ON CENTRALIZERS OF FINITE LATTICES AND SEMILATTICES

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Dedicated to the memory of Ivo Rosenberg.

Abstract. We study centralizer clones of finite lattices and semilattices. For semilattices, we give two characterizations of the centralizer and also derive formulas for the number of operations of a given essential arity in the centralizer. We also characterize operations in the centralizer clone of a distributive lattice, and we prove that the essential arity of operations in the centralizer is bounded for every finite (possibly nondistributive) lattice. Using these results, we present a simple derivation for the centralizers of clones of Boolean functions.

1. Introduction

We say that the operations $f\colon A^n \to A$ and $g\colon A^m \to A$ *commute* if

$$g\big(f(a_{11}, a_{12}, \dots, a_{1n}), f(a_{21}, a_{22}, \dots, a_{2n}), \dots, f(a_{m1}, a_{m2}, \dots, a_{mn})\big)$$
$$= f\big(g(a_{11}, a_{21}, \dots, a_{m1}), g(a_{12}, a_{22}, \dots, a_{m2}), \dots, g(a_{1n}, a_{2n}, \dots, a_{mn})\big)$$

holds for all $a_{ij} \in A$ $(1 \leq i \leq m, 1 \leq j \leq n)$. This can be visualized as follows: for every $m \times n$ matrix $Q = (a_{ij})$, first applying f to the rows of Q and then applying g to the resulting column vector yields the same result as first applying g to the columns of Q and then applying f to the resulting row vector (see Figure 1). In other words, f and g commute if and only if g is a homomorphism from $(A; f)^m$ to $(A; f)$. In particular, if g is a unary operation (i.e., $m = 1$), then g commutes with f if and only if g is an endomorphism of the algebra $(A; f)$.

Commutation of operations is one of the many areas where Ivo Rosenberg had outstanding achievements, and we would like to contribute to this memorial issue by some results that were inspired by one of his last papers [13] on this topic. We study operations commuting with (semi)lattice operations, and we aim for concrete descriptions of these operations that allow us to classify (and in some cases also count) them according to the number of their essential variables. As a "byproduct", we also obtain a simple proof for Kuznetsov's description [11] of primitive positive clones on the two-element set.

Let us introduce some notions and notation that will be used throughout the paper. An n-ary *operation* on a set A (which will always assumed to be finite) is a map $f\colon A^n \to A$. The set of all n-ary operations on A is denoted by $\mathcal{O}_A^{(n)}$, and the set of all finitary operations on A is $\mathcal{O}_A = \bigcup_{n \geq 0} \mathcal{O}_A^{(n)}$. We say that the i-th variable of $f \in \mathcal{O}_A^{(n)}$ is *essential* (or that f *depends* on its i-th variable) if there exist tuples $\mathbf{a}, \mathbf{a}' \in A^n$ differing only in their i-th component such that $f(\mathbf{a}) \neq f(\mathbf{a}')$. The number of essential variables of f is called the *essential arity* of f. To simplify notation, we often assume that operations do not have inessential variables, thus we say that f is an essentially n-ary operation on A if $f \in \mathcal{O}_A^{(n)}$ and f depends on all of its variables.

A set C of operations on A is a *clone* (notation: $C \leq \mathcal{O}_A$) if C is closed under composition and contains the *projections* $(x_1, \dots, x_n) \mapsto x_i$ for all $1 \leq i \leq n$. The

2010 *Mathematics Subject Classification.* Primary 08A40; Secondary 06A07, 06A12, 06B05, 06B25, 06D05.

Key words and phrases. centralizer, clone, primitive positive clone, commuting operations, semilattice, lattice, distributive lattice, Post lattice.

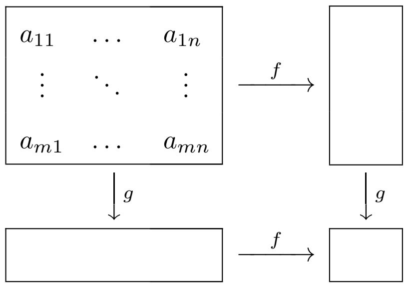

Figure 1. Commutation of f and g.

least clone containing a set $F \subseteq \mathcal{O}_A$ is called the clone *generated* by F, and it is denoted by $[F]$. The *centralizer* of F is the set F^* of all operations commuting with each member of F:

$$F^* = \{g \in \mathcal{O}_A : f \text{ commutes with } g \text{ for all } f \in F\}.$$

It is easy to verify that F^* is a clone (even if F is not), and $F^* = [F]^*$. This means that the centralizer of the clone of term operations of an algebra $(A; F)$ is F^*, which we call simply the centralizer of the algebra $(A; F)$.

Although there are countably infinitely many clones on the two-element set [14] and uncountably many clones on sets with at least three elements [9], only finitely many clones are of the form F^* on a finite set [3]. These are the so-called *primitive positive clones*; the complete list of primitive positive clones is known only for $|A| \leq 3$ [6, 11].

In Section 2 we present two different characterizations of the essentially n-ary members of the centralizer of a finite semilattice (one of them is a slight variation of a result of Larose [12]). We use these to give a general formula for the number of essentially n-ary operations commuting with the join operation of a finite lattice, and we illustrate this with the example of finite chains. This is a generalization of one of the results of [13], where this counting problem was solved for the three-element chain.

We study operations commuting with both the join and the meet operation of a lattice in Section 3. We begin with a summary of some facts about finite algebras in congruence distributive varieties, which, when applied to lattices, show that the essential arity of operations in the centralizer is bounded for every finite lattice. Therefore, we focus on the existence of essentially n-ary operations in the centralizer instead of counting them. We present some partial results about the centralizer of an arbitrary finite lattice, and then we completely determine the idempotent part of the centralizer. For distributive lattices, we provide an explicit description of the elements of the centralizer. We also prove that the centralizer of a finite distributive lattice contains an essentially n-ary operation if and only if the lattice has a sublattice (equivalently, quotient lattice) that is isomorphic to the 2^n-element Boolean lattice. As an "excuse" for the incompleteness of the results about arbitrary lattices, we present some examples showing that in general we cannot say more about the centralizer of a finite (nondistributive) lattice.

Finally, in Section 4 we describe the centralizer clones of Boolean functions. Primitive positive clones on the two-element set were described already by Kuznetsov [11], and later also in [8], and probably many of the readers of this paper have also computed these by themselves at some point. This is not a difficult task using the Post lattice (see Figure 4 in the Appendix), but it involves some case-by-case analysis. We offer a "painless" proof that covers all cases by just three general theorems. Besides presenting the list of the 25 primitive positive clones, we also give the centralizer of each Boolean clone in Table 3; we hope that it serves as a useful reference for anyone studying centralizer clones.

Some personal remarks from the second author about Ivo Rosenberg: For several years, Rosenberg's Theorem on the five types of minimal clones was my daily bread, as I was a doctoral student working on minimal clones under the supervision of Béla Csákány and Ágnes Szendrei. I met Ivo Rosenberg only once, and even then we spoke only a few words, but he still helped me a lot at the start of my career. He handled my very first paper as an editor of Algebra Universalis, and his encouragement meant a lot for me, when, to my horror, the referee found a serious gap in the proof of the main result. It took a lot of work to fill the gap, but eventually the paper was fixed and published. Later, Ivo Rosenberg was one of the reviewers of my PhD thesis. Of course, he did not fly from Montréal to Szeged for the defense, but, besides the official report, he sent a friendly hand-written letter to me (quite a curiosity nowadays!), in which he pointed out a few typos in the thesis (he was kind enough not to put them into the report) and expressed his interest in my work. Time flies quickly, the next generation is already here: now I am writing this paper with my doctoral student Endre Tóth, and we dedicate this contribution to the memory of Ivo Rosenberg.

2. Centralizers of finite semilattices

In this section we give two different characterizations of the centralizer clone of a join-semilattice $S = (S; \vee)$. Since we are interested in counting the essentially n-ary operations in the centralizer, we assume that S is finite, but some of our results are also valid for infinite complete semilattices.

2.1. **Joins of unary operations.** If S is a finite join-semilattice then it has a greatest element (denoted by 1), and if S also has a least element (denoted by 0), then there is a meet operation on S such that $(S; \vee, \wedge)$ is a lattice. In the latter case the centralizer clone $[\vee]^*$ is generated by its unary members (i.e., endomorphisms of $(S; \vee)$) together with the join operation. This was proved by B. Larose [12]; in the following theorem we reprove this result, and we extend it by providing a unique expression for any $f \in [\vee]^*$ as a join of endomorphisms, and we also determine the necessary and sufficient condition for f to depend on all of its variables.

Theorem 2.1. *Let $S = (S; \vee)$ be a finite semilattice with a least element 0 and greatest element 1. An n-ary operation $f \in \mathcal{O}_S$ belongs to the centralizer $[\vee]^*$ if and only if there exist unary operations $u_1, \dots, u_n \in [\vee]^*$ such that*

$$f(x_1, \dots, x_n) = u_1(x_1) \vee \dots \vee u_n(x_n) \text{ and } u_1(0) = u_2(0) = \dots = u_n(0).$$

The above expression for f is unique, and f depends on all of its variables if and only if none of the u_i are constant, i.e., $u_i(0) \neq u_i(1)$ for all $i \in \{1, \dots, n\}$.

Proof. The "if" part is clear: the join operation commutes with itself, hence if f can be written as a composition of $\vee$ with $u_1, \dots, u_n \in [\vee]^*$, then $f \in [\vee]^*$, as $[\vee]^*$ is closed under composition.

For the "only if" part let us assume that f is an n-ary operation in $[\vee]^*$, and define $u_1, \dots, u_n$ by $u_1(x) = f(x, 0, \dots, 0), \dots, u_n(x) = f(0, \dots, 0, x)$. Then obviously $u_1(0) = \dots = u_n(0) = f(0, \dots, 0)$; furthermore, $u_1, \dots, u_n \in [\vee]^*$, as f and the constant 0 operation belong to $[\vee]^*$. Since f commutes with $\vee$, applying the definition of commutation to the $n \times n$ matrix

$$\begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_n \end{pmatrix},$$

we can conclude that

$$\begin{aligned} f(x_1, \dots, x_n) &= f\big(x_1 \vee 0 \vee \dots \vee 0,\ \dots,\ 0 \vee \dots \vee 0 \vee x_n\big) \\ &= f(x_1, 0, \dots, 0) \vee \dots \vee f(0, \dots, 0, x_n) \\ &= u_1(x_1) \vee \dots \vee u_n(x_n). \end{aligned}$$

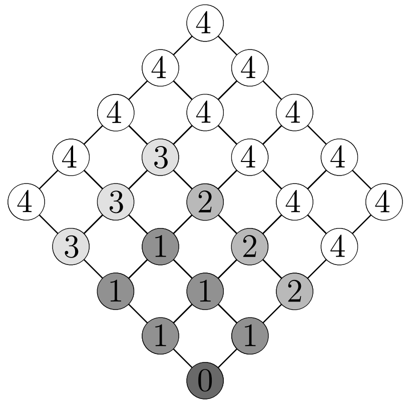

FIGURE 2. A binary operation in the centralizer of the chain $(\{0,1,2,3,4\};\vee)$

To prove uniqueness, assume that $f(x_1,\dots,x_n)=u_1(x_1)\vee\cdots\vee u_n(x_n)$ and $u_1(0)=u_2(0)=\cdots=u_n(0)$. Then we have

$$f(x_1,0,\dots,0)=u_1(x_1)\vee u_2(0)\vee\cdots\vee u_n(0)=u_1(x_1)\vee u_1(0)\vee\cdots\vee u_1(0)=u_1(x_1),$$

as u_1 is monotone. Thus u_1 is indeed uniquely determined by f, and the above equality also shows that if $u_1(0)\neq u_1(1)$, then f depends on its first variable: $f(0,0,\dots,0)=u_1(0)\neq u_1(1)=f(1,0,\dots,0)$. The statements about the uniqueness of u_i and about the essentiality of the i-th variable for $i=2,\dots,n$ can be proved in an analogous way. □

Example 2.2. Let $S=(\{0,1,2,3,4\};\vee)$ be a five-element chain (regarded as a join-semilattice), and let the unary operations u_1 and u_2 be defined by

$$u_1(0)=0,\ u_1(1)=1,\ u_1(2)=1,\ u_1(3)=3,\ u_1(4)=4;$$
$$u_2(0)=0,\ u_2(1)=1,\ u_2(2)=2,\ u_2(3)=4,\ u_2(4)=4.$$

Figure 2 shows the values of the binary operation $f(x_1,x_2)=u_1(x_1)\vee u_2(x_2)$. We can see that each of the sets $\{(a_1,a_2)\in S^2 : f(a_1,a_2)\leq b\}$ $(b=0,\dots,4)$ is a "lower rectangle". We will see later that for every join-semilattice S and every $b\in S$, the set $\{\mathbf{a}\in S^n : f(\mathbf{a})\leq b\}$ has a similar structure for all $f\in[\vee]^*$; moreover, we will characterize the operations in the centralizer in terms of these "down-sets".

The following theorem shows that the assumption about S having a least element cannot be dropped from Theorem 2.1: if a finite join-semilattice does not have a least element, then its centralizer cannot be generated by unary operations and the join operation.

Theorem 2.3. *The centralizer $[\vee]^*$ of a finite semilattice $S=(S;\vee)$ is generated by its unary part and the join operation if and only if S has a least element (i.e., if S is the join-reduct of a lattice).*

Proof. The "if" part follows from Theorem 2.1; for the "only if" part assume that $S=(S;\vee)$ is a finite semilattice without a least element. Then there are distinct minimal elements $a,b\in S$. We define a binary operation f on S by

$$f(x_1,x_2)=\begin{cases} a, & \text{if } (x_1,x_2)=(a,b);\\ b, & \text{if } (x_1,x_2)=(b,b);\\ a\vee b, & \text{otherwise.}\end{cases}$$

In order to prove that f commutes with the join operation, we need to verify the following identity:

$$f(x_1,x_2)\vee f(y_1,y_2)=f(x_1\vee y_1,x_2\vee y_2). \tag{2.1}$$

Since a is a minimal element, the only way of writing a as the join of two elements is $a = a \vee a$. Therefore, the left hand side of (2.1) is a if and only if $f(x_1, x_2) = f(y_1, y_2) = a$, and, by the definition of f, this holds only for $x_1 = y_1 = a$ and $x_2 = y_2 = b$. The right hand side of (2.1) equals a if and only if $x_1 \vee y_1 = a$ and $x_2 \vee y_2 = b$, and this is also equivalent to $x_1 = y_1 = a$ and $x_2 = y_2 = b$. Similarly, both the left hand side and the right hand side of (2.1) take the value b if and only if $x_1 = y_1 = x_2 = y_2 = b$. For all other inputs, both sides of (2.1) give $a \vee b$. Thus f belongs to the centralizer $[\vee]^*$, indeed.

If a binary operation can be obtained as a composition of the join operation and endomorphisms of S, then it can be written as $u_1(x_1) \vee u_2(x_2)$, where u_1 and u_2 are endomorphisms. (The proof of this fact is a routine term induction; we leave it to the reader.) Assume, for contradiction, that our operation f can be expressed in this form: $f(x_1, x_2) = u_1(x_1) \vee u_2(x_2)$. Then we have $a = f(a, b) = u_1(a) \vee u_2(b)$, and this implies $u_1(a) = u_2(b) = a$. On the other hand, $b = f(b, b) = u_1(b) \vee u_2(b) = u_1(b) \vee a \geq a$, which is a contradiction. $\square$

2.2. **Characterization in terms of "backwards" homomorphisms.** Next we derive another kind of characterization of the centralizer of the clone $[\vee]$, which describes, in some sense, the "distribution" of the values of an n-ary operation $f \in [\vee]^*$ on S^n, as illustrated by Figure 2. For this characterization we will not need the assumption that S has a least element. Nevertheless, we will consider the lattice $S_\perp$ obtained from S by adding a new element $\perp$ to the bottom of S. Thus let $S_\perp = S \cup \{\perp\}$, where $\perp$ is an element not contained in S, and we define the partial order on $S_\perp$ so that $\perp < a$ for all $a \in S$, and we keep the original ordering on the elements of S. Note that we add a new bottom element even if S happens to have a least element 0; in this case $\perp$ is the unique lower cover of 0.

If S is a join semilattice, then $f \in [\vee]^*$ if and only if f is a join-homomorphism from S^n to S. This motivates us to consider the set $\mathrm{Hom}_\vee(A, B)$ of all join-homomorphisms from A to B, where A and B are finite join-semilattices. We use the notation $\mathrm{Hom}_\vee^1(A, B)$ for the set of all join-homomorphisms from A to B that preserve the greatest element: $\mathrm{Hom}_\vee^1(A, B) := \{f \in \mathrm{Hom}_\vee(A, B) : f(1_A) = 1_B\}$. Similarly, if A and B have a least element, denoted by 0, then let $\mathrm{Hom}_\vee^0(A, B)$ denote the set of join-homomorphisms preserving the least element, and let $\mathrm{Hom}_\vee^{01}(A, B)$ be the set of join-homomorphisms preserving both boundary elements. Note that the least element of $A_\perp$ and $B_\perp$ is denoted by $\perp$ (not by 0), therefore in this case we will use the notation $\mathrm{Hom}_\vee^{\perp,1}(A_\perp, B_\perp)$ instead of $\mathrm{Hom}_\vee^{01}(A_\perp, B_\perp)$. For meet-semilattices A and B, the sets $\mathrm{Hom}_\wedge(A, B)$, $\mathrm{Hom}_\wedge^0(A, B)$, etc. are defined analogously.

We need to introduce one more notation: for an element a in a partially ordered set $(A; \leq)$, the *principal ideal* and the *principal filter* generated by a are defined and denoted as follows:

$$\downarrow a = \{c \in A : c \leq a\}, \qquad \uparrow a = \{c \in A : c \geq a\}.$$

Observe that in Figure 2, the elements labeled by numbers less than or equal to b form a principal ideal for any $b \in \{0, \ldots, 4\}$. This is a special case of the following lemma, which states that for all $f \in \mathrm{Hom}_\vee(A, B)$ and $b \in B$, the set $f^{-1}(\downarrow b) = \{a \in A : f(a) \leq b\}$ is a principal ideal whenever it is not empty.

Lemma 2.4. *Let $A = (A; \vee)$ and $B = (B; \vee)$ be finite semilattices. If $f\colon A \to B$ is a homomorphism, then $f^{-1}(\downarrow b) \subseteq A$ is either empty or a principal ideal for all $b \in B$.*

Proof. Assume that $f^{-1}(\downarrow b)$ is not empty. Join-homomorphisms are monotone, hence it is clear that $f^{-1}(\downarrow b)$ is an ideal, i.e., $a_1 \leq a_2 \in f^{-1}(\downarrow b)$ implies that $a_1 \in f^{-1}(\downarrow b)$. Moreover, $f^{-1}(\downarrow b)$ is closed under joins: if $a_1, a_2 \in f^{-1}(\downarrow b)$, then $f(a_1 \vee a_2) = f(a_1) \vee f(a_2) \leq b \vee b = b$, hence $a_1 \vee a_2 \in f^{-1}(\downarrow b)$. Since A is finite, we can take the join $a = \bigvee f^{-1}(\downarrow b)$ of all elements of $f^{-1}(\downarrow b)$, and from the above considerations it follows that $f^{-1}(\downarrow b)$ is the principal ideal generated by a. $\square$

In the next theorem we give a canonical bijection between the sets $\mathrm{Hom}_\vee(A,B)$ and $\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp)$, which will be the main tool for the promised characterization of the operations in the centralizer of a finite join-semilattice. Recall that $B_\perp$ and $A_\perp$ are lattices, and $\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp)$ denotes the set of all meet-homomorphisms $g\colon B_\perp\to A_\perp$ satisfying $g(\perp)=\perp$ and $g(1)=1$.

Theorem 2.5. *Let A,B be finite join-semilattices, and for every $f\in\mathrm{Hom}_\vee(A,B)$ and $g\in\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp)$, let us define the maps $f^\triangleleft\colon B_\perp\to A_\perp$ and $g^\triangleright\colon A\to B$ by*

$$f^\triangleleft(b)=\begin{cases}\bigvee f^{-1}(\downarrow b), & \text{if } f^{-1}(\downarrow b)\neq\emptyset,\\ \perp, & \text{if } f^{-1}(\downarrow b)=\emptyset,\end{cases}\qquad g^\triangleright(a)=\bigwedge g^{-1}(\uparrow a).$$

Then the following two maps are mutually inverse bijections:

$$\mathrm{Hom}_\vee(A,B)\to\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp),\quad f\mapsto f^\triangleleft,$$
$$\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp)\to\mathrm{Hom}_\vee(A,B),\quad g\mapsto g^\triangleright.$$

Proof. First let us show that if f is a join-homomorphism from A to B, then $f^\triangleleft$ is a meet-homomorphism from $B_\perp$ to $A_\perp$. By Lemma 2.4, $f^{-1}(\downarrow b)$ is either empty or a principal ideal, and in the latter case $f^\triangleleft(b)$ is the greatest element of $f^{-1}(\downarrow b)$ by the definition of $f^\triangleleft$. Therefore, we can reformulate the definition of $f^\triangleleft$ as follows:

$$\forall a\in A\ \forall b\in B_\perp\colon\ f(a)\le b\iff a\le f^\triangleleft(b). \tag{2.2}$$

(Note that if $f^{-1}(\downarrow b)=\emptyset$, then $f(a)\le b$ does not hold for any $a\in A$. In this case we have $f^\triangleleft(b)=\perp$, and the only element $a\in A_\perp$ satisfying $a\le f^\triangleleft(b)=\perp$ is $a=\perp$, thus the right hand side of (2.2) does not hold for any $a\in A$ either.) From (2.2) we can deduce the following chain of equivalences for all $a\in A$ and $b_1,b_2\in B_\perp$:

$$\begin{aligned}a\le f^\triangleleft(b_1\wedge b_2)&\iff f(a)\le b_1\wedge b_2\\&\iff f(a)\le b_1 \text{ and } f(a)\le b_2\\&\iff a\le f^\triangleleft(b_1)\text{ and } a\le f^\triangleleft(b_2)\\&\iff a\le f^\triangleleft(b_1)\wedge f^\triangleleft(b_2).\end{aligned}$$

Thus we have $a\le f^\triangleleft(b_1\wedge b_2)\iff a\le f^\triangleleft(b_1)\wedge f^\triangleleft(b_2)$ for every $a\in A$, and this implies that $f^\triangleleft(b_1\wedge b_2)=f^\triangleleft(b_1)\wedge f^\triangleleft(b_2)$, i.e., $f^\triangleleft$ is a meet-homomorphism. To verify $f^\triangleleft(1_B)=1_A$, we just need to observe that $f^{-1}(\downarrow 1_B)=f^{-1}(B)=A$, and the greatest element of A is indeed 1_A. Since $f^{-1}(\downarrow\perp)=\emptyset$, we have $f^\triangleleft(\perp)=\perp$. This completes the proof of the claim $f^\triangleleft\in\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp)$.

Now assume that $g\colon B_\perp\to A_\perp$ is a meet-homomorphism such that $g(1_B)=1_A$ and $g(\perp)=\perp$. Then $1_B\in g^{-1}(\uparrow a)$ for all $a\in A$, hence $g^{-1}(\uparrow a)$ is never empty. Therefore, the dual of Lemma 2.4 shows that $g^{-1}(\uparrow a)$ is a principal filter in $B_\perp$; moreover, $g(\perp)=\perp$ implies that $\perp\notin g^{-1}(\uparrow a)$ for every $a\in A$. This shows that the map $g^\triangleright\colon A\to B, a\mapsto\bigwedge g^{-1}(\uparrow a)$ is well defined. One can prove, by an argument similar to that of the previous paragraph, that $g^\triangleright$ is a join-homomorphism from A to B.

It remains to prove that the maps $f\mapsto f^\triangleleft$ and $g\mapsto g^\triangleright$ are inverses of each other. This follows immediately from the fact that $g=f^\triangleleft$ and $f=g^\triangleright$ are both equivalent to

$$\forall a\in A\ \forall b\in B_\perp\colon\ f(a)\le b\iff a\le g(b). \tag{2.3}$$

for all $f\in\mathrm{Hom}_\vee(A,B)$ and $g\in\mathrm{Hom}_\wedge^{\perp,1}(B_\perp,A_\perp)$. (Let us mention that a pair (f,g) of maps satisfying (2.3) is called a *monotone Galois connection.*) ☐

Remark 2.6. Let us give a categorical interpretation of Theorem 2.5. Let $\mathcal{J}$ denote the category of finite join-semilattices (with join-homomorphisms), and let $\mathcal{L}$ denote the category of finite lattices (with meet-homomorphisms). Then the following two maps are mutually inverse functors, thus $\mathcal{J}$ and $\mathcal{L}$ are isomorphic categories:

$$F\colon\mathcal{J}\to\mathcal{L},\ A\mapsto A_\perp,\ f\mapsto f^\triangleleft;$$
$$G\colon\mathcal{L}\to\mathcal{J},\ B\mapsto B\setminus\{0_B\},\ g\mapsto g^\triangleright.$$

(Here $B \setminus \{0_B\}$ is the join-semilattice obtained by removing the bottom element of the lattice B. Of course, if B is given as $B = F(A) = A_\perp$, then $0_B = \perp$.)

Theorem 2.5 can be useful if A is (much) larger than B, as in this case it might be an easier task to determine the meet-homomorphisms from $B_\perp$ to $A_\perp$ than describing the join-homomorphisms from A to B. This is the case when $A = S^n$ and $B = S$, where S is a finite join-semilattice: as mentioned before, the n-ary operations in $[\vee]^*$ are the join-homomorphisms from S^n to S, and these can be described in terms of the 1- and $\perp$-preserving meet-homomorphisms from $S_\perp$ to $(S^n)_\perp$, with the help of Theorem 2.5. We formulate this characterization in the next corollary, and we complement it with the necessary and sufficient condition for the operation to depend on all of its variables.

Corollary 2.7. *Let $S = (S; \vee)$ be a finite semilattice, and let n be a natural number. The n-ary members of $[\vee]^*$ are exactly the operations f of the form*

$$f \colon S^n \to S,\ \mathbf{x} \mapsto \bigwedge g^{-1}(\uparrow \mathbf{x}),$$

where $g \in \mathrm{Hom}_\wedge^{\perp,1}(S_\perp, (S^n)_\perp)$; here g is uniquely determined by f. The operation f depends on all of its variables if and only if for each $i \in \{1, \ldots, n\}$, the range of g contains an element of S^n whose i-th component is different from 1. If S has a least element (i.e., if S is a lattice), then the latter condition is satisfied if and only if the range of g contains a tuple from $(S \setminus \{1\})^n$.

Proof. An n-ary operation $f \in \mathcal{O}_S$ belongs to $[\vee]^*$ if and only if f is a join-homomorphism from S^n to S. Applying Theorem 2.5 with $A = S^n$ and $B = S$, we see that these operations can be uniquely written as $f(\mathbf{x}) = g^\triangleright(\mathbf{x}) = \bigwedge g^{-1}(\uparrow \mathbf{x})$ with $g \in \mathrm{Hom}_\wedge^{\perp,1}(B_\perp, A_\perp)$.

We prove that f depends on its i-th variable if and only if the range of g contains a tuple $\mathbf{s}_i \in S^n$ such that the i-th component of $\mathbf{s}_i$ is not equal to 1. First assume that f depends on the i-th variable; this means that there exist elements $\mathbf{a}, \mathbf{a}' \in S^n$ differing only in their i-th component such that $b := f(\mathbf{a})$ and $b' := f(\mathbf{a}')$ are different. We can assume without loss of generality that either $b < b'$ or b and b' are incomparable. In both cases we can conclude that $\mathbf{a} \in f^{-1}(\downarrow b)$ and $\mathbf{a}' \notin f^{-1}(\downarrow b)$. By Theorem 2.5, we have $g = f^\triangleleft$, thus $g(b) = \bigvee f^{-1}(\downarrow b)$. Therefore, $\mathbf{a} \leq g(b)$ and $\mathbf{a}' \nleq g(b)$. This implies that the i-th component of the tuple $\mathbf{s}_i := g(b) \in S^n$ (which certainly belongs to the range of g) is strictly less than 1.

Conversely, let us suppose that there is an element $b \in S$ such that the i-th component of $\mathbf{s}_i := g(b)$ is less than 1. Letting $\mathbf{s}_i'$ be the tuple obtained from $\mathbf{s}_i$ by changing its i-th component to 1, we have $\mathbf{s}_i < \mathbf{s}_i'$. Now $b \in g^{-1}(\uparrow \mathbf{s}_i)$ but $b \notin g^{-1}(\uparrow \mathbf{s}_i')$, therefore $g^{-1}(\uparrow \mathbf{s}_i) \neq g^{-1}(\uparrow \mathbf{s}_i')$. Since $f = g^\triangleright$, this implies that

$$f(\mathbf{s}_i) = g^\triangleright(\mathbf{s}_i) = \bigwedge g^{-1}(\uparrow \mathbf{s}_i) \neq \bigwedge g^{-1}(\uparrow \mathbf{s}_i') = g^\triangleright(\mathbf{s}_i') = f(\mathbf{s}_i').$$

Taking into account that $\mathbf{s}_i$ and $\mathbf{s}_i'$ differ only at the i-th component, we can conclude that f does depend on its i-th variable.

If S is a lattice, then $\mathbf{s}_1 \wedge \cdots \wedge \mathbf{s}_n$ is a tuple in the range of g, and all of its components are less than 1. □

Example 2.8. Let us illustrate Corollary 2.7 with the operation f given in Example 2.2. The corresponding operation $g = f^\triangleleft \colon S_\perp \to (S^2)_\perp$ is the following:

$$g(\perp) = \perp,\ g(0) = (0,0),\ g(1) = (2,1),\ g(2) = (2,2),\ g(3) = (3,2),\ g(4) = (4,4).$$

2.3. **Counting.** Using the characterizations presented in the previous subsection, we can determine the exact number of n-ary operations in the centralizers of certain semilattices. First we count the essentially n-ary operations commuting with the join operation of the smallest non-lattice semilattice. It will be convenient to use the convention $0^0 = 1$ (for justification, see [10]).

Proposition 2.9. *Let $S = (\{a, b, 1\}, \vee)$ be the join semilattice with $a \vee b = 1$. The number of essentially n-ary operations in the centralizer of S is $8^n - 6^n + 2 \cdot 2^n + 0^n$.*

Proof. By Corollary 2.7, we need to count the elements $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(S_\perp, (S^n)_\perp)$ such that the range of g contains a tuple from $S^{i-1} \times \{a, b\} \times S^{n-i}$ for every $i \in \{1, \dots, n\}$. For an arbitrary map $g\colon S_\perp \to (S^n)_\perp$, we have $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(S_\perp, (S^n)_\perp)$ if and only if $g(1) = 1$, $g(\perp) = \perp$ and $g(a) \wedge g(b) = \perp$; moreover, such a map is uniquely determined by $g(a)$ and $g(b)$. We distinguish four cases upon these values (we denote by $f = g^{\triangleright}$ the element of $[\vee]^*$ corresponding to g in Corollary 2.7).

(1) If $g(a) = \perp = g(b)$, then f is constant 1, hence f is essentially n-ary if and only if $n = 0$. Thus the number of essentially n-ary operations of this type is 0^n, i.e., it is 1 if $n = 0$ and 0 if $n > 0$.
(2) If $g(a) \neq \perp = g(b)$, then f depends on all of its variables if and only if $g(a) \in \{a, b\}^n$, thus the number of essentially n-ary operations $f \in [\vee]^*$ of this type is 2^n (for $n = 0$ we get the constant a function).
(3) If $g(a) = \perp \neq g(b)$, then, similarly to the previous case, we have 2^n functions (here $n = 0$ corresponds to the constant b function).
(4) If $g(a) \neq \perp \neq g(b)$, then let $\mathbf{s} := g(a) \in S^n$ and $\mathbf{t} := g(b) \in S^n$. Writing these two tuples below each other, we get the $2 \times n$ matrix $\left(\begin{smallmatrix} s_1 & \dots & s_n \\ t_1 & \dots & t_n \end{smallmatrix}\right)$. Now f depends on all of its variables if and only if no column of this matrix is $(1,1)^{\mathsf{T}}$, and there are 8^n such matrices. However, some of the corresponding maps g will violate the condition $g(a) \wedge g(b) = \perp$: we must exclude those matrices that contain neither $(a, b)^{\mathsf{T}}$ nor $(b, a)^{\mathsf{T}}$ as a column. The number of such matrices is 6^n, so we obtain $8^n - 6^n$ essentially n-ary operations $f \in [\vee]^*$ in this case.

Summing up the four cases, we see that the number of essentially n-ary operations in $[\vee]^*$ is $8^n - 6^n + 2 \cdot 2^n + 0^n$. □

Remark 2.10. It is easy to see that if the number of essentially n-ary operations in a clone C is p_n, then the number of all operations of arity n in C is $\sum_{k=0}^{n} \binom{n}{k} p_k$. Thus, by Proposition 2.9 (and by the binomial theorem), the number of n-ary operations in the centralizer of the join operation of the semilattice $(\{a, b, 1\}, \vee)$ is

$$\sum_{k=0}^{n} \binom{n}{k} (8^k - 6^k + 2 \cdot 2^k + 0^k) = 9^n - 7^n + 2 \cdot 3^n + 1^n.$$

Next we provide a general formula for the number of essentially n-ary operations commuting with the join operation of a finite lattice, and then we apply it to the case of finite chains.

Theorem 2.11. *Let $S = (S; \vee, \wedge)$ be a finite lattice, and let n be a natural number. The number of essentially n-ary operations in $[\vee]^*$ is*

$$\sum_{b \in S} (|\mathrm{Hom}_{\vee}^{0}(S, \uparrow b)| - 1)^n = \sum_{b \in S} (|\mathrm{Hom}_{\wedge}^{1}(\uparrow b, S)| - 1)^n.$$

Proof. According to Theorem 2.1, the essentially n-ary members of the centralizer are in a one-to-one correspondence with the tuples $(u_1, \dots, u_n) \in \mathrm{Hom}_{\vee}(S, S)^n$ such that $u_1(0) = \dots = u_n(0)$ and none of the u_i are constant. Let $b := u_1(0)$, then each u_i maps S to $\uparrow b$ in such a way that the least element of S is mapped to the least element of $\uparrow b$, i.e., $u_i \in \mathrm{Hom}_{\vee}^{0}(S, \uparrow b)$ for $i = 1, \dots, n$. However, we need to exclude the constant b function, hence the number of choices for each u_i is $|\mathrm{Hom}_{\vee}^{0}(S, \uparrow b)| - 1$, and this gives the first formula. (Note that for $b = 1$, the principal filter $\uparrow b$ has just one element, thus $|\mathrm{Hom}_{\vee}^{0}(S, \uparrow b)| - 1 = 0$. Therefore, the contribution of $b = 1$ to the sum is 0^n, and this could be omitted if $n > 0$. However, for $n = 0$, we need to keep the term $0^0 = 1$ in order to get the correct number of nullary operations, which is clearly $|S|$.)

The second formula follows from the first one by applying Theorem 2.5 to $A = S$ and $B = \uparrow b$. If $f \in \mathrm{Hom}_{\vee}(S, \uparrow b)$ and $g \in \mathrm{Hom}_{\wedge}^{\perp,1}((\uparrow b)_\perp, S_\perp)$ correspond to each other under the bijections of Theorem 2.5, then $f(0) = b$ if and only if $g(0) \neq \perp$ (i.e., g does not take the value $\perp$ except for $g(\perp) = \perp$). Therefore, we obtain a bijection from $\mathrm{Hom}_{\vee}^{0}(S, \uparrow b)$ to $\mathrm{Hom}_{\wedge}^{1}(\uparrow b, S)$ by restricting $f^{\triangleleft}$ to the set $\uparrow b$ for each $f \in \mathrm{Hom}_{\vee}^{0}(S, \uparrow b)$. □

Remark 2.12. The second formula of the above theorem can be also derived directly from Corollary 2.7 as follows. We need to count the meet-homomorphisms $g \in \mathrm{Hom}_{\wedge}^{\perp,1}(S_{\perp}, (S^n)_{\perp})$ whose range satisfies the conditions of Corollary 2.7. If S is a lattice and g is such a meet-homomorphism, then there is a least element $b \in S$ such that $g(b) \neq \perp$. Restricting g to $\uparrow b$, we get a 1-preserving meet-homomorphism from $\uparrow b$ to S^n, which can be viewed as an n-tuple $(g_1, \ldots, g_n)$ of 1-preserving meet-homomorphisms from $\uparrow b$ to S. The range of g contains a tuple from $(S \setminus \{1\})^n$ if and only if $g(b) \in (S \setminus \{1\})^n$, which holds if and only if none of the g_i are constant 1. Therefore, there are exactly $(|\mathrm{Hom}_{\wedge}^{1}(\uparrow b, S)| - 1)^n$ such tuples $(g_1, \ldots, g_n)$, and this proves the second formula of Theorem 2.11. This argument does not work if S is not a lattice, even though Corollary 2.7 holds in that case, too. The problem is that there may be no least element b with $g(b) \neq \perp$; in fact, $g^{-1}(S^n)$ is not necessarily closed under meets. Nevertheless, as we have seen in Proposition 2.9, Corollary 2.7 can be used to count the essentially n-ary operations in a semilattice even if it is not a lattice.

Proposition 2.13. *The number of essentially n-ary operations commuting with the join-operation of a chain of cardinality ℓ is*

$$\sum_{i=1}^{\ell} \left[\binom{\ell + i - 2}{\ell - 1} - 1 \right]^n.$$

Proof. First, as an auxiliary result, let us count the join-homomorphisms from an r-element chain $A = \{a_1 < \cdots < a_r\}$ to an s-element chain $B = \{b_1 < \cdots < b_s\}$. Clearly, the join-homomorphisms in this case are just the monotone maps, thus an element of $\mathrm{Hom}_{\vee}(A, B)$ can be given by a nondecreasing sequence $f(a_1) \leq \cdots \leq f(a_r)$ in B. These sequences can be viewed as r-combinations with repetitions from the elements $b_1, \ldots, b_s$, and the number of such combinations is $\binom{s+r-1}{r} = \binom{s+r-1}{s-1}$.

Now let S be a chain of cardinality ℓ, and let $b \in S$. The 0-preserving join-homomorphisms from S to $\uparrow b$ are in a one-to-one correspondence with the join-homomorphisms from $S \setminus \{0\}$ to $\uparrow b$ (by restricting to $S \setminus \{0\}$). Note that $S \setminus \{0\}$ is a chain of size $\ell - 1$, and if b is the i-th element from the top in S, then $\uparrow b$ is an i-element chain. Thus, by the considerations made in the first paragraph, we have

$$|\mathrm{Hom}_{\vee}^{0}(S, \uparrow b)| = |\mathrm{Hom}_{\vee}(S \setminus \{0\}, \uparrow b)| = \binom{\ell + i - 2}{\ell - 1}.$$

Applying Theorem 2.11 completes the proof: we just need to substitute the above formula into the first sum of Theorem 2.11 (replacing the summation variable b by i). □

Remark 2.14. For an arbitrary poset A, the number of monotone maps from A to an s-element chain is a polynomial in s, called the *order polynomial* of A. As we have seen in the proof of the above proposition, the order polynomial of the r-element chain is $\binom{s+r-1}{r}$. This fact, and much more about order polynomials can be found in [15].

Remark 2.15. Similarly to Remark 2.10, we can derive from Proposition 2.13 that the number of n-ary operations in the centralizer of the join (or meet) operation of an ℓ element chain is

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i=1}^{\ell} \left[\binom{\ell + i - 2}{\ell - 1} - 1 \right]^k = \sum_{i=1}^{\ell} \sum_{k=0}^{n} \binom{n}{k} \left[\binom{\ell + i - 2}{\ell - 1} - 1 \right]^k = \sum_{i=1}^{\ell} \binom{\ell + i - 2}{\ell - 1}^n.$$

Example 2.16. For the two-element chain (regarded as a semilattice), Proposition 2.13 gives the formula $1^n + 0^n$ for the number of essentially n-ary operations (recall that 0^0 is defined as 1), and Remark 2.15 gives $2^n + 1^n$ for the number of all n-ary operations in the centralizer (which are of course also easily verified without our results). Similarly, for the three-element chain, we have $5^n + 2^n + 0^n$ essentially n-ary operations and $6^n + 3^n + 1^n$ operations of arity n in the centralizer of the join operation; for the four-element chain we get the numbers $19^n + 9^n + 3^n + 0^n$ and $20^n + 10^n + 4^n + 1^n$, etc. A formula for the number of n-ary operations in the centralizer of the meet operation of the three-element chain appeared already in [13]:

$$3^{n+1} + \sum_{\substack{0 \leq p < n \\ 0 \leq q \leq n-p}} \binom{n}{p} \binom{n-p}{q} (3^p 2^q - 1),$$

This can be simplified to $6^n + 3^n + 1^n$ with the help of the binomial theorem.

3. Centralizers of finite lattices

In this section $L = (L; \vee, \wedge)$ denotes a finite lattice with greatest element 1 and least element 0. To avoid degenerate cases, we always assume that L is not trivial, i.e., $|L| \geq 2$, thus $0 \neq 1$. We use the symbol $\mathbf{2}$ for the two-element lattice $\{0, 1\}$, and $\mathcal{P}(U)$ denotes the lattice of all subsets of a set U. Note that $\mathcal{P}(\{1, \dots, n\}) \cong \mathbf{2}^n$.

3.1. **Finite algebras in congruence distributive varieties.** First we recall some general facts about algebras generating a congruence distributive variety. These results certainly apply to lattices, as the variety of lattices is congruence distributive.

Definition 3.1 ([2, Definition IV§11.4])**.** Let $A_1, \dots, A_n$ be algebras of the same type and let $\vartheta_i \in \mathrm{Con}(A_i)$ for all $i \in \{1, \dots, n\}$. The *product congruence* $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$ on $A_1 \times \cdots \times A_n$ is defined by $\big((a_1, \dots, a_n), (b_1, \dots, b_n)\big) \in \vartheta \Leftrightarrow \forall i \in \{1, \dots, n\} \colon (a_i, b_i) \in \vartheta_i$.

Lemma 3.2 ([2, Lemma IV§11.10])**.** *Let $\mathcal{V}$ be a congruence distributive variety, $A_1, \dots, A_n \in \mathcal{V}$ and $\vartheta \in \mathrm{Con}(A_1 \times \cdots \times A_n)$. Then there exist $\vartheta_i \in \mathrm{Con}(A_i)$ for all $i \in \{1, \dots, n\}$ such that $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$.*

The next theorem and its corollary is implicit in [16, Theorem 2]. (Similar ideas also appear in [4, 5].)

Lemma 3.3 ([16, Theorem 2])**.** *Let $\mathcal{V}$ be a congruence distributive variety and $A \in \mathcal{V}$. Then the following are equivalent:*

(i) *There exists an essentially n-ary operation $f \in \mathcal{O}_A$ that is a homomorphism from A^n to A.*
(ii) *There exist $\vartheta_i \in \mathrm{Con}(A)$ such that $\vartheta_i \neq A^2$ for all $i \in \{1, \dots, n\}$, and $A_1/\vartheta_1 \times \cdots \times A_n/\vartheta_n$ embeds into A.*

Proof. (i) $\Rightarrow$ (ii): Let $f \colon A^n \to A$ be a homomorphism. Then by Lemma 3.2 there exist $\vartheta_1, \dots, \vartheta_n \in \mathrm{Con}(A)$ such that $\ker(f) = \vartheta_1 \times \cdots \times \vartheta_n$. Since f is essentially n-ary, we have that $\vartheta_i \neq A^2$ for all $i \in \{1, \dots, n\}$. By the homomorphism theorem we have $A^n/\ker(f) = A^n/(\vartheta_1 \times \cdots \times \vartheta_n) = A/\vartheta_1 \times \cdots \times A/\vartheta_n \cong f(A^n)$, and since $f(A^n)$ is a subalgebra of A, it is clear that $A_1/\vartheta_1 \times \cdots \times A_n/\vartheta_n$ is embeddable into A.

(ii) $\Rightarrow$ (i): Let $\varphi \colon A/\vartheta_1 \times \cdots \times A/\vartheta_n \hookrightarrow A$ be an embedding, let $\vartheta \in \mathrm{Con}(A^n)$ be the product congruence $\vartheta = \vartheta_1 \times \cdots \times \vartheta_n$ and let ν denote the natural homomorphism from A^n to $A^n/\ker \varphi = A^n/\vartheta$. Then we define the operation $f \in \mathcal{O}_A^{(n)}$ as $f(x_1, \dots, x_n) = \varphi(x_1/\vartheta_1, \dots, x_n/\vartheta_n) = \varphi(\nu(x_1, \dots, x_n))$. Thus $f = \varphi \circ \nu$ is a homomorphism, and since $\vartheta_i \neq A^2$ for all i, we have that f is essentially n-ary. $\square$

Corollary 3.4 ([16, Theorem 2])**.** *Let $\mathcal{V}$ be a congruence distributive variety, let $A \in \mathcal{V}$ be a finite algebra and let C denote the clone of term operations of A. If there is an essentially n-ary operation in C^*, then we have $n \leq \log_2 |A|$. In other words, the essential operations in C^* are at most $\log_2 |A|$-ary.*

Proof. By Lemma 3.3, if we have an essentially n-ary operation in C^*, then there exist $\vartheta_1, \dots, \vartheta_n \in \mathrm{Con}(A)$ such that $A/\vartheta_1 \times \cdots \times A/\vartheta_n$ is embeddable into A. Therefore we have $|A/\vartheta_1 \times \cdots \times A/\vartheta_n| = |A/\vartheta_1| \cdot \ldots \cdot |A/\vartheta_n| \leq |A|$, and since for every i we have $\vartheta_i \neq A^2$, it follows that $2^n = 2 \cdot \ldots \cdot 2 \leq |A/\vartheta_1| \cdot \ldots \cdot |A/\vartheta_n| \leq |A|$. Thus $n = \log_2(2^n) \leq \log_2(|A|)$. $\square$

Remark 3.5. Let, as above, A be a finite algebra in a congruence distributive variety, and let C denote the clone of term operations of A. If all constants belong to C, then every homomorphism $f\colon A^n \to A$ (such as the one considered in the proof of (i) $\Rightarrow$ (ii) in Lemma 3.3) is idempotent, hence surjective. Therefore, in this case each essentially n-ary operation in the centralizer gives rise to a direct product decomposition of A. Conversely, assume that $A = A_1 \times \cdots \times A_n$, where each A_i is a nontrivial algebra. Then the following essentially n-ary operation f clearly belongs to the centralizer of A (as it is just a tuple of projections):

$$f((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn})) = (a_{11}, \ldots, a_{nn}) \qquad (a_{i,j} \in A_j,\ i, j = 1, \ldots, n).$$

Such operations are called *diagonal operations*. In fact, this is not merely an example of an operation in the centralizer: since $(A; C^*)$ is an idempotent algebra with a finite upper bound on the essential arity of term operations, a result of Urbanik [19, Theorem 3] implies that every element of C^* is a diagonal operation. Thus we see that if A is a finite algebra such that every constant is a term operation of A, then the essentially n-ary operations in the centralizer correspond to decompositions of A into a direct product of n nontrivial algebras.

3.2. **Finite lattices.** We give a necessary condition for an operation $f \in \mathcal{O}_L^{(n)}$ to belong to the centralizer of the clone of a finite lattice L. We will show in Theorem 3.10 that for distributive lattices this condition is also sufficient. For nondistributive lattices this may not be the case (see Example 3.14), but for simple lattices the converse of Proposition 3.6 happens to be true (see Remark 3.7).

Proposition 3.6. *If $L = (L; \vee, \wedge)$ is a finite lattice and $f \in \mathcal{O}_L^{(n)}$ belongs to the centralizer of L, then there exist unary operations $u_1, \ldots, u_n \in [\vee, \wedge]^*$ such that $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and $u_i(1) \wedge u_j(1) = u_1(0) = \cdots = u_n(0)$ for all $i, j \in \{1, \ldots, n\}, i \neq j$. Furthermore, the operation f depends on all of its variables if and only if none of the unary operations u_i are constant.*

Proof. As in the proof of Theorem 2.1, we define the unary operations $u_1, \ldots, u_n$ as $u_1(x) = f(x, 0, \ldots, 0), \ldots, u_n(x) = f(0, \ldots, 0, x)$. By that theorem, for these unary operations we have $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and $u_1(0) = \cdots = u_n(0) = f(0, \ldots, 0)$. Since f and the constant 0 operation belong to $[\vee, \wedge]^*$, we have $u_1, \ldots, u_n \in [\vee, \wedge]^*$. It only remains to show that for all $i, j \in \{1, \ldots, n\}, i \neq j$ we have $u_i(1) \wedge u_j(1) = f(0, \ldots, 0)$. For notational simplicity, let us assume that $i = 1$ and $j = 2$; the proof of the general case is similar. Using that f commutes with the operation $\wedge$, with the help of the 2 by n matrix $\left(\begin{smallmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{smallmatrix}\right)$ we can conclude that the following equality holds:

$$\begin{aligned} f(0, \ldots, 0) &= f\big(1 \wedge 0,\ 0 \wedge 1,\ 0 \wedge 0,\ \ldots,\ 0 \wedge 0) \\ &= f(1, 0, 0, \ldots, 0) \wedge f(0, 1, 0, \ldots, 0) = u_1(1) \wedge u_2(1). \end{aligned} \qquad \square$$

Remark 3.7. Let us verify that the converse of Proposition 3.6 is true for simple lattices. Indeed, assume that L is a finite simple lattice, $u_1, \ldots, u_n$ are endomorphisms of L such that $u_1(0) = \cdots = u_n(0)$ and $u_i(1) \wedge u_j(1) = u_1(0)$ whenever $i \neq j$, and let $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$. Simplicity of L implies that the kernel of every u_i is one of the two relations L^2 or $\{(a, a) \mid a \in L\}$; therefore, every u_i is either a constant operation or an automorphism of L. We distinguish three cases on the number of automorphisms occurring in $u_1(x_1) \vee \cdots \vee u_n(x_n)$.

(i) If each u_i is constant, then f is also constant, hence $f \in [\vee, \wedge]^*$.
(ii) If u_i and u_j are automorphisms with $i \neq j$, then $1 = u_i(1) \wedge u_j(1) = u_i(0) = 0$, which is a contradiction.
(iii) In the remaining cases we can assume without loss of generality that u_1 is an automorphism and $u_2, \ldots, u_n$ are all constants. Then $u_1(0) = 0$, thus $u_2(0) = \cdots = u_n(0) = 0$ and $f(x_1, \ldots, x_n) = u_1(x_1)$.

Therefore, f depends on at most one variable, and f is equivalent to an automorphism or a constant, hence $f \in [\vee, \wedge]^*$. Thus Proposition 3.6 gives a complete description of the centralizer for finite simple (nondistributive) lattices; moreover, every operation in $[\vee, \wedge]^*$ depends on at most one variable in this case.

The variety of lattices is congruence distributive, thus if the centralizer of a finite lattice L contains an essentially n-ary operation, then, by Lemma 3.3, a direct product of n nontrivial lattices embeds into L. Since every nontrivial lattice contains the two-element lattice $\mathbf{2}$ as a sublattice, it follows that $\mathbf{2}^n$ embeds into L. This motivates us to investigate connections among the following three conditions for a finite lattice L:

(Ess) there exists an essentially n-ary operation in $[\vee, \wedge]^*$;
(Sub) there exists a sublattice of L that is isomorphic to $\mathbf{2}^n$;
(Quo) there exists a congruence ϑ of L such that L/ϑ is isomorphic to $\mathbf{2}^n$.

We will see in Section 3.4 that these conditions are equivalent if L is distributive. For arbitrary finite lattices the only valid implications are (Ess) $\Rightarrow$ (Sub) and (Sub)&(Quo) $\Rightarrow$ (Ess); we prove these below, and we provide counterexamples for all other implications in Section 3.5.

Proposition 3.8. *If $L = (L; \vee, \wedge)$ is a finite lattice, then we have* (Ess) $\Rightarrow$ (Sub) *and* (Sub)&(Quo) $\Rightarrow$ (Ess).

Proof. As explained above, the implication (Ess) $\Rightarrow$ (Sub) follows immediately from Lemma 3.3, as the variety of lattices is congruence distributive and every nontrivial lattice has a two-element sublattice.

Suppose now that (Sub) and (Quo) hold for a finite lattice L. Then by (Quo), there is a quotient of L that is isomorphic to $\mathbf{2}^n$, and since $\mathbf{2}$ is a homomorphic image of $\mathbf{2}^n$, we have that $\mathbf{2}$ is a homomorphic image of L. Let φ denote a surjective homomorphism $\varphi\colon L \to \mathbf{2}$. Then obviously $\ker(\varphi) \neq L^2$ and by (Sub) we have that $L/\ker(\varphi) \times \cdots \times L/\ker(\varphi) = \big(L/\ker(\varphi)\big)^n \cong \mathbf{2}^n$ embeds into L. Therefore, by Lemma 3.3, there exists an essentially n-ary operation in $\mathcal{O}_L$. □

3.3. **Finite lattices with constants.**

As mentioned in Remark 3.5, the idempotent part of the centralizer of a finite lattice consists of diagonal operations. As an application of Proposition 3.6, we provide a short proof for this special case of the characterization of idempotent algebras with term operations of bounded essential arity obtained by Urbanik [19, Theorem 3] .

Theorem 3.9. *Let $L = (L; \vee, \wedge)$ be a finite lattice, and let C be the clone generated by the lattice operations $\vee, \wedge$ together with all constant operations. An essentially n-ary operation $f \in \mathcal{O}_L^{(n)}$ belongs to the centralizer C^* if and only if f is a diagonal operation corresponding to a direct decomposition $L = L_1 \times \cdots \times L_n$ of L into nontrivial sublattices $L_1, \ldots, L_n$.*

Proof. It is clear that every diagonal operation belongs to the centralizer (cf. Remark 3.5). Conversely, let $f \in C^*$ be an essentially n-ary operation. Then f commutes with the lattice operations $\vee$ and $\wedge$, hence, by Proposition 3.6, f can be written as $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$, where each u_i is an endomorphism of L. Since C includes the constants, f is idempotent:

$$x = f(x, \ldots, x) = u_1(x) \vee \cdots \vee u_n(x). \tag{3.1}$$

We claim that $L \cong L_1 \times \cdots \times L_n$, where $L_i = u_i(L)$ denotes the range of u_i. Let $\ker(f) = \vartheta_1 \times \cdots \times \vartheta_n$ as in the proof of (i) $\Rightarrow$ (ii) in Lemma 3.3. From (3.1) we can infer $0 = u_1(0) \vee \cdots \vee u_n(0)$, hence $0 = u_1(0) = \cdots = u_n(0)$. Therefore, we have $f(0, \ldots, 0, x_i, 0, \ldots, 0) = u_i(x_i)$, which implies that $\ker(u_i) = \vartheta_i$, and consequently, $L_i \cong L/\ker(u_i) = L/\vartheta_i$ for $i = 1, \ldots, n$. Since the homomorphism $f\colon L^n \to L$ is surjective by (3.1), the homomorphism theorem gives the desired isomorphism:

$$L \cong L^n/\ker(f) = L^n/(\vartheta_1 \times \cdots \times \vartheta_n) = L/\vartheta_1 \times \cdots \times L/\vartheta_n \cong L_1 \times \cdots \times L_n.$$

To prove that f is the diagonal operation corresponding to this direct product decomposition, let us consider the following two maps:

$$\varphi\colon L \to L_1 \times \cdots \times L_n,\ \ x \mapsto (u_1(x), \ldots, u_n(x));$$
$$\psi\colon L_1 \times \cdots \times L_n \to L,\ \ (z_1, \ldots, z_n) \mapsto z_1 \vee \cdots \vee z_n.$$

It is clear that φ is a homomorphism, and from (3.1) it follows that $\psi(\varphi(x)) = x$. Taking into account that L is finite, we can conclude that φ and ψ are mutually inverse isomorphisms. In particular, we have $\varphi(\psi(z_1, \ldots, z_n)) = (z_1, \ldots, z_n)$, which implies

$$u_i(z_1 \vee \cdots \vee z_n) = z_i \text{ for all } i \in \{1, \ldots, n\} \text{ and } (z_1, \ldots, z_n) \in L_1 \times \cdots \times L_n.$$

This means exactly that f becomes a diagonal operation when "transported" from L to $L_1 \times \cdots \times L_n$ by the isomorphisms φ and ψ. □

3.4. **Finite distributive lattices.** First let us prove that in the case of distributive lattices, Proposition 3.6 gives a complete description of the centralizer.

Theorem 3.10. *Let $L = (L; \vee, \wedge)$ be a finite distributive lattice and $f \in \mathcal{O}_L^{(n)}$. Then the following are equivalent:*

(1) *$f \in [\vee, \wedge]^*$;*
(2) *there exist unary operations $u_1, \ldots, u_n \in [\vee, \wedge]^*$ such that $f(x_1, \ldots, x_n) = u_1(x_1) \vee \cdots \vee u_n(x_n)$ and for all $i, j \in \{1, \ldots, n\}$, $i \neq j$ we have $u_i(1) \wedge u_j(1) = u_1(0) = \cdots = u_n(0)$.*

Furthermore, the operation f given by (2) depends on all of its variables if and only if none of the unary operations u_i are constant.

Proof. The implication (1) ⇒ (2) is just a special case of Proposition 3.6, If (2) holds, then $f \in [\vee]^*$, since f is a composition of the operations $u_1, \ldots, u_n$ and $\vee$, all of which commute with $\vee$. Therefore, to complete the proof we have to show that f commutes with $\wedge$:

$$f(x_1, \ldots, x_n) \wedge f(y_1, \ldots, y_n) = f(x_1 \wedge y_1, \ldots, x_n \wedge y_n).$$

Thus we need to prove that for all $x_1, \ldots, x_n, y_1, \ldots, y_n$ we have

$$\big(u_1(x_1) \vee \cdots \vee u_n(x_n)\big) \wedge \big(u_1(y_1) \vee \cdots \vee u_n(y_n)\big) = u_1(x_1 \wedge y_1) \vee \cdots \vee u_n(x_n \wedge y_n).$$

Using the notation $c_i := u_i(x_i), d_i := u_i(y_i) (i = 1, \ldots, n)$, the above equality can be written as

$$(c_1 \vee \cdots \vee c_n) \wedge (d_1 \vee \cdots \vee d_n) = u_1(x_1 \wedge y_1) \vee \cdots \vee u_n(x_n \wedge y_n).$$

Since L is distributive, we have

$$(c_1 \vee \cdots \vee c_n) \wedge (d_1 \vee \cdots \vee d_n) = \bigvee_{i,j=1}^{n} (c_i \wedge d_j). \tag{3.2}$$

From $u_1, \ldots, u_n \in [\vee, \wedge]^*$ it follows that these operations are monotone, hence $u_1(0) = u_i(0) \leq c_i, d_i \leq u_i(1)$ for every i; moreover, (2) also implies that for all $i \neq j$, we have

$$u_1(0) = u_i(0) \wedge u_j(0) \leq c_i \wedge d_j \leq u_i(1) \wedge u_j(1) = u_1(0).$$

Thus $c_i \wedge d_j = u_1(0) \leq c_i \wedge d_i$ whenever $i \neq j$, so we can omit $c_i \wedge d_j$ from the join on the right hand side of (3.2), and using the fact that each u_i commutes with $\wedge$ we obtain the desired equality:

$$\begin{aligned}(c_1 \vee \cdots \vee c_n) \wedge (d_1 \vee \cdots \vee d_n) &= \bigvee_{i=1}^{n} (c_i \wedge d_i) \\ &= \bigvee_{i=1}^{n} (u_i(x_i) \wedge u_i(y_i)) \\ &= \bigvee_{i=1}^{n} (u_i(x_i \wedge y_i)).\end{aligned}$$

□

Our next goal is to prove that the conditions (Ess), (Sub) and (Quo) defined in Section 3.2 are all equivalent for finite distributive lattices. The equivalence of (Sub) and (Quo) follows from the description of projective and injective distributive lattices [1], but we provide a short proof.

Lemma 3.11. *Let $L = (L; \vee, \wedge)$ be a finite distributive lattice. Then the following are equivalent:*

(Sub) *there exists a sublattice of L that is isomorphic to $\mathbf{2}^n$;*
(Quo) *there exists a congruence ϑ of L such that L/ϑ is isomorphic to $\mathbf{2}^n$.*

Proof. Instead of $\mathbf{2}^n$, it will be more convenient to use the lattice $K_n := \mathcal{P}(\{1, ..., n\})$, which is clearly isomorphic to $\mathbf{2}^n$. To prove (Sub) $\Rightarrow$ (Quo), assume that L has a sublattice that is isomorphic to $\mathbf{2}^n$. Identifying this sublattice with K_n, we may assume without loss of generality that K_n itself is a sublattice of L. For any $i \in \{1, \ldots, n\}$, the principal ideal generated by $\{1, \ldots, n\} \setminus \{i\}$ does not contain $\{i\}$, hence, by the prime ideal theorem for distributive lattices, there is a prime ideal P_i of L that does not contain $\{i\}$ (see Corollary 116 in [7]). Consequently, there is a homomorphism $\varphi_i \colon L \to \mathbf{2}$ mapping P_i to 0 and $L \setminus P_i$ to 1. In particular, we have $\varphi_i(\{i\}) = 1$ and $\varphi_i(\{j\}) = 0$ for all $j \neq i$. Combining these maps we obtain a homomorphism $\varphi \colon L \to \mathbf{2}^n,\ x \mapsto (\varphi_1(x_1), \ldots, \varphi_n(x_n))$. We have $\varphi(\{i\}) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears in the i-th coordinate. These elements generate $\mathbf{2}^n$, hence φ is surjective, and this proves (Sub).

For (Quo) $\Rightarrow$ (Sub), let us suppose that ϑ is a congruence of L such that L/ϑ is isomorphic to K_n. For every $I \in K_n$, let C_I denote the congruence class of ϑ corresponding to I at this isomorphism. Let a be the greatest element of $C_\emptyset$, and let b_i be the least element of $C_{\{i\}}$ for all $i \in \{1, \ldots, n\}$. Then $c_i := a \vee b_i$ belongs to $C_{\{i\}}$, and $c_i \wedge c_j$ belongs to $C_\emptyset$ whenever $i \neq j$. Moreover, $c_i \wedge c_j = (a \vee b_i) \wedge (a \vee b_j) \geq a$, hence $c_i \wedge c_j = a$, as a is the greatest element of $C_\emptyset$. From this it is not hard to deduce using distributivity that $\mathcal{P}(\{1, \ldots, n\}) \hookrightarrow L,\ I \mapsto \bigvee\{c_i : i \in I\}$ is an embedding. (Alternatively, one can verify with the help of Theorem 360 of [7] that $\{c_1, \ldots, c_n\}$ is an independent set in the sublattice $\uparrow a$, hence it generates a sublattice isomorphic to $\mathcal{P}(\{1, \ldots, n\})$.) $\square$

Theorem 3.12. *Let $L = (L; \vee, \wedge)$ be a finite distributive lattice. Then the conditions* (Ess), (Sub) *and* (Quo) *are equivalent.*

Proof. This follows from the implications (Ess) $\Rightarrow$ (Sub) and (Sub)&(Quo) $\Rightarrow$ (Ess) (Proposition 3.8) and the equivalence (Sub) $\Leftrightarrow$ (Quo) (Lemma 3.11). $\square$

Corollary 3.13. *For a finite distributive lattice $L = (L; \vee, \wedge)$ the following are equivalent:*

- *every operation in $[\vee, \wedge]^*$ is essentially at most unary;*
- *L is a chain.*

3.5. **Finite nondistributive lattices: counterexamples.** We have seen in Section 3.4 that if L is a finite distributive lattice, then conditions (1) and (2) of Theorem 3.10 are equivalent, and that conditions (Ess), (Sub) and (Quo) are also equivalent. In this section we provide counterexamples showing that for arbitrary finite lattices only the implications proved in Section 3.2 are valid. First we present a finite lattice for which the implication (2) $\Rightarrow$ (1) of Theorem 3.10 is not true.

Example 3.14. Let L be the lattice shown on Figure 3. Let us define the operations u_1 and u_2 as

$$u_1(x) = \begin{cases} 0, & \text{if } x \in \{0, b\}, \\ a, & \text{if } x \in \{a, d\}, \\ b, & \text{if } x \in \{c, e\}, \\ d, & \text{if } x \in \{1\}; \end{cases} \qquad u_2(x) = \begin{cases} 0 & \text{if } x \in \{0, a, b, d\}, \\ c & \text{if } x \in \{c, e, 1\}. \end{cases}$$

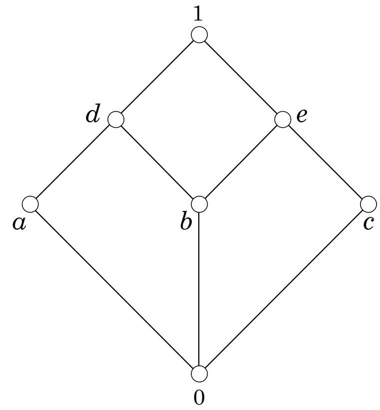

FIGURE 3. A nondistributive counterexample to the implication (2) $\Rightarrow$ (1) of Theorem 3.10.

It is easy to check that $\ker(u_1)$ is a congruence of L, and u_1 establishes an isomorphism from the quotient lattice $L/\ker(u_1) = \{\{0,b\},\{a,d\},\{c,e\},\{1\}\}$ to the sublattice $\{0,a,b,d\}$. Therefore u_1 is an endomorphism of L. Similarly, one can verify that $u_2 \in [\vee,\wedge]^*$. Let $f(x_1,x_2) = u_1(x_1) \vee u_2(x_2)$, we will show that f does not belong to $[\wedge]^*$. Let us suppose that f commutes with $\wedge$. Then applying the definition of commutation to the 2 by 2 matrix $\left(\begin{smallmatrix} a & c \\ c & c \end{smallmatrix}\right)$, we have the following equality:

$$f(a \wedge c, c \wedge c) = f(a,c) \wedge f(c,c);$$

that is,

$$u_1(a \wedge c) \vee u_2(c \wedge c) = (u_1(a) \vee u_2(c)) \wedge (u_1(c) \vee u_2(c)).$$

However, the left hand side evaluates to c, and the value of the right hand side is e;

$$c = 0 \vee c = u_1(0) \vee u_2(c) = (a \vee c) \wedge (b \vee c) = 1 \wedge e = e$$

which is a contradiction.

Next we consider the conditions (Ess), (Sub) and (Quo). In general, (Sub) $\Rightarrow$ (Ess) does not hold: partition lattices provide counterexamples to this implication. Indeed, every finite lattice (in particular, $\mathbf{2}^n$) embeds into a large enough finite partition lattice (see Theorem 413 in [7]), and partition lattices are simple (see Theorem 404 in [7]), hence by Remark 3.7, (Ess) holds only for $n \leq 1$.

Now we show that for arbitrary lattices neither (Sub) $\Rightarrow$ (Quo) nor (Quo) $\Rightarrow$ (Sub) holds. Using partition lattices again, we can give counterexamples to (Sub) $\Rightarrow$ (Quo). Since $\mathbf{2}^n$ embeds into a large enough finite partition lattice and partition lattices are simple, we have that $\mathbf{2}^n$ is not a homomorphic image of a partition lattice.

To disprove (Quo) $\Rightarrow$ (Sub), let $K_n = \mathcal{P}(\{1,\ldots,n\}) \cong \mathbf{2}^n$ for some $n \geq 4$, and define a partial order on the set $L := K_n \times \{0,1\}$ as follows. For $(a,i),(b,j) \in K_n$, let $(a,i) \leq (b,j)$ iff either $a \leq b$, or $a = b$ and $i \leq j$. Note that this is the lexicographic order on L, and this makes L a lattice with the following lattice operations (here $\|$ stands for incomparability in K_n):

$$(a,i)\vee(b,j) = \begin{cases} (a, i \vee j), & \text{if } a = b, \\ (a,i), & \text{if } a > b, \\ (b,j), & \text{if } a < b, \\ (a \vee b, 0), & \text{if } a \parallel b; \end{cases} \qquad (a,i)\wedge(b,j) = \begin{cases} (a, i \wedge j), & \text{if } a = b, \\ (b,j), & \text{if } a > b, \\ (a,i), & \text{if } a < b, \\ (a \wedge b, 1), & \text{if } a \parallel b. \end{cases}$$

Now K_n is a homomorphic image of L under the homomorphism $L \to K_n,\ (a,i) \mapsto a$. To see that K_n does not occur as a sublattice of L, note that $\{1,2\}$ is a doubly reducible element in K_n, i.e., it can be written as a join as well as a meet of two incomparable elements: $\{1,2\} = \{1\} \vee \{2\} = \{1,2,3\} \wedge \{1,2,4\}$. However, there is no doubly reducible element in L, since a nontrivial join in L is always of the form $(a,0)$, and a nontrivial meet is always of the form $(a,1)$. (It is not necessary to double each element of K_n: with a more careful argument, one can construct a counterexample of only 2^n+1 elements.)

Note that the assumption $n \geq 4$ was essential in the construction of this counterexample, since for $n \leq 3$, there are no doubly reducible elements in K_n. In fact, one can prove that (Quo) $\Rightarrow$ (Sub) holds for all lattices (distributive or not) for $n \leq 3$ (see Lemma 73 in [7]).

Summarizing the results up to this point we know that (Ess) $\Rightarrow$ (Sub) holds, but none of the implications (Sub) $\Rightarrow$ (Ess), (Sub) $\Rightarrow$ (Quo) or (Quo) $\Rightarrow$ (Sub) hold for arbitrary finite lattices. This immediately implies that (Quo) $\Rightarrow$ (Ess) can not hold in general. The lattice $L = M_3^n$ shows that (Ess) $\Rightarrow$ (Quo) does not hold, either. It is straightforward to verify that the operation $f \in \mathcal{O}_L^{(n)}$ defined by

$$f\big((x_{11},\dots,x_{1n}),\dots,(x_{n1},\dots,x_{nn})\big) = (x_{11},\dots,x_{n1})$$

commutes with $\vee$ and $\wedge$ and depends on all of its variables, hence (Ess) holds for L. By Lemma 3.2, every quotient of M_3^n is isomorphic to a product of quotients of M_3. Since M_3 is simple, L only has the quotients $M_3, M_3^2, \dots, M_3^n$, and therefore $\mathbf{2}^n$ does not appear as the quotient algebra of L.

Finally, let us investigate whether any two of the three conditions (Ess), (Sub) and (Quo) imply the third one. We have seen in Proposition 3.8 that (Sub)&(Quo) $\Rightarrow$ (Ess), and it also follows from that proposition that (Ess)&(Quo) $\Rightarrow$ (Sub). On the other hand, (Ess)&(Sub)$\not\Rightarrow$(Quo), since we have (Ess)$\Rightarrow$(Sub) and (Ess)$\not\Rightarrow$(Quo).

4. Centralizer clones over the two-element set

As promised in the introduction, we are going to determine the centralizer of each clone on $\{0,1\}$ in a fairly simple way. We use the Post lattice and the notation for Boolean clones from the Appendix (see Figure 4 and Table 2). First let us record two entirely obvious facts, just for reference:

Fact 4.1. *For any clones $C_1, C_2 \leq \mathcal{O}_A$ we have $(C_1 \vee C_2)^* = C_1^* \wedge C_2^*$. (Here $\vee$ and $\wedge$ denote the join and meet operations of the clone lattice over A, i.e., $C_1 \vee C_2 = [C_1 \cup C_2]$ and $C_1 \wedge C_2 = C_1 \cap C_2$.) This implies that if $C_1 \leq C_2$ then $C_2^* \leq C_1^*$.*

Fact 4.2. *By the definition of the clones Ω_0, Ω_1 and S, for any clone $C \leq \mathcal{O}_{\{0,1\}}$ we have*

- $0 \in C^* \iff C \leq \Omega_0$,
- $1 \in C^* \iff C \leq \Omega_1$,
- $\neg \in C^* \iff C \leq S$.

We will see that all centralizers over $\{0,1\}$ can be computed using three tools: Theorem 2.1, Corollary 3.4, and Proposition 4.3 below. This proposition can be found in [17, Proposition 2.1], but, for the reader's convenience, we include a proof, which is very similar to the proof of Theorem 2.1.

Proposition 4.3. *Let $A = (A;+)$ be an Abelian group, and let $m(x,y,z) = x-y+z$. An n-ary operation $f \in \mathcal{O}_A$ belongs to the centralizer $[m]^*$ if and only if there exist unary operations $u_1,\dots,u_n \in [m]^*$ such that*

$$f(x_1,\dots,x_n) = u_1(x_1) + \dots + u_n(x_n).$$

Proof. It is easy to see that the addition commutes with m, hence if f is a sum of endomorphisms of $(A;m)$, then $f \in [m]^*$.

Conversely, assume that f is an n-ary operation in $[m]^*$, and define $u_1,\dots,u_n$ the same way as in the proof of Theorem 2.1: $u_1(x) = f(x,0,\dots,0),\dots,u_n(x) = f(0,\dots,0,x)$. Then we have $u_1,\dots,u_n \in [m]^*$, as f and the constant 0 operation commute with m. Let us consider the $(n+1)$-ary operation $g(x_1,\dots,x_n,y) = x_1 + \dots + x_n - (n-1)y$. The following expression shows that $g \in [m]$:

$$\begin{aligned} g(x_1,\dots,x_n,y) &= x_1 - y + x_2 - y + x_3 - \dots - y + x_n \\ &= m(\cdots m(m(x_1,y,x_2),y,x_3),\dots,y,x_n). \end{aligned}$$

(Actually, it is well known and also easy to verify that the elements of $[m]$ are the operations of the form $a_1x_1 + \cdots + a_nx_n$ ($n \in \mathbb{N}^+, a_i \in \mathbb{Z}, \sum a_i = 1$), but for the purposes of this proof we only need the operation g above.) Since $g \in [m]$ and $f \in [m]^*$, the operations f and g commute. Applying the definition of commutation to the $(n+1) \times n$ matrix

$$\begin{pmatrix} x_1 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_n \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

we can conclude that

$$\begin{aligned} f(x_1,\dots,x_n) &= f\big(g(x_1,0,\dots,0,0),\ \dots,\ g(0,\dots,0,x_n,0)\big) \\ &= g\big(f(x_1,0,\dots,0),\dots,f(0,\dots,0,x_n),f(0,\dots,0)\big) \\ &= u_1(x_1) + \cdots + u_n(x_n) - (n-1)\cdot f(0,\dots,0). \end{aligned}$$

This is almost the required form of f; we only need to deal with the constant term $(n-1)\cdot f(0,\dots,0)$. However, since m is idempotent, every constant commutes with m, thus $u_n'(x_n) := u_n(x_n) - (n-1)\cdot f(0,\dots,0)$ belongs to $[m]^*$. Then we can write f as $f(x_1,\dots,x_n) = u_1(x_1) + \cdots + u_n'(x_n)$, and this completes the proof. □

Theorem 4.4. *The centralizers of the clones of Boolean functions are as indicated in Table 3 in the Appendix. The clones are grouped by their centralizer clones; the first column shows the 25 primitive positive clones over the two-element set, and the second column lists all clones having the given primitive positive clone as their centralizer.*

Proof. Let us recall that a variety $\mathcal{V}$ is congruence distributive if and only if it has, for some n, a sequence of terms $J_0(x,y,z),\dots,J_n(x,y,z)$ satisfying the following identities:

$$\begin{aligned} &J_0(x,y,z) = x, \\ &J_n(x,y,z) = z, \\ &J_i(x,y,x) = x \text{ for each } 0 \leq i \leq n, \\ &J_i(x,x,y) = J_{i+1}(x,x,y) \text{ if } i \text{ is even}, \\ &J_i(x,y,y) = J_{i+1}(x,y,y) \text{ if } i \text{ is odd}. \end{aligned}$$

These terms J_i are called *Jónsson terms*. If $C \leq \mathcal{O}_{\{0,1\}}$ is a clone, then the existence of a sequence of Jónsson terms in the clone C guarantees that the variety generated by the algebra $(\{0,1\}; C)$ is congruence distributive. By Corollary 3.4, this implies that $C^* \leq \Omega^{(1)}$.

The clone SM of self-dual monotone Boolean functions is generated by the majority operation $\mu(x,y,z) = xy + xz + yz$, which immediately gives us a sequence of Jónsson terms with $J_0(x,y,z) = x, J_1(x,y,z) = \mu(x,y,z)$ and $J_2(x,y,z) = z$. We provide a sequence of Jónsson terms in $U_{01}^{\infty}M$ in Table 1; the duals of these operations are Jónsson terms in $W_{01}^{\infty}M$. Thus if C contains at least one of the three clones $U_{01}^{\infty}M, W_{01}^{\infty}M$ and SM as a subclone, then C^* contains only essentially at most unary functions by Corollary 3.4, and then C^* is easy to find using Fact 4.2. This covers the first six rows of Table 3.

After having determined clones with essentially unary centralizers, there are finitely many clones left to investigate. It is easy to see that these clones appear as joins of some of the clones $[0],[1],[\neg],V_{01},\Lambda_{01}$ and L_{01}. According to Fact 4.1, it suffices to determine the centralizers of these six clones. It follows immediately from the definition of Ω_0, Ω_1 and S that $[0]^* = \Omega_0$, $[1]^* = \Omega_1$ and $[\neg]^* = S$.

Theorem 2.1 gives us the centralizer of V_{01}: every operation in V_{01}^* is of the form $u_1(x_1) \vee \cdots \vee u_n(x_n)$, where $u_i(x_i) = x_i$ or u_i is constant for all $i = 1,\dots,n$. Thus, we have $V_{01}^* = [\vee,0,1] = V$, and dually, $\Lambda_{01}^* = [\wedge,0,1] = \Lambda$. Finally, Proposition 4.3

x	y	z	$x = J_0$	J_1	J_2	J_3	$J_4 = z$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	1
0	1	0	0	0	0	0	0
0	1	1	0	0	0	1	1
1	0	0	1	0	0	0	0
1	0	1	1	1	1	1	1
1	1	0	1	1	0	0	0
1	1	1	1	1	1	1	1

Table 1. A sequence of Jónsson terms in the clone $U_{01}^{\infty} M$.

shows that the centralizer of $L_{01} = [x - y + z] = [x + y + z]$ consists of sums of unary functions, hence $L_{01}^* = \{x_1 + x_2 + \cdots + x_n + c \mid c \in \{0, 1\}, n \in \mathbb{N}_0\} = L$. $\square$

The following remark makes it easier to remember the centralizers of all Boolean clones.

Remark 4.5. We can group the clones on $\{0, 1\}$ by the "type" of their centralizers. These groups give a partition of the Post lattice into five blocks:

- $\mathcal{C}_{\mathrm{cd}} := \{C \leq \mathcal{O}_{\{0,1\}} \mid U_{01}^{\infty} M \leq C \text{ or } W_{01}^{\infty} M \leq C \text{ or } SM \leq C\}$;
- $\mathcal{C}_{\vee} := \{V, V_0, V_1, V_{01}\}$;
- $\mathcal{C}_{\wedge} := \{\Lambda, \Lambda_0, \Lambda_1, \Lambda_{01}\}$;
- $\mathcal{C}_{\mathrm{lin}} := \{L, L_0, L_1, L_{01}, SL\}$;
- $\mathcal{C}_{\mathrm{un}} := \{C \leq \mathcal{O}_{\{0,1\}} \mid C \leq \Omega^{(1)}\}$.

The "centralizing" operation $C \mapsto C^*$ preserves this partition: for every $C \leq \mathcal{O}_{\{0,1\}}$, we have

- if $C \in \mathcal{C}_{\mathrm{cd}}$ then $C^* \in \mathcal{C}_{\mathrm{un}}$ (i.e., $C^* \leq \Omega^{(1)}$);
- if $C \in \mathcal{C}_{\vee}$ then $C^* \in \mathcal{C}_{\vee}$;
- if $C \in \mathcal{C}_{\wedge}$ then $C^* \in \mathcal{C}_{\wedge}$;
- if $C \in \mathcal{C}_{\mathrm{lin}}$ then $C^* \in \mathcal{C}_{\mathrm{lin}}$;
- if $C \in \mathcal{C}_{\mathrm{un}}$, then $C^* \geq S_{01}$ (and thus $C \in \mathcal{C}_{\mathrm{cd}}$).

Figure 4 in the Appendix shows the above partition of the Post lattice (the five blocks are indicated by different symbols) with primitive positive clones marked by a symbol having an outline. Observe that primitive positive clones belonging to the same block have different unary parts most of the time, the only exception being $\Omega_{01} \cap \Omega^{(1)} = S_{01} \cap \Omega^{(1)} = [x]$. Thus the observations above together with Fact 4.2 allow us to find the centralizer of any clone with ease.

Acknowledgements

The authors are grateful to Gábor Czédli and László Zádori for helpful discussions as well as to the anonymous referee for suggestions that significantly improved the paper.

This research was partially supported by the National Research, Development and Innovation Office of Hungary under grants KH126581 and K128042, by the ÚNKP-19-3-SZTE-305 new national excellence program and by grant TUDFO/47138-1/2019-ITM of the Ministry for Innovation and Technology, Hungary.

Appendix: Clones on the two-element set

The lattice of all clones on the set $\{0, 1\}$ is shown in Figure 4. Different symbols are used according to the partition defined in Remark 4.5; primitive positive clones are indicated by a symbol having an outline, while the gray circles without an outline indicate clones that are not primitive positive. In Table 2 we give the definitions of the clones that are labeled on the diagram; the remaining clones can be obtained as intersections of some of these clones.

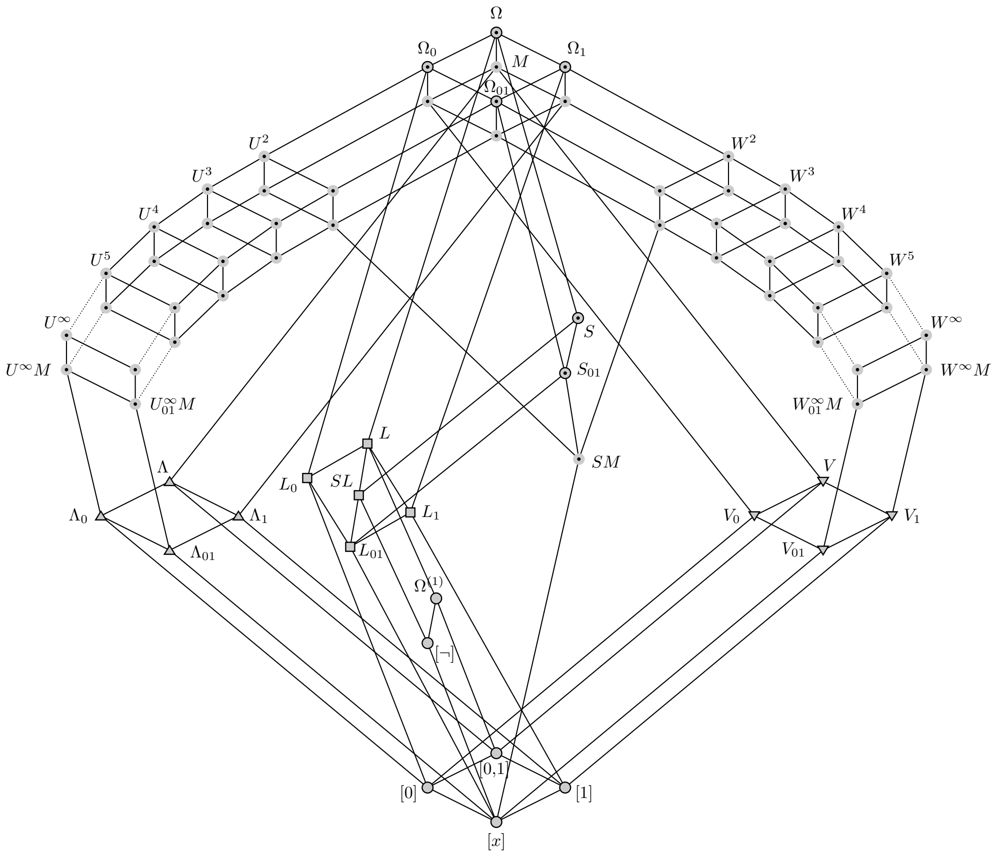

FIGURE 4. The Post lattice.

Ω is the clone of all Boolean functions: $\Omega = \mathcal{O}_{01}$.
$\Omega_0 = \{f \in \Omega \mid f(0,\dots,0) = 0\}$ is the clone of 0-preserving functions. $\Omega_1 = \{f \in \Omega \mid f(1,\dots,1) = 1\}$ is the clone of 1-preserving functions. $\Omega_{01} = \Omega_0 \cap \Omega_1$ is the clone of idempotent functions. In general, if C is a clone, then $C_0 = C \cap \Omega_0$, $C_1 = C \cap \Omega_1$, and $C_{01} = C_0 \cap C_1$.
$\Omega^{(1)}$ is the clone of all essentially at most unary functions: $\Omega^{(1)} = [x, \neg x, 0, 1]$. $[x]$ is the trivial clone containing only projections. $[0]$, $[1]$ and $[0,1]$ are the clones generated by constant operations. $[\neg]$ is the clone generated by negation.
$M = \{f \in \Omega \mid \mathbf{x} \leq \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})\}$ is the clone of monotone functions.
$U^\infty = \{f \in \Omega^{(n)} \mid n \in \mathbb{N}_0, \exists k \in \{1,\dots,n\} \colon f(\mathbf{x}) = 1 \implies x_k = 1\}$, and $U^\infty M = U^\infty \cap M$, and $U^\infty_{01} M = U^\infty \cap \Omega_{01} \cap M$.
$W^\infty = \{f \in \Omega^{(n)} \mid n \in \mathbb{N}_0, \exists k \in \{1,\dots,n\} \colon f(\mathbf{x}) = 0 \implies x_k = 0\}$, and $W^\infty M = W^\infty \cap M$ and $W^\infty_{01} M = W^\infty \cap \Omega_{01} \cap M$.
$S = \{f \in \Omega \mid \neg f(\neg \mathbf{x}) = f(\mathbf{x})\}$ is the clone of self-dual functions. $SM = S \cap M = [\mu]$ where $\mu(x,y,z)$ is the majority function on $\{0,1\}$.
$\Lambda = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}^+\} \cup [0,1] = [\wedge, 0, 1]$. $\Lambda_0 = \Lambda \cap \Omega_0 = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}^+\} \cup [0] = [\wedge, 0]$. $\Lambda_1 = \Lambda \cap \Omega_1 = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}^+\} \cup [1] = [\wedge, 1]$. $\Lambda_{01} = \Lambda \cap \Omega_{01} = \{x_1 \wedge \cdots \wedge x_n \mid n \in \mathbb{N}^+\} = [\wedge]$.
$V = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}^+\} \cup [0,1] = [\vee, 0, 1]$. $V_0 = V \cap \Omega_0 = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}^+\} \cup [0] = [\vee, 0]$. $V_1 = V \cap \Omega_1 = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}^+\} \cup [1] = [\vee, 1]$. $V_{01} = V \cap \Omega_{01} = \{x_1 \vee \cdots \vee x_n \mid n \in \mathbb{N}^+\} = [\vee]$.
$L = \{x_1 + \cdots + x_n + c \mid c \in \{0,1\}, n \in \mathbb{N}_0\} = [x + y, 1]$. $L_0 = L \cap \Omega_0 = \{x_1 + \cdots + x_n \mid n \in \mathbb{N}_0\} = [x + y]$. $L_1 = L \cap \Omega_1 = \{x_1 + \cdots + x_n + (n + 1 \bmod 2) \mid n \in \mathbb{N}_0\} = [x + y + z, 1]$. $L_{01} = L \cap \Omega_{01} = \{x_1 + \cdots + x_n \mid n \text{ is odd}\} = [x + y + z]$. $SL = S \cap L = \big\{x_1 + \cdots + x_n + c \mid n \text{ is odd and } c \in \{0,1\}\big\} = [x + y + z, x + 1]$.

TABLE 2. Definitions of some clones of Boolean functions.

P	all clones $C \leq \mathcal{O}_{\{0,1\}}$ such that $C^* = P$
$[x]$	Ω, M
$[0]$	$\Omega_0, M_0, U^k, U^\infty, U^k M, U^\infty M$ (for all $k \in \mathbb{N}^+$)
$[1]$	$\Omega_1, M_1, W^k, W^\infty, W^k M, W^\infty M$ (for all $k \in \mathbb{N}^+$)
$[0,1]$	$\Omega_{01}, M_{01}, U_{01}^k, U_{01}^\infty, U_{01}^k M, U_{01}^\infty M, W_{01}^k, W_{01}^\infty, W_{01}^k M, W_{01}^\infty M$ (for all $k \in \mathbb{N}^+$)
$[\neg]$	S
$\Omega^{(1)}$	S_{01}, SM
L_{01}	L
L_0	L_0
L_1	L_1
L	L_{01}
SL	SL
Λ_{01}	Λ
Λ_0	Λ_0
Λ_1	Λ_1
Λ	Λ_{01}
V_{01}	V
V_0	V_0
V_1	V_1
V	V_{01}
S_{01}	$\Omega^{(1)}$
S	$[\neg]$
Ω_{01}	$[0,1]$
Ω_0	$[0]$
Ω_1	$[1]$
Ω	$[x]$

TABLE 3. The centralizers of all clones of Boolean functions.

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