & 2 \\ & 0 & 1 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 1 & \longrightarrow & 2{=}2*x \end{array} \quad , \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 1 & \longrightarrow & 2 \\ & 0 & 1 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 1 & \longrightarrow & 1{=}2*x \end{array}$$

. □

Lemma 5.14. *The partial operation*

$$\varphi : \begin{cases} (1,1) \mapsto 1 \\ (1,2) \mapsto 1 \\ (2,1) \mapsto 1 \\ (2,2) \mapsto 2 \end{cases}$$

is a non-extendable local polymorphism of the groupoid (354).

Proof. Let us suppose that φ is extendable, let $\hat{\varphi}$ denote the extension. By the following tables we must have $\hat{\varphi}(0,2) = 1$:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 2 & \longrightarrow & 2 \\ & 0 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 1 & 2 & \longrightarrow & 1{=}2*x \end{array}, \quad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 1 & \longrightarrow & 1 \\ & 0 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 1 & \longrightarrow & 1{=}1*x \end{array}.$$

But then the table

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & 1 \\ & 0 & 2 & \longrightarrow & 1 \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 2 & \longrightarrow & 2 \neq 1 \end{array}$$

gives us a contradiction. □

Lemma 5.15. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 2 \\ (2,0) \mapsto 2 \\ (2,2) \mapsto 0 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (1012), (1084), (1151), (1153), (1176), (1200), (1219), (1221), (1242), (1321), (1433), (1437), (1481), (2102), (2104).

Proof. As in the proof of Lemma 4.15, it helps to observe that all of the listed groupoids have their basic operation defined on $\{0,2\} \times \{0,2\}$ as

$$\begin{array}{c|cc} * & 0 & 2 \\ \hline 0 & 0 & 2 \\ 2 & 2 & 0. \end{array}$$

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. If the investigated groupoid satisfies $0 * 1 = 0$, then we must have $\hat{\varphi}(1,2) = \hat{\varphi}(2,1) = 2$, shown by the following tables:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 0 & 0 & \longrightarrow & 0 \\ & 1 & 2 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 2 & \longrightarrow & 2{=}0*x \end{array} \qquad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 0 & 0 & \longrightarrow & 0 \\ & 2 & 1 & \longrightarrow & x \\ * & \downarrow & \downarrow & & \downarrow \\ & 2 & 0 & \longrightarrow & 2{=}0*x \end{array}$$

And if the groupoid also satisfies $1 * 2 = 0$ or $2 * 1 = 0$, then we have a contradiction:

$$\begin{array}{ccccc} & & & \hat{\varphi} & \\ & 1 & 2 & \longrightarrow & 2 \\ & 2 & 2 & \longrightarrow & 0 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0 \neq 2 \end{array} \qquad \begin{array}{ccccc} & & & \hat{\varphi} & \\ & 2 & 2 & \longrightarrow & 0 \\ & 1 & 2 & \longrightarrow & 2 \\ * & \downarrow & \downarrow & & \downarrow \\ & 0 & 0 & \longrightarrow & 0 \neq 2 \end{array}$$

This shows that for groupoids (1012), (1084), (1151), (1153), (1219), (1221), (1242), (1321), (1433), (1437) and (1481) φ is non-extendable.

Now suppose that the investigated groupoid satisfies $1 * 1 = 2$; then we must have $\hat{\varphi}(1,0) = \hat{\varphi}(0,1) = 1$, shown by the following tables:

$$\begin{array}{ccccc}
 & & & \hat{\varphi} & \\
 & 1 & 0 & \longrightarrow & x \\
 & 1 & 0 & \longrightarrow & x \\
* & \downarrow & \downarrow & & \downarrow \\
 & 2 & 0 & \longrightarrow & 2{=}x*x
\end{array}
\qquad
\begin{array}{ccccc}
 & & & \hat{\varphi} & \\
 & 0 & 1 & \longrightarrow & x \\
 & 0 & 1 & \longrightarrow & x \\
* & \downarrow & \downarrow & & \downarrow \\
 & 0 & 2 & \longrightarrow & 2{=}x*x
\end{array}$$

And if the groupoid also satisfies $0*1=1*0=0$, then we have a contradiction:

$$\begin{array}{ccccc}
 & & & \hat{\varphi} & \\
 & 1 & 0 & \longrightarrow & 1 \\
 & 0 & 1 & \longrightarrow & 1 \\
* & \downarrow & \downarrow & & \downarrow \\
 & 0 & 0 & \longrightarrow & 0 \neq 2
\end{array} \quad .$$

This shows that for groupoids (1176) and (1200) φ is non-extendable.

Lastly, suppose that the investigated groupoid satisfies $1*2=1$; then we must have $\hat{\varphi}(1,2)=\hat{\varphi}(1,0)=1$, shown by the following tables:

$$\begin{array}{ccccc}
 & & & \hat{\varphi} & \\
 & 1 & 2 & \longrightarrow & x \\
 & 2 & 0 & \longrightarrow & 2 \\
* & \downarrow & \downarrow & & \downarrow \\
 & 1 & 2 & \longrightarrow & x{=}x*2
\end{array}
\qquad
\begin{array}{ccccc}
 & & & \hat{\varphi} & \\
 & 1 & 0 & \longrightarrow & x \\
 & 2 & 0 & \longrightarrow & 2 \\
* & \downarrow & \downarrow & & \downarrow \\
 & 1 & 0 & \longrightarrow & x{=}x*2
\end{array}$$

And if the groupoid also satisfies $1*1=0$, then we have a contradiction:

$$\begin{array}{ccccc}
 & & & \hat{\varphi} & \\
 & 1 & 2 & \longrightarrow & 1 \\
 & 1 & 0 & \longrightarrow & 1 \\
* & \downarrow & \downarrow & & \downarrow \\
 & 0 & 2 & \longrightarrow & 2 \neq 0
\end{array} \quad .$$

This shows that for groupoids (2102) and (2104) φ is non-extendable. □

Lemma 5.16. *The partial operation*

$$\varphi : \begin{cases} (0,0) \mapsto 0 \\ (0,2) \mapsto 0 \\ (1,1) \mapsto 1 \\ (2,0) \mapsto 0 \\ (2,2) \mapsto 2 \end{cases}$$

is a non-extendable local polymorphism of the groupoids (2654), (2686), (2698), (2702).

Proof. As in the proof of Lemma 4.16, it helps to observe that all of the listed groupoids have their basic operation defined on $\{0,2\}\times\{0,2\}$ as

*	0	2
0	1	1
2	1	1.

Let us suppose that for each groupoid there exists an extension $\hat{\varphi}$ that is a total polymorphism of the groupoid. We will prove that such extension can not exists by showing that $\hat{\varphi}$ can not be defined on $(1,2)$. Note that all listed groupoids satisfy $1*0=0*1=0$. Let us suppose first that $\hat{\varphi}(1,2)=0$. By the following table we must have $\hat{\varphi}(0,1) = 1$: $\begin{array}{lcr} 1\ 2 & \overset{\hat{\varphi}}{\longrightarrow} & 0 \\ 0\ 2 & \longrightarrow & 0 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 0\ 1 & \longrightarrow & \hat{\varphi}(0,1){=}1 \end{array}$. And then we must also have $\hat{\varphi}(1,0) = 0$: $\begin{array}{lcr} 0\ 1 & \overset{\hat{\varphi}}{\longrightarrow} & 1 \\ 0\ 0 & \longrightarrow & 0 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 1\ 0 & \longrightarrow & \hat{\varphi}(1,0){=}0 \end{array}$.

The next two tables show that in this case the groupoid can not satisfy neither $1{*}1 = 2$, nor $2*1 = 2$ (respectively): $\begin{array}{lcr} 1\ 1 & \overset{\hat{\varphi}}{\longrightarrow} & 1 \\ 0\ 1 & \longrightarrow & 1 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 0\ v & \longrightarrow & \hat{\varphi}(0,v){=}v \end{array}$, $\begin{array}{lcr} 2\ 2 & \overset{\hat{\varphi}}{\longrightarrow} & 2 \\ 0\ 1 & \longrightarrow & 1 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 1\ v & \longrightarrow & \hat{\varphi}(1,v){=}v \end{array}$.

Let us suppose next that $\hat{\varphi}(1,2) = 1$. By the following table we must have $\hat{\varphi}(0,1) = 0$: $\begin{array}{lcr} 1\ 2 & \overset{\hat{\varphi}}{\longrightarrow} & 1 \\ 0\ 2 & \longrightarrow & 0 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 0\ 1 & \longrightarrow & \hat{\varphi}(0,1){=}0 \end{array}$. And then we must also have $\hat{\varphi}(1,0) = 1$: $\begin{array}{lcr} 0\ 1 & \overset{\hat{\varphi}}{\longrightarrow} & 0 \\ 0\ 0 & \longrightarrow & 0 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 1\ 0 & \longrightarrow & \hat{\varphi}(1,0){=}1 \end{array}$. The next table shows that in this case the groupoid can not satisfy $1 * 1 = 2$: $\begin{array}{lcr} 1\ 1 & \overset{\hat{\varphi}}{\longrightarrow} & 1 \\ 1\ 0 & \longrightarrow & 1 \\ {*}\downarrow\ \downarrow & & \downarrow \\ u\ 0 & \longrightarrow & \hat{\varphi}(u,0){=}u \end{array}$, And the next two tables show that the investigated groupoid also can not satisfy $2 * 1 = 2$: $\begin{array}{lcr} 2\ 2 & \overset{\hat{\varphi}}{\longrightarrow} & 2 \\ 1\ 0 & \longrightarrow & 1 \\ {*}\downarrow\ \downarrow & & \downarrow \\ u\ 1 & \longrightarrow & \hat{\varphi}(u,1){=}u \end{array}$, $\begin{array}{lcr} 2\ 0 & \overset{\hat{\varphi}}{\longrightarrow} & 0 \\ 1\ 0 & \longrightarrow & 1 \\ {*}\downarrow\ \downarrow & & \downarrow \\ u\ 1 & \longrightarrow & \hat{\varphi}(u,1){=}0 \end{array}$.

And lastly, suppose that $\hat{\varphi}(1,2) = 2$. By the following table we must have $\hat{\varphi}(0,1) = 1$: $\begin{array}{lcr} 1\ 2 & \overset{\hat{\varphi}}{\longrightarrow} & 2 \\ 0\ 2 & \longrightarrow & 0 \\ {*}\downarrow\ \downarrow & & \downarrow \\ 0\ 1 & \longrightarrow & \hat{\varphi}(0,1){=}1 \end{array}$. And then we must also have $\hat{\varphi}(1,0) = 0$:

$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 0\ 1 & \longrightarrow & 1 \\ & 0\ 0 & \longrightarrow & 0 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 1\ 0 & \longrightarrow & \hat{\varphi}(1,0)=0 \end{array}$$

. The next two tables show that in this case the groupoid can not satisfy neither $1 * 1 = 2$, nor $2 * 1 = 2$ (respectively):

$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 1 & \longrightarrow & 1 \\ & 0\ 1 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ v & \longrightarrow & \hat{\varphi}(0,v)=v \end{array},$$

$$\begin{array}{cccc} & & \hat{\varphi} & \\ & 1\ 2 & \longrightarrow & 2 \\ & 0\ 1 & \longrightarrow & 1 \\ * & \downarrow\ \downarrow & & \downarrow \\ & 0\ v & \longrightarrow & \hat{\varphi}(0,v)=v \end{array}.$$

In all cases we got that the investigated groupoid can not satisfy $1 * 1 = 2$ or $2 * 1 = 2$. This shows that for all listed groupoids φ is non-extendable. □

Acknowledgments

The author is grateful to Tamás Waldhauser and Mike Behrisch for helpful discussions.

This research was partially supported by the National Research, Development and Innovation Office of Hungary under grants K128042 and K138892, and by the ÚNKP-21-3-SZTE-465 New National Excellence Program Of The Ministry for Innovation And Technology from the source of the National Research, Development and Innovation Fund.

Appendix A. Non-polymorphism-homogeneous three-element groupoids: non-extendable local polymorphisms

A partial operation φ	The groupoids G (up to clone equivalence) such that φ is a non-extendable local polymorphism of G
$\varphi:$ $(0,0) \mapsto 0$ $(0,1) \mapsto 1$ $(1,0) \mapsto 1$ $(1,1) \mapsto 1$	(2), (4), (5), (6), (8), (10), (11), (12), (13), (15), (16), (18), (22), (26), (30), (31), (32), (33), (34), (35), (36), (37), (38), (39), (40), (41), (42), (43), (44), (45), (46), (48), (49), (51), (52), (53), (59), (61), (63), (65), (66), (67), (69), (73), (376), (377), (378), (379), (380), (381), (382), (384), (385), (387), (388), (390), (391), (405), (407), (410), (417), (434), (436), (437), (439), (1014), (1038), (1040), (1066)
$\varphi:$ $0 \mapsto 2$ $2 \mapsto 0$	(21), (24), (47), (50), (72), (75), (78), (96), (99), (119), (141), (144), (162), (165), (168), (182), (185), (188), (201), (204), (218), (221), (235), (252) (255), (1227), (1233)
$\varphi:$ $(0,0) \mapsto 0$ $(0,2) \mapsto 0$ $(2,0) \mapsto 0$ $(2,2) \mapsto 2$	(60), (147)

$\varphi:$ $(0,0)\mapsto 0$ $(0,1)\mapsto 0$ $(1,0)\mapsto 0$ $(1,1)\mapsto 1$	(79), (81), (82), (83), (85), (87), (88), (89), (90), (91), (93), (97), (101), (111), (112), (113), (115), (117), (120), (121), (122), (123), (124), (125), (130), (132), (134), (136), (137), (138), (142), (143), (406), (454), (455), (456), (457), (458), (459), (460), (462), (463), (465), (469), (483), (484), (485), (487), (493), (494), (495), (496), (512), (513), (514), (515), (517), (519), (522), (1086), (1107), (1108), (1133)
$\varphi:$ $(0,0)\mapsto 0$ $(0,2)\mapsto 2$ $(1,0)\mapsto 1$	(80), (100), (102), (239), (241), (1132)
$\varphi:$ $1\mapsto 2$ $2\mapsto 1$	(116), (135), (178), (194), (203), (281), (308), (316), (488), (520), (565), (758), (780), (984), (1793), (1799), (1818), (1962), (2430), (2436), (2539), (2545), (2636)
$\varphi:$ $(0,0)\mapsto 0$ $(0,1)\mapsto 0$ $(0,2)\mapsto 0$ $(1,0)\mapsto 0$ $(1,1)\mapsto 1$ $(2,0)\mapsto 0$ $(2,2)\mapsto 2$	(149), (175)
$\varphi:$ $(1,1)\mapsto 2$ $(1,2)\mapsto 1$ $(2,1)\mapsto 1$ $(2,2)\mapsto 2$	(176), (198), (306), (320), (563), (756)
$\varphi:$ $(0,0)\mapsto 0$ $(1,1)\mapsto 1$ $(1,2)\mapsto 2$ $(2,1)\mapsto 2$ $(2,2)\mapsto 2$	(216), (284), (1708), (1837), (2088), (2472), (2558)
$\varphi:$ $0\mapsto 0$ $1\mapsto 2$	(219), (222), (2090)
$\varphi:$ $0\mapsto 1$ $1\mapsto 0$	(257), (258), (259), (260), (261), (262), (263), (265), (266), (267), (268), (269), (270), (271), (272), (274), (278), (282), (677), (678), (679), (680), (682), (684), (687), (690), (691), (693), (695), (696), (697), (698), (704), (705), (707), (710), (712), (1271), (1277), (1281), (2460), (2461), (2462), (2463), (2464), (2467), (2476), (2478), (2479), (2480), (2483), (2486), (2487), (2493), (2739)
$\varphi:$ $(0,0)\mapsto 0$ $(0,1)\mapsto 1$ $(1,0)\mapsto 1$ $(1,1)\mapsto 0$	(273), (681), (2116)

$\varphi:$ $(1,1)\mapsto 1$, $(1,2)\mapsto 2$, $(2,1)\mapsto 2$, $(2,2)\mapsto 1$	(283), (287), (353), (356), (359)
$\varphi:$ $(1,1)\mapsto 1$, $(1,2)\mapsto 1$, $(2,1)\mapsto 1$, $(2,2)\mapsto 2$	(354)
$\varphi:$ $(0,0)\mapsto 0$, $(0,2)\mapsto 2$, $(2,0)\mapsto 2$, $(2,2)\mapsto 0$	(1012), (1084), (1151), (1153), (1176), (1200), (1219), (1221), (1242), (1321), (1433), (1437), (1481), (2102), (2104)
$\varphi:$ $(0,0)\mapsto 0$, $(0,2)\mapsto 0$, $(1,1)\mapsto 1$, $(2,0)\mapsto 0$, $(2,2)\mapsto 2$	(2654), (2686), (2698), (2702)

Table 1: Partial operations proving that the corresponding groupoids are not polymorphism-homogeneous.

Appendix B. Non-polymorphism-homogeneous three-element groupoids: operation tables

(2)	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

(4)	0	1	2
0	0	0	0
1	0	0	0
2	0	1	0

(5)	0	1	2
0	0	0	0
1	0	0	0
2	0	1	1

(6)	0	1	2
0	0	0	0
1	0	0	0
2	0	1	2

(8)	0	1	2
0	0	0	0
1	0	0	0
2	0	2	1

(10)	0	1	2
0	0	0	0
1	0	0	0
2	1	0	0

(11)	0	1	2
0	0	0	0
1	0	0	0
2	1	0	1

(12)	0	1	2
0	0	0	0
1	0	0	0
2	1	0	2

(13)	0	1	2
0	0	0	0
1	0	0	0
2	1	1	0

(15)	0	1	2
0	0	0	0
1	0	0	0
2	1	1	2

(16)	0	1	2
0	0	0	0
1	0	0	0
2	1	2	0

(18)	0	1	2
0	0	0	0
1	0	0	0
2	1	2	2

(21)	0	1	2
0	0	0	0
1	0	0	0
2	2	0	2

(22)	0	1	2
0	0	0	0
1	0	0	0
2	2	1	0

(24)	0	1	2
0	0	0	0
1	0	0	0
2	2	1	2

(26)	0	1	2
0	0	0	0
1	0	0	0
2	2	2	1

(30)	0	1	2
0	0	0	0
1	0	0	1
2	0	1	0

(31)	0	1	2
0	0	0	0
1	0	0	1
2	0	1	1

(32)	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

(33)	0	1	2
0	0	0	0
1	0	0	1
2	0	2	0

(34)	0	1	2
0	0	0	0
1	0	0	1
2	0	2	1

(35)	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

(36)	0	1	2
0	0	0	0
1	0	0	1
2	1	0	0

(37)	0	1	2
0	0	0	0
1	0	0	1
2	1	0	1

(38)	0	1	2
0	0	0	0
1	0	0	1
2	1	0	2

(39)	0	1	2
0	0	0	0
1	0	0	1
2	1	1	0

(40)	0	1	2
0	0	0	0
1	0	0	1
2	1	1	1

(41)	0	1	2
0	0	0	0
1	0	0	1
2	1	1	2

(42)	0	1	2
0	0	0	0
1	0	0	1
2	1	2	0

(43)	0	1	2
0	0	0	0
1	0	0	1
2	1	2	1

(44)	0	1	2
0	0	0	0
1	0	0	1
2	1	2	2

(45)	0	1	2
0	0	0	0
1	0	0	1
2	2	0	0

(46)	0	1	2
0	0	0	0
1	0	0	1
2	2	0	1

(47)	0	1	2
0	0	0	0
1	0	0	1
2	2	0	2

(48)	0	1	2
0	0	0	0
1	0	0	1
2	2	1	0

(49)	0	1	2
0	0	0	0
1	0	0	1
2	2	1	1

(50)	0	1	2
0	0	0	0
1	0	0	1
2	2	1	2

(51)	0	1	2
0	0	0	0
1	0	0	1
2	2	2	0

(52)	0	1	2
0	0	0	0
1	0	0	1
2	2	2	1

(53)	0	1	2
0	0	0	0
1	0	0	1
2	2	2	2

(59)	0	1	2
0	0	0	0
1	0	0	2
2	0	2	1

(60)	0	1	2
0	0	0	0
1	0	0	2
2	0	2	2

(61)	0	1	2
0	0	0	0
1	0	0	2
2	1	0	0

(63)	0	1	2
0	0	0	0
1	0	0	2
2	1	0	2

(65)	0	1	2
0	0	0	0
1	0	0	2
2	1	1	1

(66)	0	1	2
0	0	0	0
1	0	0	2
2	1	1	2

(67)	0	1	2
0	0	0	0
1	0	0	2
2	1	2	0

(69)	0	1	2
0	0	0	0
1	0	0	2
2	1	2	2

(72)	0	1	2
0	0	0	0
1	0	0	2
2	2	0	2

(73)	0	1	2
0	0	0	0
1	0	0	2
2	2	1	0

(75)	0	1	2
0	0	0	0
1	0	0	2
2	2	1	2

(78)	0	1	2
0	0	0	0
1	0	0	2
2	2	2	2

(79)	0	1	2
0	0	0	0
1	0	1	0
2	0	0	1

(80)	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

(81)	0	1	2
0	0	0	0
1	0	1	0
2	0	1	1

(82)	0	1	2
0	0	0	0
1	0	1	0
2	0	1	2

(83)	0	1	2
0	0	0	0
1	0	1	0
2	0	2	1

(85)	0	1	2
0	0	0	0
1	0	1	0
2	1	0	0

(87)	0	1	2
0	0	0	0
1	0	1	0
2	1	0	2

(88)	0	1	2
0	0	0	0
1	0	1	0
2	1	1	0

(89)	0	1	2
0	0	0	0
1	0	1	0
2	1	1	1

(90)	0	1	2
0	0	0	0
1	0	1	0
2	1	1	2

(91)	0	1	2
0	0	0	0
1	0	1	0
2	1	2	0

(93)	0	1	2
0	0	0	0
1	0	1	0
2	1	2	2

(96)	0	1	2
0	0	0	0
1	0	1	0
2	2	0	2

(97)	0	1	2
0	0	0	0
1	0	1	0
2	2	1	0

(99)	0	1	2
0	0	0	0
1	0	1	0
2	2	1	2

(100)	0	1	2
0	0	0	0
1	0	1	0
2	2	2	0

(101)	0	1	2
0	0	0	0
1	0	1	0
2	2	2	1

(102)	0	1	2
0	0	0	0
1	0	1	0
2	2	2	2

(111)	0	1	2
0	0	0	0
1	0	1	1
2	1	1	0

(112)	0	1	2
0	0	0	0
1	0	1	1
2	1	1	1

(113)	0	1	2
0	0	0	0
1	0	1	1
2	1	1	2

(115)	0	1	2
0	0	0	0
1	0	1	1
2	1	2	1

(116)	0	1	2
0	0	0	0
1	0	1	1
2	1	2	2

(117)	0	1	2
0	0	0	0
1	0	1	1
2	2	0	0

(119)	0	1	2
0	0	0	0
1	0	1	1
2	2	0	2

(120)	0	1	2
0	0	0	0
1	0	1	1
2	2	1	0

(121)	0	1	2
0	0	0	0
1	0	1	1
2	2	1	1

(122)	0	1	2
0	0	0	0
1	0	1	1
2	2	1	2

(123)	0	1	2
0	0	0	0
1	0	1	1
2	2	2	0

(124)	0	1	2
0	0	0	0
1	0	1	1
2	2	2	1

(125)	0	1	2
0	0	0	0
1	0	1	1
2	2	2	2

(130)	0	1	2
0	0	0	0
1	0	1	2
2	1	0	0

(132)	0	1	2
0	0	0	0
1	0	1	2
2	1	0	2

(134)	0	1	2
0	0	0	0
1	0	1	2
2	1	1	1

(135)	0	1	2
0	0	0	0
1	0	1	2
2	1	1	2

(136)	0	1	2
0	0	0	0
1	0	1	2
2	1	2	0

(137)	0	1	2
0	0	0	0
1	0	1	2
2	1	2	1

(138)	0	1	2
0	0	0	0
1	0	1	2
2	1	2	2

(141)	0	1	2
0	0	0	0
1	0	1	2
2	2	0	2

(142)	0	1	2
0	0	0	0
1	0	1	2
2	2	1	0

(143)	0	1	2
0	0	0	0
1	0	1	2
2	2	1	1

(143)	0	1	2
0	0	0	0
1	0	1	2
2	2	1	2

(147)	0	1	2
0	0	0	0
1	0	1	2
2	2	2	2

(149)	0	1	2
0	0	0	0
1	0	2	0
2	0	1	1

(162)	0	1	2	(165)	0	1	2	(168)	0	1	2	(175)	0	1	2	(176)	0	1	2	(178)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	0	1	0	2	0	1	0	2	0	1	0	2	1	1	0	2	1	1	0	2	1
2	2	0	2	2	2	1	2	2	2	2	2	2	1	1	1	2	1	1	2	2	1	2	1

(182)	0	1	2	(185)	0	1	2	(188)	0	1	2	(194)	0	1	2	(198)	0	1	2	(201)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	1	1	0	2	1	1	0	2	1	1	0	2	2	1	0	2	2	1	0	2	2
2	2	0	2	2	2	1	2	2	2	2	2	2	1	1	1	2	1	2	2	2	2	0	2

(203)	0	1	2	(204)	0	1	2	(216)	0	1	2	(218)	0	1	2	(219)	0	1	2	(221)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	2	2	1	0	2	2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
2	2	1	1	2	2	1	2	2	2	0	0	2	2	0	2	2	2	1	0	2	2	1	2

(222)	0	1	2	(235)	0	1	2	(239)	0	1	2	(241)	0	1	2	(252)	0	1	2	(255)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	2	1	1	0	2
2	2	2	0	2	2	0	2	2	2	2	0	2	2	2	2	2	2	0	2	2	2	1	2

(257)	0	1	2	(258)	0	1	2	(259)	0	1	2	(260)	0	1	2	(261)	0	1	2	(262)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0
2	1	0	0	2	1	0	1	2	1	0	2	2	1	1	0	2	1	1	1	2	1	1	2

(263)	0	1	2	(265)	0	1	2	(266)	0	1	2	(267)	0	1	2	(268)	0	1	2	(269)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0
2	1	2	0	2	1	2	2	2	2	0	1	2	2	0	2	2	2	1	1	2	2	1	2

(270)	0	1	2	(271)	0	1	2	(272)	0	1	2	(273)	0	1	2	(274)	0	1	2	(278)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
2	2	2	1	2	2	2	2	2	1	0	0	2	1	0	2	2	1	2	0	2	1	0	2

(281)	0	1	2	(282)	0	1	2	(283)	0	1	2	(284)	0	1	2	(287)	0	1	2	(306)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	1	2	1	1	2	1
2	1	1	2	2	1	2	0	2	1	2	1	2	1	2	2	2	2	1	2	2	1	1	2

(308)	0	1	2	(316)	0	1	2	(320)	0	1	2	(353)	0	1	2	(354)	0	1	2	(356)	0	1	2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	2	1	1	1	2	2	1	1	2	2	1	2	1	1	1	2	1	1	1	2	1	1
2	1	2	1	2	1	1	1	2	1	2	2	2	1	1	1	2	1	1	2	2	1	2	2

(359)	0	1	2	(376)	0	1	2	(377)	0	1	2	(378)	0	1	2	(379)	0	1	2	(380)	0	1	2
0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
1	2	1	2	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	2	2	1	0	0	2	1	0	1	2	1	0	2	2	1	1	0	2	1	1	1

(381)	0	1	2	(382)	0	1	2	(384)	0	1	2	(385)	0	1	2	(387)	0	1	2	(388)	0	1	2
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	2	2	1	2	0	2	1	2	2	2	2	0	0	2	2	0	2	2	2	1	0

(390)	0	1	2
0	0	0	1
1	0	0	0
2	2	1	2

(391)	0	1	2
0	0	0	1
1	0	0	0
2	2	2	0

(405)	0	1	2
0	0	0	1
1	0	0	1
2	1	1	0

(406)	0	1	2
0	0	0	1
1	0	0	1
2	1	1	1

(407)	0	1	2
0	0	0	1
1	0	0	1
2	1	1	2

(410)	0	1	2
0	0	0	1
1	0	0	1
2	1	2	2

(417)	0	1	2
0	0	0	1
1	0	0	1
2	2	2	0

(434)	0	1	2
0	0	0	1
1	0	0	2
2	1	2	0

(436)	0	1	2
0	0	0	1
1	0	0	2
2	1	2	2

(437)	0	1	2
0	0	0	1
1	0	0	2
2	2	0	0

(439)	0	1	2
0	0	0	1
1	0	0	2
2	2	0	2

(454)	0	1	2
0	0	0	1
1	0	1	0
2	1	0	0

(455)	0	1	2
0	0	0	1
1	0	1	0
2	1	0	1

(456)	0	1	2
0	0	0	1
1	0	1	0
2	1	0	2

(457)	0	1	2
0	0	0	1
1	0	1	0
2	1	1	0

(458)	0	1	2
0	0	0	1
1	0	1	0
2	1	1	1

(459)	0	1	2
0	0	0	1
1	0	1	0
2	1	1	2

(460)	0	1	2
0	0	0	1
1	0	1	0
2	1	2	0

(462)	0	1	2
0	0	0	1
1	0	1	0
2	1	2	2

(463)	0	1	2
0	0	0	1
1	0	1	0
2	2	0	0

(465)	0	1	2
0	0	0	1
1	0	1	0
2	2	0	2

(469)	0	1	2
0	0	0	1
1	0	1	0
2	2	2	0

(483)	0	1	2
0	0	0	1
1	0	1	1
2	1	1	0

(484)	0	1	2
0	0	0	1
1	0	1	1
2	1	1	1

(485)	0	1	2
0	0	0	1
1	0	1	1
2	1	1	2

(487)	0	1	2
0	0	0	1
1	0	1	1
2	1	2	1

(488)	0	1	2
0	0	0	1
1	0	1	1
2	1	2	2

(493)	0	1	2
0	0	0	1
1	0	1	1
2	2	1	1

(494)	0	1	2
0	0	0	1
1	0	1	1
2	2	1	2

(495)	0	1	2
0	0	0	1
1	0	1	1
2	2	2	0

(496)	0	1	2
0	0	0	1
1	0	1	1
2	2	2	1

(512)	0	1	2
0	0	0	1
1	0	1	2
2	1	2	0

(513)	0	1	2
0	0	0	1
1	0	1	2
2	1	2	1

(514)	0	1	2
0	0	0	1
1	0	1	2
2	1	2	2

(515)	0	1	2
0	0	0	1
1	0	1	2
2	2	0	0

(517)	0	1	2
0	0	0	1
1	0	1	2
2	2	0	2

(519)	0	1	2
0	0	0	1
1	0	1	2
2	2	1	1

(520)	0	1	2
0	0	0	1
1	0	1	2
2	2	1	2

(522)	0	1	2
0	0	0	1
1	0	1	2
2	2	2	1

(563)	0	1	2
0	0	0	1
1	0	2	1
2	1	1	2

(565)	0	1	2
0	0	0	1
1	0	2	1
2	1	2	1

(677)	0	1	2
0	0	0	1
1	1	1	0
2	0	0	0

(678)	0	1	2
0	0	0	1
1	1	1	0
2	0	0	1

(679)	0	1	2
0	0	0	1
1	1	1	0
2	0	0	2

(680)	0	1	2
0	0	0	1
1	1	1	0
2	0	1	0

(681)	0	1	2
0	0	0	1
1	1	1	0
2	0	1	2

(682)	0	1	2
0	0	0	1
1	1	1	0
2	0	2	0

(684)	0	1	2
0	0	0	1
1	1	1	0
2	0	2	2

(687)	0	1	2
0	0	0	1
1	1	1	0
2	1	2	0

(690)	0	1	2
0	0	0	1
1	1	1	0
2	2	2	0

(691)	0	1	2
0	0	0	1
1	1	1	2
2	0	0	0

(693)	0	1	2
0	0	0	1
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2	0	0	2

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2	0	1	1

(696)	0	1	2
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2	0	1	2

(697)	0	1	2
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2	0	2	0

(698)	0	1	2
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2	0	2	1

(704)	0	1	2
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2	1	1	1

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2	2	1	1

(756)	0	1	2
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2	1	1	1

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2	1	1	1

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2	2	0	0

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2	2	1	0

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2	2	2	1

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2	2	0	0

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2	2	1	0

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2	2	1	0

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2	2	1	2

(1132)	0	1	2
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2	2	2	0

(1133)	0	1	2
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1	0	1	2
2	2	2	1

(1151)	0	1	2
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2	2	0	0

(1153)	0	1	2
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2	2	1	0

(1176)	0	1	2
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2	2	1	0

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2	2	2	0

(1219)	0	1	2
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2	2	0	0

(1221)	0	1	2
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2	2	1	0

(1227)	0	1	2
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1	1	0	1
2	0	0	2

(1233)	0	1	2
0	0	0	2
1	1	0	1
2	0	2	2

(1242)	0	1	2
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2	2	0	0

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2	0	0	1

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2	1	0	0

(1281)	0	1	2
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1	1	1	2
2	2	2	0

(1321)	0	1	2
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1	1	2	1
2	2	0	0

(1433)	0	1	2
0	0	0	2
1	2	1	0
2	2	0	0

(1437)	0	1	2
0	0	0	2
1	2	1	0
2	2	2	0

(1481)	0	1	2
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2	2	2	0

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2	2	0	0

(1793)	0	1	2
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1	1	2	1
2	1	2	1

(1799)	0	1	2
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1	1	2	1
2	2	2	1

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1	1	2	2
2	2	1	1

(1837)	0	1	2
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1	2	0	0
2	2	0	0

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1	2	2	2
2	1	1	1

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1	1	0	0
2	2	0	0

(2090)	0	1	2
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1	1	0	0
2	2	1	0

(2102)	0	1	2
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2	2	1	0

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2	2	2	0

(2116)	0	1	2
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1	1	0	2
2	2	2	1

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2	0	2	1

(2436)	0	1	2
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2	1	2	1

(2460)	0	1	2
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2	0	0	0

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2	0	0	1

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2	0	1	0

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1	1	0	0
2	0	1	1

(2464)	0	1	2
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2	0	2	0

(2467)	0	1	2
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2	1	0	1

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2	2	0	0

(2476)	0	1	2
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2	2	2	0

(2478)	0	1	2
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1	1	0	1
2	0	0	0

(2479)	0	1	2
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2	0	0	1

(2480)	0	1	2
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2	0	1	0

(2483)	0	1	2
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2	1	0	0

(2486)	0	1	2
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2	2	2	0

(2487)	0	1	2
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2	0	0	0

(2493)	0	1	2
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2	1	0	0

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2	1	1	1

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2	2	1	1

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1	2	0	0
2	2	0	0

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2	1	1	1

(2654)	0	1	2
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2	1	2	1

(2686)	0	1	2
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2	1	2	1

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2	1	0	1

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1	0	2	0
2	1	2	1

(2739)	0	1	2
0	1	0	1
1	1	0	0
2	0	0	1

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