COMPLETENESS CRITERIA FOR COGNATES OF SŁUPECKI'S CLONE

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Abstract. ???

1. Introduction and preliminaries

Let A be a finite set of cardinality ≥ 3. Słupecki's clone on A is the clone consisting of all operations that are either essentialy unary or nonsurjective. We will denote Słupecki's clone by $\mathcal{S}$. The subclone of $\mathcal{S}$ that consists of all projections and all nonsurjective operations will be denoted by $\mathcal{S}^-$. Thus, every operation in $\mathcal{S} \setminus \mathcal{S}^-$ is an essentially unary operations that is a nonidentity permutation of A. It follows that the interval $\mathfrak{I} = [\mathcal{S}^-, \mathcal{S}]$ in the lattice of clones on A is isomorphic to the lattice of subgroups of the symmetric group on A. Hence the interval $\mathfrak{I}$ is finite. Moreover, it is not hard to see that every clone in $\mathfrak{I}$ is finitely generated.

It is well known (see, e.g., []) that for every finitely generated clone $\mathcal{C}$ on A the lattice of all subclones of $\mathcal{C}$ is dually atomic with finitely many dual atoms; that is, there exists a finite set $\mathfrak{M}_\mathcal{C}$ of proper clones of $\mathcal{C}$ such that every proper subclone of $\mathcal{C}$ is contained in some member of $\mathfrak{M}_\mathcal{C}$. Therefore

$$\mathfrak{M}_\mathfrak{I} := \Big(\bigcup_{\mathcal{C}\in\mathfrak{I}} \mathfrak{M}_\mathcal{C}\Big) \setminus \mathfrak{I}$$

is a finite family of clones on A such that the following two conditions hold for $\mathfrak{M} = \mathfrak{M}_\mathfrak{I}$:

(1) every clone in $\mathfrak{M}$ is a subclone of Słupecki's clone $\mathcal{S}$, but is outside the interval $\mathfrak{I}$, and
(2) every subclone of $\mathcal{S}$ that is outside the interval $\mathfrak{I}$ is contained in some member of $\mathfrak{M}$.

Our goal in this paper is to explicitly describe a manageable (finite) set $\mathfrak{M}$ of subclones of Słupecki's clone $\mathcal{S}$ such that $\mathfrak{M}$ satisfies conditions (1)–(2). This yields a test for checking whether a clone $\mathcal{C}$ on A belongs to the interval $\mathfrak{I}$; namely:

$$\mathcal{C} \in \mathfrak{I} \quad \iff \quad \mathcal{C} \not\subseteq \mathcal{M} \text{ for all } \mathcal{M} \in \mathfrak{M}.$$

Optimality?
Maximal subclones of Słupecki's clone; other submaximal clones
Notation: T^-, $\mathcal{S}_r$ ($r = 0$ ess. unary, $r = 1$ Burle, $r = 2, \ldots, k$ range $\leq r$), $\mathcal{S}_r(T)$ ($r = 0, \ldots, k-1$); so $\mathcal{S} = S_{k-1}$, $\mathcal{S}^- = \mathcal{S}_{k-1}(T^-)$.
ι_r^A (on set A), ι_r (if A clear from context) ($3 \leq r \leq |A|$)
define $\iota_1 := \emptyset$
trivial relations

This material is based upon work supported by the Hungarian National Foundation for Scientific Research (OTKA) grant no. K77409.

clone
relational clone
Galois connection, description of clones bt relations
almost central relation (define so that it is not central)
pr_I
$\mathbf{a}$
nontrivial
Δ, Δ_ε, $\Delta_{12|3}$, etc.
$\mathsf{m} = \{1, \dots, m\}$

2. The main theorem

We will assume throughout that A is a finite set with $k = |A|$ elements, and $k \geq 3$.

thm-main

Theorem 2.1. *Let A be a k-element set ($k \geq 3$). The following conditions on a clone $\mathcal{C}$ on A are equivalent*

(a) $\mathcal{S}^- \not\subseteq \mathcal{C}$;
(b) $\mathcal{C} \subseteq \{\rho\}^\perp$ *for one of the relations ρ on A listed below (m is the arity of ρ):*
  (1) *a bounded partial order;*
  (2) *a prime permutation;*
  (3) *a prime affine relation;*
  (4) *a nontrivial equivalence relation;*
  (5) *a central relation ($1 \leq m \leq k-1$);*
  (6) *an m-regular relation ($3 \leq m \leq k-1$);*
  (7) *an almost central relation ($2 \leq m \leq k-2$);*
  (8) *an almost m-regular relation ($3 \leq m \leq k-1$);*
  (9) *Burle's relation β if $k = 3$.*

3. Proof

lm-suff-cond's

Lemma 3.1. *Let A be a k-element set ($k \geq 3$), and let ρ be an arbitrary relation on A. We have $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$ whenever one of the following conditions holds for ρ:*

(1) *ρ is an n-ary relation ($n \leq k$) such that ρ is not totally reflexive, but contains an n-tuple whose coordinates are pairwise distinct;*
(2) *ρ is a nontrivial totally reflexive relation of arity $n < k$;*
(3) *ρ is Burle's relation β;*
(4) *ρ is the ternary relation $\rho = \Delta_{12|3} \cup \Delta_{13|2}$.*

Proof. In case (3), when $\rho = \beta$, then $\{\beta\}^\perp$ is Burle's clone (see [PK]), a proper subclone of $\mathcal{S}_2 = \{\iota_3\}^\perp$, which fails to contain some operations with range of size 2 from $\mathcal{S}_2$. This implies that $\mathcal{S}^- \not\subseteq \{\beta\}^\perp = \{\rho\}^\perp$.

In case (4), when $\rho = \Delta_{12|3} \cup \Delta_{13|2}$, then $\{\rho\}^\perp$ is the clone of all essentially unary operations (see []; or: it is in the Słupecki–Burle chain, but does not contain Burle's clone), therefore again $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$.

To prove $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$ in cases (1)–(2) the following claim will be useful.

clm-suff

Claim 3.2. *Let ρ be an n-ary relation such that $T^- \subseteq \{\rho\}^\perp$ and ρ contains an n-tuple whose coordinates are pairwise distinct. Then*

- *$\rho = A^n$, if $n < k$, and*

- *ρ is totally reflexive, if $n = k$.*

For the proof of the claim, let $(a_1, \ldots, a_n)$ be an n-tuple in ρ whose coordinates are pairwise distinct, and assume that $T^- \subseteq \{\rho\}^\perp$. Then $n \le k$, and $(a_1, \ldots, a_n) \in \rho$ implies that $\big(f(a_1), \ldots, f(a_n)\big) \in \rho$ for all nonsurjective unary operations f. Thus the conclusions stated in the claim follow. $\diamond$

Claim 3.2 immediately implies that if (1) holds for ρ, then $T^- \subseteq \{\rho\}^\perp$ must fail. Hence $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$.

Now assume that (2) holds for ρ. If ρ contains an n-tuple whose coordinates are pairwise distinct, then the assumptions that $n < k$ and ρ is nontrivial, combined with Claim 3.2 show that $T^- \not\subseteq \{\rho\}^\perp$. As before, it follows that $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$. If, in turn, such a tuple does not exist in ρ, then $\rho \subseteq \iota_n$. The reverse inclusion also holds, since ρ is totally reflexive. Thus $\rho = \iota_n$. Since ρ is nontrivial, we must have $n \ge 3$, hence $3 \le n < k$. It is well known (see e.g. [PK]) that for $3 \le n \le k$, $\{\iota_n\}^\perp$ is the clone $\mathcal{S}_{n-1}$ Therefore, in view of $n < k$, $\mathcal{S}^- \not\subseteq \{\iota_n\}^\perp = \{\rho\}^\perp$. $\square$

Corollary 3.3. *For every relation ρ in ... [Main Theorem] we have that $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$.*

prop-nec-rels

Proposition 3.4. *Let A be a k-element set ($k \ge 3$). For a clone $\mathcal{C}$ on A we have $\mathcal{S}^- \not\subseteq \mathcal{C}$ if and only if there exists a relation ρ on A such that $\mathcal{C} \subseteq \{\rho\}^\perp$ and one of the following conditions is satisfied:*

(1) *ρ is a k-ary relation such that ρ is not totally reflexive, but contains a k-tuple whose coordinates are pairwise distinct;*
(2) *$k > 4$ and $\rho = \iota_{k-1}$;*
(3) *$k = 3$ and ρ is Burle's relation β.*

Proof. Sufficiency. Assume that $\mathcal{C} \subseteq \{\rho\}^\perp$ and that one of conditions (1)–(3) holds. Then, by Lemma 3.1, we get that $\mathcal{S}^- \not\subseteq \{\rho\}^\perp$, and hence $\mathcal{S}^- \not\subseteq \mathcal{C}$.

Necessity. Let $\mathcal{C}$ be a clone such that $\mathcal{S}^- \not\subseteq \mathcal{C}$. Let $\mathcal{C}^{(1)}$ denote the set of unary operations in $\mathcal{C}$. Since $\mathcal{C}^{(1)}$ is a set of functions $A \to A$ and $|A| = k$, $\mathcal{C}^{(1)}$ is a k-ary relation on A. It is well known and easy to check that $\mathcal{C} \subseteq \{\mathcal{C}^{(1)}\}^\perp$, and that the clone $\{\mathcal{C}^{(1)}\}^\perp$ contains the same unary operations as $\mathcal{C}$. If $T^- \not\subseteq \mathcal{C}^{(1)}$, then the k-ary relation $\mathcal{C}^{(1)}$ is not totally reflexive, but contains a tuple with pairwise distinct coordinates, namely the identity function $A \to A$. Thus $\rho := \mathcal{C}^{(1)}$ satisfies condition (1) and $\mathcal{C} \subseteq \{\rho\}^\perp$, finishing the proof in this case.

Now assume that $T^- \subseteq \mathcal{C}^{(1)}$. By a result of Szabó [], if $\mathcal{C}$ is such a clone and $\mathcal{C}$ is not the clone of all operations, then there exist an integer $r < k$ and a transformation monoid $M \supseteq T^-$ such that $\mathcal{C} = \mathcal{S}_r(M)$. In particular, $\mathcal{S}^- = \mathcal{S}_{k-1}(T^-)$. Therefore the assumption $\mathcal{S}^- \not\subseteq \mathcal{C}$ is equivalent to $\mathcal{S}_{k-1}(T^-) \not\subseteq \mathcal{S}_r(M)$, and hence implies that $r < k-1$. Thus $\mathcal{C} = \mathcal{S}_r(M) \subseteq \mathcal{S}_{k-2}$. As was mentioned in the proof of Lemma 3.1, if $k \ge 4$, then $\mathcal{S}_{k-2} = \{\iota_{k-1}\}^\perp$, while if $k = 3$, then $\mathcal{S}_{k-2} = \{\beta\}^\perp$. Thus $\mathcal{C} \subseteq \{\rho\}^\perp$ holds for $\rho = \iota_{k-1}$ or $\rho = \beta$, respectively, and ρ satisfies condition (2) or (3).

The proof of the proposition is complete. $\square$

Let $\mathcal{R}_0 := (\mathcal{S}^-)^\perp$.

cor-first-approx

Corollary 3.5. *Let A be a k-element set ($k \geq 3$), and let $\mathcal{R}$ be a relational clone on A such that $\mathcal{R} \not\subseteq \mathcal{R}_0$. Then $\mathcal{R}$ contains a relation ρ satisfying one of conditions (1)–(3) in Proposition 3.4. Moreover, $\rho \notin \mathcal{R}_0$ holds for all such relations ρ.*

Proof. This is a restatement of ... □

thm-non-trts

Theorem 3.6. *Let A be a k-element set ($k \geq 3$), and let $\mathcal{R}$ be a relational clone on A such that $\mathcal{R} \not\subseteq \mathcal{R}_0$. If $\mathcal{R}$ does not contain any nontrivial totally reflexive, totally symmetric relations of arity $< k$, and for $|A| = 3$, $\mathcal{R}$ does not contain Burle's relation β either, then $\mathcal{R}$ contains one of the following relations:*

1. *a prime permutation,*
2. *a bounded partial order,*
3. *a prime affine relation.*

Proof. Throughout the proof m will denote the smallest positive integer such that $\mathcal{R}$ contains a nontrivial m-ary relation. We start with some auxiliary claims.

clm-m-less-k

Claim 3.7. $2 \leq m \leq k$. *Moreover,*

1. *$\mathcal{R}$ contains no nontrivial totally reflexive relation of arity n with $2 < n < k$, but*
2. *if $2 < m \leq k$, then $\mathcal{R}$ contains a nontrivial m-ary relation that is not totally reflexive.*

We will start by proving statement (1), which is independent of m. Let σ be a totally reflexive relation of arity n ($2 < n < k$) in $\mathcal{R}$. Then the relation σ' obtained by intersecting all relations arising from σ by permuting coordinates is both totally reflexive and totally symmetric. Furthermore, $\sigma' \in \mathcal{R}$. Since $n < k$, our assumptions on $\mathcal{R}$ force σ' to be a trivial relation. Since σ' is totally reflexive of arity $n > 2$, it follows that $\sigma' = A^n$. Hence $\sigma = A^n$ and σ is a trivial relation, proving (1).

Towards the proof of $2 \leq m \leq k$ notice first that unary relations are totally reflexive and totally symmetric, therefore our assumptions on $\mathcal{R}$ imply that $m \geq 2$. By Corollary 3.5, $\mathcal{R}$ contains a relation ρ satisfying one of conditions (1)–(3) in Proposition 3.4. However, our current assumptions on $\mathcal{R}$ (i.e., the assumptions of Theorem 3.6) exclude cases (2) and (3), therefore $\mathcal{R}$ contains a relation ρ satisfying condition (1) in Proposition 3.4. Since ρ is a relation of arity k in $\mathcal{R} \setminus \mathcal{R}_0$, it is nontrivial, and therefore implies that $m \leq k$.

By condition (1) in Proposition 3.4, this relation ρ is not totally reflexive, therefore if $m = k$, then this proves the existence in $\mathcal{R}$ of a nontrivial m-ary relation that is not totally reflexive, and proves statement (2). If $2 < m < k$, then let δ be any nontrivial m-ary relation in $\mathcal{R}$. It follows from statement (1) that δ is not totally reflexive. This proves statement (2) for the case when $m < k$, and completes the proof of Claim 3.7. ◇

clm-proj's

Claim 3.8. *For every nontrivial m-ary relation $\delta \in \mathcal{R}$ and $(m-1)$-element subset I of $\{1, \ldots, m\}$ we have that $\mathrm{pr}_I(\delta) = A^{m-1}$.*

To prove the claim let δ be a nontrivial relation in $\mathcal{R}$, and let I be an $(m-1)$-element subset of $\{1, \ldots, m\}$. By the minimality of m, $\mathrm{pr}_I(\delta)$ is a trivial relation. Clearly, $\mathrm{pr}_I(\delta) \neq \emptyset$, as $\delta \neq \emptyset$. Since $\emptyset$ and A are the only trivial unary relations, if $m = 2$, then we must have $\mathrm{pr}_I(\delta) = A = A^{m-1}$. From now on let $m \geq 3$, and assume that out claim fails and $\mathrm{pr}_I(\delta) \neq A^{m-1}$. Then there exist $i < j$ in I such that $\mathrm{pr}_{i,j}\big(\mathrm{pr}_I(\delta)\big) = \Delta$. The left hand side is equal to $\mathrm{pr}_{i,j}(\delta)$, therefore we get that $\mathrm{pr}_{i,j}(\delta) = \Delta$. Again by the minimality of m, $\mathrm{pr}_{\mathsf{m}\setminus\{i\}}(\delta)$ is a trivial relation, and therefore by $\mathrm{pr}_{i,j}(\delta) = \Delta$ so is δ. This contradiction proves that $\mathrm{pr}_I(\delta) = A^{m-1}$. $\diamond$

clm-diag's

Claim 3.9. *Let δ be a nontrivial m-ary relation in $\mathcal{R}$, and let $1 \le i < j \le m$ be arbitrary integers.*

1. *Either $\delta \cap \Delta_{[ij]} = \emptyset$, or $\delta \cap \Delta_{[ij]} = \Delta_\varepsilon$ for some equivalence relation ε of m such that $\varepsilon \supseteq [ij]$.*
2. *$\delta \cap \iota_m$ is a union of trivial relations (the union may be empty).*
3. *If $m \ge 3$, then for each $r \in \mathsf{m}\setminus\{i,j\}$ there exists a two-element set $\{r, r'\} \subseteq \mathsf{m} \setminus \{i\}$ such that $\delta \cap \Delta_{[ij]} \supseteq \Delta_{[ij]\vee[rr']}$.*
4. *If $m \ge 3$ and $\Delta_{[ij]} \not\subseteq \delta$, then $\delta \cap \Delta_{[ij]} = \Delta_{[ij]\vee[uv]}$ for a two-element subset $\{u, v\}$ of $\mathsf{m} \setminus \{i\}$ such that $\mathsf{m} \setminus \{i, j\} \subseteq \{u, v\}$.*

Let δ be a nontrivial relation of arity m in $\mathcal{R}$, and let $1 \le i < j \le m$. Then $\mathrm{pr}_{\mathsf{m}\setminus\{i\}}(\delta \cap \Delta_{[ij]})$ is an $(m-1)$-ary relation in $\mathcal{R}$, so by the minimality of m it is a trivial relation. This proves (1).

(2) follows immediately from (1), using the fact that $\iota_m = \bigcup_{1\le i<j\le m} \Delta_{[ij]}$.

To prove (3) let us assume that $m \ge 3$, and consider the intersection $\delta \cap \Delta_{[ij]}$. For arbitrary $r \in \mathsf{m}\setminus\{i,j\}$ we have $\mathrm{pr}_{\mathsf{m}\setminus\{r\}} = A^{m-1}$ by Claim 3.8, therefore for each $(m-1)$-tuple $(a_1, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n) \in A^{m-1}$ with $a_i = a_j$ there exists $b \in A$ such that $(a_1, \ldots, a_{r-1}, b, a_{r+1}, \ldots, a_n) \in \delta$. Hence $|\delta \cap \Delta_{[ij]}| \ge |A|^{m-2}$. Combining this with statement (1) we get that either $\delta \cap \Delta_{[ij]} = \Delta_{[ij]}$, or there exists $r' \ne r$ such that $\delta \cap \Delta_{[ij]} \supseteq \Delta_{[ij]\vee[rr']}$. Clearly, the latter inclusion will be true for some r' even if $\delta \cap \Delta_{[ij]} = \Delta_{[ij]}$. Furthermore, since $[ij] \vee [ri] = [ij] \vee [rj]$, we can choose r' so that $r' \ne i$. This proves statement (3).

For (4), let us assume that $\Delta_{[ij]} \not\subseteq \delta$. Thus, by part (1), $\delta \cap \Delta_{[ij]} = \Delta_\varepsilon$ for an equivalence relation ε on m such that $\varepsilon \supsetneq [ij]$. However, by part (3), $|\delta \cap \Delta_{[ij]}| \ge |A|^{m-2}$, therefore $\varepsilon = [ij] \vee [uv]$ for some two-element subset $\{u, v\}$ of $\mathsf{m} \setminus \{i\}$, and hence $\delta \cap \Delta_{[ij]} = \Delta_{[ij]\vee[uv]}$. According to (3), for each $r \in \mathsf{m} \setminus \{i, j\}$ there exists a two-element set $\{r, r'\} \subseteq \mathsf{m} \setminus \{i\}$ such that

$$\Delta_{[ij]\vee[rr']} \subseteq \delta \cap \Delta_{[ij]} = \Delta_{[ij]\vee[uv]}.$$

Hence $\mathsf{m} \setminus \{i, j\} \subseteq \{u, v\}$. This completes the proof of statement (4). $\diamond$

clm-m-less-4

Claim 3.10. $2 \le m \le 4$.

Recall that it was established in Claim 3.7 that $m \ge 2$, therefore we need only to prove that $m \not\ge 5$. Assume that $m \ge 5$. We know from Claim 3.7 (2) that $\mathcal{R}$ contains a nontrivial m-ary relation δ such that δ is not totally reflexive. Therefore there exist $1 \le i < j \le m$ such that $\Delta_{[ij]} \not\subseteq \delta$. Hence the conclusions of Claim 3.9 (4) have to hold. However, if $m \ge 5$, then there is no two-element set $\{u, v\}$ such that $\mathsf{m} \setminus \{i, j\} \subseteq \{u, v\}$. This shows that $m \ge 5$ is impossible, and thereby completes the proof of Claim 3.10. $\diamond$

Now we will look at the cases $m = 2,\ 3, 4$ separately.

clm-binary-prep

Claim 3.11. *If $m = 2$, then for every nontrivial binary relation $\delta \in \mathcal{R}$,*

1. *either δ is areflexive or δ is reflexive and antisymmetric,*
2. *either $\delta^{-1} \circ \delta = A^2$ or δ is (the graph of) a permutation,*
3. *if $\delta^{-1} \circ \delta = A^2$, then there exists an element $c \in A$ such that $(c, a) \in \delta$ for all $a \in A$.*

It follows from Claim 3.9 (1) that $\mathrm{pr}_1(\delta\cap\Delta)=\emptyset$ or A, so δ is either areflexive or reflexive. In the latter case $\delta\cap\delta^{-1}$ is a reflexive, symmetric relation in $\mathcal{R}$, so it must be trivial. Since $\delta\neq A^2$, it must be the case that $\delta\cap\delta^{-1}=\Delta$, i.e., δ is antisymmetric. This proves (1).

To establish (2) notice first that, by Claim 3.8, $\mathrm{pr}_1(\delta)=\mathrm{pr}_2(\delta)=A$. Moreover,

$$\delta^{-1}\circ\delta=\{(a_1,a_2)\in A^2 : \text{there exists } b\in A \text{ such that } (b,a_1),(b,a_2)\in\delta\}$$

is a symmetric binary relation in $\mathcal{R}$, and by $\mathrm{pr}_2(\delta)=A$ it is also reflexive. Thus, by our assumptions on $\mathcal{R}$, $\delta^{-1}\circ\delta$ must be a trivial reflexive, symmetric relation. Hence either $\delta^{-1}\circ\delta=A^2$ or $\delta^{-1}\circ\delta=\Delta$. In the first case there is nothing more to prove, so assume that $\delta^{-1}\circ\delta=\Delta$. This equality, together with $\mathrm{pr}_1(\delta)=A$, implies that for each $b\in A$ there is exactly one $a\in A$ such that $(b,a)\in\delta$. Thus δ is the graph of a function $A\to A$. The equality $\mathrm{pr}_2(\delta)=A$ shows that this function is onto, so by the finiteness of A, it is a permutation of A.

Finally, we prove (3). For each t $(2\leq t\leq k-1)$ we define a relation α_t as follows:

$$\alpha_t := \{(a_1,\dots,a_t)\in A^t : \text{there exists } b\in A \text{ such that } (b,a_i)\in\delta \text{ for all } i\ (1\leq i\leq t)\}.$$

It is clear from the definition that $\alpha_t\in\mathcal{R}$ for all t. Moreover, $\alpha_2=\delta^{-1}\circ\delta$. Hence the assumption $\delta^{-1}\circ\delta=A^2$ yields that $\alpha_2=A^2$.

Next we want to show that $\alpha_{k-1}=A^{k-1}$. Suppose not, and let t be the smallest integer such that $\alpha_t\neq A^t$. Then $2<t\leq k-1$, and by the minimality of t we have that $\alpha_{t-1}=A^{t-1}$. This implies that α_t contains all t-tuples with fewer than t distinct coordinates, so α_t is totally reflexive. It is clear from its definition that α_t is also totally symmetric. Since our assumption on $\mathcal{R}$ is that $\mathcal{R}$ contains no nontrivial totally reflexive, totally symmetric relations of arity $<k$, we get that α_t is trivial. The arity of α_t is $t>2$, therefore $\alpha_t=A^t$, which contradicts the choice of t. This proves that $\alpha_{k-1}=A^{k-1}$.

Let $\mathfrak{D}$ denote the set of $(k-1)$-element subsets of A, and for each $D\in\mathfrak{D}$ let $A\setminus D=\{d_D\}$. The equality $\alpha_{k-1}=A^{k-1}$ means that for each $D\in\mathfrak{D}$ there exists $c_D\in A$ such that $(c_D,a)\in\delta$ for all $a\in D$. If $(c_D,d_D)\in\delta$ also holds for some $D\in\mathfrak{D}$, in particular, if $c_{D'}=c_D$ for some $D'\neq D$ in $\mathfrak{D}$, then $(c_D,a)\in\delta$ for all $a\in A$, and we are done. Otherwise, we have that $(c_D,d_D)\notin\delta$ for all $D\in\mathfrak{D}$, and the elements c_D are distinct for all $D\in\mathfrak{D}$. Since the elements d_D $(D\in\mathfrak{D})$ are also distinct and A is finite, we get that $\{c_D : D\in\mathfrak{D}\}=A=\{d_D : D\in\mathfrak{D}\}$ and the assignment $A\to A$, $c_D\mapsto d_D$ $(D\in\mathfrak{D})$ defines a permutation π of A. Moreover, we have that for each $a\in A$ and $D\in\mathfrak{D}$, $(c_D,a)\in\delta$ if and only if $a\neq d_D$. Therefore $\delta=A^2\setminus\pi$.

To complete the proof we have to show that this case cannot occur. Suppose first that $\delta=A^2\setminus\pi$ is areflexive. Then $\pi=\Delta$, so δ is the relation $\neq$. It is well known (see e.g. [PK]) that $\{\neq\}^{\perp}$ is the essentially unary clone where the unary operations are all permutations. Hence $\{\neq\}^{\perp}\subseteq\mathcal{S}_1\subseteq\mathcal{S}_{k-2}$. Therefore $\iota_{k-1}\in\langle\neq\rangle\subseteq\mathcal{R}$ if $k\geq 4$, and $\beta\in\langle\neq\rangle\subseteq\mathcal{R}$ if $k=3$. Each of these conclusions contradicts our assumptions on $\mathcal{R}$, and hence proves that δ is not areflexive.

By part (1), it must be the case that $\delta=A^2\setminus\pi$ is reflexive and antisymmetric. Thus $\delta\cap\delta^{-1}=\Delta$. But $\delta\cap\delta^{-1}=A^2\setminus(\pi\cup\pi^{-1})$, therefore $|\delta\cap\delta^{-1}|\geq k^2-2k>k=|\Delta|$, unless $k=3$. Hence $k=3$, and since $\delta=A^2\setminus\pi$ is reflexive, π has no fixed points. We may assume without loss of generality that $A = \{0, 1, 2\}$ and π is the 3-cycle (0 1 2). Thus

$$\delta = \{(0,0),\ (1,1),\ (2,2),\ (0,2),\ (1,0),\ (2,1)\}.$$

It is straightforward to check that the constants and the powers of π are in the clone $\{\delta\}^{\perp}$. We want to show that these are the only unary operations in $\{\delta\}^{\perp}$. Let $f \in \{\delta\}^{\perp}$. Replacing $f(x)$ by $\pi^{3-f(0)}\big(f(x)\big)$ we may assume that $f(0) = 0$. Since f preserves δ, $(1,0) \in \delta$ forces that $(f(1), 0) = (f(1), f(0)) \in \delta$, therefore $f(1) \in \{0, 1\}$. We get similarly that $f(2) \in \{0, 2\}$. Two of these possible operations f are the constant 0 and π^0. The other two do not preserve δ; indeed, if $f(1) = 0$, $f(2) = 2$, then $(2,1) \in \delta$ but $(f(2), f(1)) = (2,0) \notin \delta$, while if $f(1) = 1$, $f(2) = 0$, then $(2,1) \in \delta$ but $(f(2), f(1)) = (0,1) \notin \delta$. Thus $\mathcal{C} := \{\delta\}^{\perp}$ is a clone on a 3-element set A whose unary part $\mathcal{C}^{(1)}$ contains all constants and the nonconstant operations in $\mathcal{C}^{(1)}$ are all permutations. By Pálfy's theorem [] such a clone is either essentially unary or is the clone of polynomial operations of a vector space. For our clone $\mathcal{C}$ the latter condition fails, because a 3-element vector space has 6 permutations among its polynomial operations. Therefore $\mathcal{C} = \{\delta\}^{\perp}$ is an essentially unary clone. Hence $\{\delta\}^{\perp} \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_{k-2}$, and we get a contradiction the same way as in the preceding paragraph.

This completes the proof of Claim 3.11. $\diamond$

clm-binary

Claim 3.12. *If $m = 2$, then $\mathcal{R}$ contains either*

(1) *a prime permutation, or*
(2) *a bounded partial order.*

To prove the claim assume first that $\mathcal{R}$ contains a nontrivial binary relation π which is a permutation. Since π is nontrivial, it is not the identity permutation. Hence some power δ of π is of prime order (and therefore still not the identity permutation). Thus δ is not reflexive, which implies by Claim 3.11 (1) that δ is areflexive. Consequently, δ has no fixed points, and so δ is a prime permutation in $\mathcal{R}$.

Now assume that $\mathcal{R}$ contains a nontrivial binary relation δ which is not a permutation. Thus Claim 3.11 (2) implies that $\delta^{-1} \circ \delta = A^2$, and hence by Claim 3.11 (3)

eq-least-el

(3.1) there exists an element $c \in A$ such that $(c, a) \in \delta$ for all $a \in A$.

Since δ^{-1} is not a permutation either, we can apply the same argument to δ^{-1} in place of δ to conclude that eq-greatest-el

(3.2) there exists an element $u \in A$ such that $(a, u) \in \delta$ for all $a \in A$.

In particular, $(u, u) \in \delta$, hence δ is not areflexive. It follows from Claim 3.11 (1) that eq-refl-antis

(3.3) δ is reflexive and antisymmetric.

We may assume without loss of generality that δ is maximal, with respect to inclusion, among the nontrivial binary relations in $\mathcal{R}$ that satisfy conditions (3.1)–(3.3).

Now consider $\delta \circ \delta \in \mathcal{R}$. First we will show $\delta \circ \delta \neq A^2$ by arguing that $(u, a) \in \delta \circ \delta$ holds only if $a = u$. Indeed, if $(u, a) \in \delta \circ \delta$, then $(u, a') \in \delta$ and $(a', a) \in \delta$ for some $a' \in A$. By (3.3) and (3.2), δ is antisymmetric and $(a', u) \in \delta$, therefore $a' = u$, and hence $(u, a) = (a', a) \in \delta$. Repeating the same argument yields that $a = u$, as claimed. Thus $\delta \circ \delta \neq A^2$. Now observe that since δ is reflexive, we have that $\delta \subseteq \delta \circ \delta$. This implies that (3.1)–(3.2) hold for $\delta \circ \delta$ in place of δ. The inclusion $\delta \subseteq \delta \circ \delta$, combined with $\delta \circ \delta \neq A^2$, also shows that $\delta \circ \delta$ is nontrivial and reflexive. Thus, by Claim 3.11 (1), $\delta \circ \delta$ is reflexive and antisymmetric. By the maximality of δ we get that $\delta \circ \delta = \delta$, that is, δ is transitive. This, together with properties (3.1)–(3.3) implies that δ is a bounded partial order, completing the proof of the claim. $\diamond$

clm-ternary-prep

Claim 3.13. *If $m = 3$, then $\mathcal{R}$ contains a ternary relation δ satisfying one of the following conditions:*

(I) $\delta \cap \iota_3 = \Delta_{123}$,
(II) $\delta \cap \iota_3 = \Delta_{12|3}$,
(III) $\delta \cap \iota_3 = \Delta_{12|3} \cup \Delta_{13|2}$.

Moreover, we have $\delta \notin \mathcal{R}_0$ for each such nontrivial δ.

Let $m = 3$. By Claim 3.7 (2), $\mathcal{R}$ contains a nontrivial ternary relation δ such that δ is not totally reflexive. Thus $\Delta_{[ij]} \not\subseteq \delta$ for some $1 \leq i < j \leq 3 = m$. Let $\{i, j, s\} = \{1, 2, 3\}$. Hence, by Claim 3.9 (4), there exists a two-element set $\{u, v\} \subseteq \{j, s\}$ such that $\delta \cap \Delta_{[ij]} = \Delta_{[ij] \vee [uv]}$. Thus $\{u, v\} = \{j, s\}$ and $\Delta_{[ij] \vee [uv]} = \Delta_{123}$, so $\delta \cap \Delta_{[ij]} = \Delta_{123}$. Furthermore, by Claim 3.9 (2), $\delta \cap \iota_3$ is a union of trivial relations. Therefore it follows that, up to a permutaion of coordinates, one of conditions (I)–(III) holds for δ.

For the last statement of the claim assume that δ is a nontrivial ternary relation satisfying one of conditions (I)–(III). Thus, δ is not totally reflexive. Moreover, since δ is nontrivial, either δ contains a triple with distinct coordinates, or $\delta = \Delta_{12|3} \cup \Delta_{13|2}$. Accordingly, $\delta \notin \mathcal{R}_0$ follows from Lemma 3.1 (1) and (4), respectively. $\diamond$

We will look at the cases (I)–(III) separately.

clm-ternary-II

Claim 3.14. *There is no nontrivial ternary relation of type* (II) *in $\mathcal{R}$.*

To prove the claim assume that $\delta \in \mathcal{R}$ is a nontrivial ternary relation which satisfies condition (II). For $2 \leq t < k$ we define a t-ary relation as follows:

$$\gamma_t := \{(a_1, \ldots, a_t) \in A^t : \text{there exist } b, c \in A \text{ such that } (a_i, b, c) \in \delta \text{ for all } 1 \leq i \leq t\}.$$

It is clear from the definition that $\gamma_t \in \mathcal{R}$ for all t. Our goal is to prove that $\gamma_{k-1} = A^{k-1}$.

First we will argue that $\gamma_2 = A^2$. Let $(a_1, a_2) \in A^2$ be arbitrary. By Claim 3.8 we have that $(a_1, a_2) \in \mathrm{pr}_{1,2}(\delta)$, therefore there exists $c \in A$ such that $(a_1, a_2, c) \in \delta$. The assumption $\Delta_{12|3} \subseteq \delta$ of (II) implies that $(a_2, a_2, c) \in \delta$, which together with $(a_1, a_2, c) \in \delta$ implies $(a_1, a_2) \in \gamma_2$, establishing the equality $\gamma_2 = A^2$. Assume now that $\gamma_{k-1} \neq A^{k-1}$, and let t be the smallest integer such that $\gamma_t \neq A^t$. Then $2 < t \leq k - 1$, and by the minimality of t we have that $\gamma_{t-1} = A^{t-1}$. This implies that γ_t contains all t-tuples with fewer than t distinct coordinates, so γ_t is totally reflexive. It is clear from its definition that γ_t is also totally symmetric. Since our assumption on $\mathcal{R}$ is that $\mathcal{R}$ contains no nontrivial totally reflexive, totally symmetric relations of arity $< k$, we get that γ_t is trivial. The arity of γ_t is $t > 2$, therefore $\gamma_t = A^t$, which contradicts the choice of t. This proves that $\gamma_{k-1} = A^{k-1}$.

As before, let $\mathfrak{D}$ denote the set of all $(k-1)$-element subsets of A, and for $D \in \mathfrak{D}$ let $D = A \setminus \{d_D\}$. The equality $\gamma_{k-1} = A^{k-1}$ implies that for each $D \in \mathfrak{D}$ there exist $b_D, c_D \in A$ such that $(a, b_D, c_D) \in \delta$ for all $a \in D = A \setminus \{d_D\}$. It must be the case that $c_D = d_D$, because otherwise we would have $c_D \neq d_D$, $c_D \in D$, and hence $(c_D, b_D, c_D) \in \delta$. By the assumption $\delta \cap \Delta_{13|2} = \Delta_{123}$ this would imply that $b_D = c_D$, and hence $(a, c_D, c_D) \in \delta$ for all $a \in D$. Since $|D| = k-1 > 1$, this concusion contradicts the assumption $\delta \cap \Delta_{23|1} = \Delta_{123}$ in (II), and hence proves that $c_D = d_D$. Thus $(a, b_D, d_D) \in \delta$ for all $a \in D = A \setminus \{d_D\}$. Since $\{d_D : D \in \mathfrak{D}\} = A$, this yields a function $\varphi\colon A \to A$, $d_D \mapsto b_D$ $(D \in \mathfrak{D})$ such that eq-xxx

$$(3.4) \qquad (x, \varphi(d), d) \in \delta \quad \text{for all } x \in A \setminus \{d\}.$$

To finish the proof of (2) we will look at the unary operations in the clone $\{\delta\}^\perp$. Let f be a unary operation on A that is not a permutation and preserves δ. Since f is not a permutation, there exist distinct elements $a, d \in A$ such that $f(a) = f(d)$. Applying (3.4) and the assumption that f preserves δ we get that eq-yyy

$$(3.5) \qquad \big(f(x), f(\varphi(d)), f(d)\big) \in \delta \quad \text{for all } x \in A \setminus \{d\};$$

in particular, for $x = a$, we have that $\big(f(a), f(\varphi(d)), f(d)\big) \in \delta$. Since $f(a) = f(d)$ and $\delta \cap \Delta_{13|2} = \Delta_{123}$ by (II), we conclude that $f(\varphi(d)) = f(d)$. But then (3.5), combined with the condition $\delta \cap \Delta_{23|1} = \Delta_{123}$ from (II), yields that $f(x) = f(d)$ for all $x \in A \setminus \{d\}$. Thus f is constant. This shows that every unary operation that is not a permutation but preserves δ is constant. Clearly, all constant operations preserve δ, since δ is reflexive. Therefore the clone $\mathcal{C} := \{\delta\}^\perp$ has the property that its unary part $\mathcal{C}^{(1)}$ contans all constant operations, and every nonconstant operation in $\mathcal{C}^{(1)}$ is a permutation. By Pálfy's theorem [] either $\mathcal{C}$ is essentially unary or $\mathcal{C}$ is the clone of polynomial operations of a vector space on A. If $\mathcal{C} = \{\delta\}^\perp$ is an essentially unary clone, then $\mathcal{C} \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_{k-2}$, and we get a contradiction the same way as in the proof of Claim 3.11. Therefore $\mathcal{C}$ is the clone of polynomial operations of a vector space on A, and hence $\mathcal{C}$ contains a Mal'tsev operation g. Thus g preserves δ. Since δ is nontrivial and satisfies condition (II), there is a triple $(a, b, c) \in \delta$ such that a, b, c are distinct. The assumption $\Delta_{12|3} \subseteq \delta$ implies that $(b, b, c) \in \delta$ and $(b, b, b) \in \delta$. Thus $(a, b, b) = \big(g(a, b, b), g(b, b, b), g(c, c, b)\big) \in \delta$ where $a \neq b$, which contradicts (II).

This contradiction completes the proof of Claim 3.14. ◇

clm-ternary-I

Claim 3.15. *If $m = 3$ and $\delta \in \mathcal{R}$ is a nontrivial ternary relation for which condition* (I) *holds, then δ is the graph of a quasigroup operation on A.*

To prove the claim let $\delta \in \mathcal{R}$ be a nontrivial ternary relation for which condition (I) holds. We define a ternary relation η on A as follows:

$$\eta := \{(b, c, c') \in A^3 : \text{there exists } a \in A \text{ such that } (a, b, c) \in \delta \text{ and } (a, b, c') \in \delta\}.$$

Clearly, $\eta \in \mathcal{R}$. Since $\mathrm{pr}_{\{2,3\}}\delta = A^2$ holds by Claim 3.8, we get that $(b, c, c) \in \eta$ for all $b, c \in A$, i.e., $\Delta_{23|1} \subseteq \eta$. Now we will show that $\eta \cap \iota_3 = \Delta_{23|1}$. Since η is symmetric in its last two variables, it suffices to argue that $\eta \cap \Delta_{12|3} = \Delta_{123}$. The inclusion $\supseteq$ being obvious, let us assume that $(c, c, c') \in \eta$ for some $c, c' \in A$. Then there exists $a \in A$ such that (i) $(a, c, c) \in \delta$ and (ii) $(a, c, c') \in \delta$. But $\delta \cap \iota_3 = \Delta_{123}$, therefore we get $a = c$ from (i), and then $c = c'$ from (ii). The equality $\eta \cap \iota_3 = \Delta_{23|1}$ just proved shows that were η nontrivial, a relation obtained from it by permuting coordinates would be a nontrivial relation of type (II) in $\mathcal{R}$, which is impossible by Claim 3.14. Thus η is a trivial relation, and hence $\eta = \Delta_{23|1}$.

This equality shows that δ is the graph of a function $\mathrm{pr}_{\{1,2\}}\delta \to A$. Since we have $\mathrm{pr}_{\{1,2\}}\delta = A^2$ by Claim 3.8, δ is in fact the graph of a binary operation $*$ on A; that is,

$$\delta = \{(a, b, a * b) : a, b \in A\}.$$

Every relation obtained from δ by permuting coordinates is also a nontrivial relation of type (I) in $\mathcal{R}$, and is therefore the graph of a binary operation on A. In particular, there exist binary operations $/$ and $\backslash$ on A such that

$$\delta = \{(a, a\backslash c, c) : a, c \in A\}, \text{and}$$
$$\delta = \{(c \,/\, b, b, c) : b, c \in A\}.$$

It is clear from these descriptions of $*$, $/$, and $\backslash$ that $a * (a\backslash c) = c$, $(c \,/\, b) * b = c$, and $a \backslash (a * b) = b$, $(a * b) \,/\, b = a$ hold for all $a, b, c \in A$. Hence $*$ is a quasigroup operation with left and right divisions $\backslash$ and $/$. $\diamond$

clm-ternary-III

Claim 3.16. *If $m = 3$ and $\delta \in \mathcal{R}$ is a nontrivial ternary relation for which condition* (III) *holds, then $k > 3$ and $\mathcal{R}$ contains a quaternary relation λ such that*

eq-lambda-prop's

$$(3.6)\qquad \begin{cases} \mathrm{pr}_I\lambda = A^3 \textit{ for all } 3\textit{-element sets } I \subseteq \{1, 2, 3, 4\}, \\ \lambda \textit{ is totally symmetric, and} \\ \lambda \cap \iota_4 = \beta. \end{cases}$$

To prove the claim let $\delta \in \mathcal{R}$ be a nontrivial ternary relation for which condition (III) holds, and let δ' be obtained from δ by switching the first two coordinates, i.e., $\delta' := \{(a, b, c) \in A^3 : (b, a, c) \in \delta\}$. Then $\delta' \cap \iota_3 = \Delta_{12|3} \cup \Delta_{23|1}$. Hence $\delta \cap \delta'$ is a relation in $\mathcal{R}$ such that $(\delta \cap \delta') \cap \iota_3 = \Delta_{12|3}$. Were $\delta \cap \delta'$ nontrivial, it would be a nontrivial relation of type (II) in $\mathcal{R}$, which is impossible by Claim 3.14. Therefore $\delta \cap \delta'$ is trivial, namely $\delta \cap \delta' = \Delta_{12|3}$. This equality means that

$$(a, b, c),\ (b, a, c) \in \delta \implies a = b$$

for all $a, b, c \in A$, that is, δ is antisymmetric in its first two coordinates.

Now let λ be the intersection of all quaternary relations $\xi \in \mathcal{R}$ such that $\beta \subseteq \xi$. Clearly, $\beta \subseteq \lambda$ and $\lambda \in \mathcal{R}$. Also, λ is totally symmetric, because the family of ξ's intersected contains with each ξ all relations obtained from ξ by permuting coordinates. Our next goal is to show that $\lambda \cap \iota_4 = \beta$.

To this end, let ζ be the quaternary relation on A defined as follows:

$$\zeta := \{(a, b, c, d) \in A^4 : \text{there exist } u, v \in A \text{ such that } (a, b, u),\ (u, c, d) \in \delta,\ (b, a, v),\ (v, c, d) \in \delta\}.$$

First we want to show that $\beta \subseteq \zeta$. For arbitrary elements $a, b \in A$ we have $(a, a, b, b) \in \zeta$, because

$$(a, a, b),\ (b, b, b) \in \delta,$$
$$(a, a, b),\ (b, b, b) \in \delta,$$

and $(a, b, a, b) \in \zeta$, because

$$(a, b, a),\ (a, a, b) \in \delta,$$
$$(b, a, b),\ (b, a, b) \in \delta.$$

Thus $\Delta_{12|34} \subseteq \zeta$ and $\Delta_{13|24} \subseteq \zeta$. Since ζ is invariant under permuting its last two coordinates, the latter inclusion also implies that $\Delta_{14|23} \subseteq \zeta$. Hence $\beta \subseteq \zeta$.

Next we will show that $\zeta \cap \Delta_{[34]} = \Delta_{12|34}$. So, assume that $(a, b, c, c) \in \zeta$ for some $a, b, c \in A$. Thus there exist $u, v \in A$ such that

$$\text{(i)}\ (a, b, u) \in \delta, \quad \text{(ii)}\ (u, c, c) \in \delta,$$
$$\text{(iii)}\ (b, a, v) \in \delta, \quad \text{(iv)}\ (v, c, c) \in \delta.$$

By assumption (III), $\delta \cap \Delta_{23|1} = \Delta_{123}$, therefore we get $u = c = v$ from (ii) and (iv). Hence, by (i) and (iii), we have that (a, b, c), $(b, a, c) \in \delta$. Since δ is antisymmetric in its first two coordinates, we conclude that $a = b$. This proves that $\zeta \cap \Delta_{[34]} = \Delta_{12|34}$.

Since $\beta \subseteq \zeta$, therefore ζ is one of the relations intersected to obtain λ, so $\lambda \subseteq \zeta$. But then the equality $\zeta \cap \Delta_{[34]} = \Delta_{12|34}$ established in the preceding paragraph implies that $\lambda \cap \Delta_{[34]} \subseteq \Delta_{12|34}$. Since λ is totally symmetric, we get tha $\lambda \cap \iota_4 \subseteq \beta$. Because of $\lambda \supseteq \beta$ the reverse inclusion is clear, therefore the proof of the equality $\lambda \cap \iota_4 = \beta$ is complete.

In view of Claim 3.9 we have that $k \geq m = 3$. Assume that $k = 3$. Then every quaternary relation on A is contained in ι_4, hence in particular, $\lambda = \lambda \cap \iota_4 = \beta$. This is impossible, because by our assumptions on $\mathcal{R}$, $\beta \notin \mathcal{R}$ for $k = 3$. Therefore it must be the case that $k > 3$. For every 3-element subset I of $\{1, 2, 3, 4\}$, $\mathrm{pr}_I\lambda$ is a ternary totally reflexive, totally symmetric relation in $\mathcal{R}$. Since $k > 3$, our assumptions on $\mathcal{R}$ imply that $\mathrm{pr}_I\lambda$ must be trivial. This forces that $\mathrm{pr}_I\lambda = A^3$, and completes the proof of Claim 3.16. ◇

clm-quaternary-prep

Claim 3.17. *If $m = 4$, then $\mathcal{R}$ contains a quaternary relation λ which satisfies all conditions in* (3.6). *For each such relation λ we have $\lambda \notin \mathcal{R}_0$.*

Let $m = 4$. By Claim 3.7 (2), $\mathcal{R}$ contains a nontrivial quaternary relation δ such that δ is not totally reflexive. Thus $\Delta_{[ij]} \not\subseteq \delta$ for some $1 \leq i < j \leq 4 = m$. Let $\{i, j, s, t\} = \{1, 2, 3, 4\}$. Applying Claim 3.9 (4) we get that there exists a two-element set $\{u, v\} \subseteq \{j, s, t\}$ such that $\delta \cap \Delta_{[ij]} = \Delta_{[ij]\vee[uv]}$ and $\{s, t\} \subseteq \{u, v\}$. Thus $\{u, v\} = \{s, t\}$ and

$$\delta \cap \Delta_{ij|u|v} = \delta \cap \Delta_{[ij]} = \Delta_{ij|uv}.$$

This implies that $\Delta_{iju|v} \not\subseteq \delta$, and therefore $\Delta_{[iu]} = \Delta_{iu|j|v} \not\subseteq \delta$. Since any two two-element subsets of $\{1, 2, 3, 4\}$ can be connected by overlapping two-element sets, we get that $\Delta_{[ij]} \not\subseteq \delta$ for all two-element subsets $\{i, j\}$ of $\{1, 2, 3, 4\}$, and

$$\delta \cap \iota_4 = \delta \cap \bigcup_{1\leq i<j\leq 4} \Delta_{[ij]} = \Delta_{12|34} \cup \Delta_{13|24} \cup \Delta_{14|23} = \beta.$$

Now let λ be the relation obtained from δ by intersecting all relations arising from δ by permuting coordinates. Clearly, $\lambda \in \mathcal{R}$, $\lambda \cap \iota_4 = \beta$, and λ is totally symmetric. Since $m = 4$, Claim 3.8 shows that $\mathrm{pr}_I\lambda = A^3$ holds for all three-element sets $I \subseteq \{1, 2, 3, 4\}$. Thus λ satisfies all conditions in (3.6).

For the last statement of the claim assume that λ satisfies all conditions in (3.6). Then either λ contains a quadruple with distinct coordinates, or $\lambda = \beta$. Accordingly, $\lambda \notin \mathcal{R}_0$ follows from Lemma 3.1 (1) and (3), respectively. ◇

We will use the following observation from [ASz]:

lm-affine-op's

Lemma 3.18. *Let M be a Mal'tsev operation on a set A. The following conditions on M are equivalent:*

(a) *there exists an abelian group* $(A;+,-,0)$ *such that*

$$M(x,y,z) = x - y + z \quad \text{for all } x,y,z \in A;$$

eq-maltsev1

(b) M *satisfies*

$$M(x,y,z) = M(z,y,x) \quad \text{for all } x,y,z \in A, \text{ and} \tag{3.7}$$

$$M(x,y,z) = M(M(x,y,u),u,z) \quad \text{for all } x,y,z,u \in A. \tag{3.8}$$

eq-maltsev2

clm-ter-quater-aux

Claim 3.19. *If* $k > 4$ *and* μ *is a quaternary relation in* $\mathcal{R}$ *such that*

- μ *is the graph of a Mal'tsev operation* M *on* A*, and*
- μ *is invariant under the cyclic permutation of its coordinates and under switching its first and third coordinates,*

then μ *is a prime affine relation.*

Suppose that μ and M satisfy the assumptions of the claim. Since μ is invariant under switching its first and third coordinates, M satisfies condition (3.7). Since μ is invariant under the cyclic permutation (1 2 3 4) of its coordinates as well as the transposition (1 3) of its coordinates, it is invariant under all permutations of its coordinates in the dihedral group on $\{1,2,3,4\}$. In particular, it is invariant under the permutation (1 4)(2 3) of its coordinates, i.e.,

$$\mu = \{\big(M(a,b,c),c,b,a\big) : a,b,c \in A\}.$$

eq-compose-maltsev

Hence

$$M\big(M(a,b,c),c,b\big) = a \quad \text{for all } a,b,c \in A. \tag{3.9}$$

To prove that M also satisfies condition (3.8), we need to show that the relation ξ defined below is equal to A^4:

$$\xi := \{(a,b,c,d) \in A^4 : M\big(M(a,b,d),d,c\big) = M(a,b,c)\}.$$

Since the graph μ of M belongs to $\mathcal{R}$, it follows that $\xi \in \mathcal{R}$. If $a = b$, $b = d$, or $c = d$, then $(a,b,c,d) \in \xi$ follows from the Mal'tsev identities. If $b = c$, then $(a,b,c,d) \in \xi$ follows from (3.9). Thus

$$\Delta_{12|3|4} \cup \Delta_{24|1|3} \cup \Delta_{34|1|2} \cup \Delta_{23|1|4} \subseteq \xi.$$

Hence for $i = 3$ and $i = 4$, $\mathrm{pr}_{2,3,4}(\xi \cap \Delta_{[1i]})$ is a totally reflexive ternary relation in $\mathcal{R}$. Since $k > 4$, therefore by Claim 3.7 (1), $\mathcal{R}$ contains no nontrivial totally reflexive ternary relation, so $\mathrm{pr}_{2,3,4}(\xi \cap \Delta_{[1i]}) = A^3$. This yields that $\Delta_{[1i]} \subseteq \xi$ for $i = 3, 4$. Combining this with the inclusion in last displayed formula, we get that ξ is totally reflexive. Since $k > 4$, applying Claim 3.7 (1) again we get that $\mathcal{R}$ contains no nontrivial totally reflexive quaternary relation, hence it must be the case that $\xi = A^4$. This proves that condition (3.8) holds for M.

Hence, by Lemma 3.18, M is the Mal'tsev operation $x - y + z$ of some abelian group $\mathbf{A} = (A;+,-,0)$. It follows that the clone $\{\mu\}^\perp$ consists of all polynomial operations of the module ${}_{\mathrm{End}(\mathbf{A})}\mathbf{A}$, i.e., $\mathbf{A}$ considered as a module over its endomorphism ring $\mathrm{End}(\mathbf{A})$. If $\mathbf{A}$ is not an elementary abelian p-group (p prime), then the module ${}_{\mathrm{End}(\mathbf{A})}\mathbf{A}$ is not simple, so there is a nontrivial equivalence relation θ in $(\{\mu\}^\perp)^\perp = \langle\mu\rangle$. However, we have $\langle\mu\rangle \subseteq \mathcal{R}$, and by our assumptions on $\mathcal{R}$, there is no nontrivial equivalence relation in $\mathcal{R}$. Therefore the module ${}_{\mathrm{End}(\mathbf{A})}\mathbf{A}$ must be simple, and hence $\mathbf{A}$ must be elementary abelian.

This completes the proof that μ is a prime affine relation. ◇

clm-ter-quater

Claim 3.20. *If $3 \leq m \leq 4$, then $\mathcal{R}$ contains a prime affine relation.*

Let $m = 3$ or $m = 4$. It follows from Claims 3.13–3.17 and Claim 3.7 that either

(†) $k \geq 4$ and $\mathcal{R}$ contains a quaternary relation λ satisfying all conditions in (3.6), or

(‡) $\mathcal{R}$ contains a ternary relation δ of type (I) which is the graph of a quasigroup operation.

Suppose first that (†) holds. We will argue that λ is the graph of ternary operation. To this end define a ternary relation η on A as follows:

$$\eta := \{(c, d, d') \in A^3 : \text{there exist } a, b \in A \text{ such that } (a, b, c, d) \in \lambda \text{ and } (a, b, c, d') \in \lambda\}.$$

Clearly, $\eta \in \mathcal{R}$. As a consequence of the first condition in (3.6), $\mathrm{pr}_{3,4}\lambda = A^2$ holds for λ, therefore we get that $(c, d, d) \in \eta$ for all $b, c, d \in A$, i.e., $\Delta_{23|1} \subseteq \eta$. Now we will show that $\eta \cap \iota_3 = \Delta_{23|1}$. Since η is symmetric in its last two variables, it suffices to argue that $\eta \cap \Delta_{12|3} = \Delta_{123}$. The inclusion $\supseteq$ being obvious, let us assume that $(d, d, d') \in \eta$ for some $d, d' \in A$. Then there exist $a, b \in A$ such that (i) $(a, b, d, d) \in \lambda$ and (ii) $(a, b, d, d') \in \lambda$. But the last condition in (3.6) forces that $\lambda \cap \Delta_{34|1|2} = \Delta_{12|34} = \lambda \cap \Delta_{12|3|4}$, therefore we get $a = b$ from (i), and then $d = d'$ from (ii). The equality $\eta \cap \iota_3 = \Delta_{23|1}$ just proved shows that were η nontrivial, a relation obtained from it by permuting coordinates would be a nontrivial relation of type (II) in $\mathcal{R}$, which is impossible by Claim 3.14. Thus η is a trivial relation, and hence $\eta = \Delta_{23|1}$.

The equality $\eta = \Delta_{23|1}$ shows that λ is the graph of a function $\mathrm{pr}_{\{1,2,3\}}\lambda \to A$. Since we have $\mathrm{pr}_{\{1,2,3\}}\lambda = A^3$ by (3.6), λ is in fact the graph of a ternary operation M on A; that is,

$$\lambda = \big\{\big(a, b, c, M(a, b, c)\big) : a, b, c \in A\big\}.$$

By (3.6), λ is also totally symmetric. The equality $\lambda \cap \iota_4 = \beta$ from (3.6) implies that $M(a, a, c) = M(a, c, a) = M(c, a, a) = c$ for all $a, c \in A$. Hence M is a minority (so a Mal'tsev) operation. Thus M and its graph λ satisfy all assumptions on Claim 3.19. So, if $k > 4$, then Claim 3.19 yields that λ is a prime affine relation, and we are done.

If $k = 4$, then using again the equality $\lambda \cap \iota_4 = \beta$ from (3.6) we get that for arbitrary distinct elements $a, b, c \in A$ we have that $M(a, b, c) \neq a, b, c$. This condition, together with the fact that M is a minority operation, uniquely determines M, so M is the operation $x - y + z$ of any elementary abelian 2-group $(A; +, -, 0)$ on the 4-element set A. This finishes the proof of Claim 3.20 in the case when $\mathcal{R}$ satisfies condition (†).

From now on we will assume that condition (†) fails for $\mathcal{R}$. As we remarked at the beginning of the proof, in this case (‡) holds for $\mathcal{R}$. Let $*$ denote the quasigroup operation whose graph is δ. We saw in the proof of Claim 3.15 that the graphs of the left and right divisions $\backslash$, $/$ of $*$ are obtained from δ by permuting coordinates, so they are also members of $\mathcal{R}$. It follows that any operation composed from $*$, $\backslash$, and $/$ has its graph in $\mathcal{R}$. In particular, the Mal'tsev operation

$$M(x, y, z) := (x \,/\, x) \setminus ((x \,/\, y) * z)$$

has its graph μ in $\mathcal{R}$.

Since the graph δ of $*$ satisfies $\delta \cap \iota_3 = \Delta_{123}$, we get that $*$ is idempotent, i.e., $a * a = a$ for all $a \in A$, and for distinct $a, b \in A$ we have that $a * b \neq a, b$. If $k = 3$, then there is only one such quasigroup operation on the 3-element set A, namely $2x + 2y$ of any group $(A; +, -, 0)$ on A. Hence $*$, $\backslash$, $/$ all coincide with this operation, and M is the Mal'tsev operation $x - y + z$ of the 3-element cyclic group $(A; +, -, 0)$. Similarly, if $k = 4$, then there are exactly two idempotent quasigroup operations on a 4-element set: the term operations $ax + (a+1)y$ and $(a+1)x + ay$ of a one-dimensional vector space over the 4-element field $\mathrm{GF}(4) = \{0, 1, a, a+1\}$ with $a^2 = a + 1$. In both cases M is the Mal'tsev operation $x - y + z$ of the vertor space, and hence of its underlying elementary abelian 2-group. This proves that if $k = 3, 4$, then μ is a prime affine relation.

From now on we will assume that $k > 4$. In view of Claim 3.19, it will follow that μ is a prime affine relation if we can show that μ is invariant under the cyclic permutation $(1\ 2\ 3\ 4)$ of its coordinates, and under the transposition $(1\ 3)$ of its coordinates. Since M is a Mal'tsev operation, it contains all tuples of the form (a, a, b, b) and (a, b, b, a) $(a, b \in A)$; i.e,, $\Delta_{12|34} \cup \Delta_{14|23} \subseteq \mu$.

Let μ' be the intersection of all relations in $\mathcal{R}$ which contain $\Delta_{12|34} \cup \Delta_{14|23}$. Clearly, $\mu' \in \mathcal{R}$ and $\mu' \subseteq \mu$. Since the set $\Delta_{12|34} \cup \Delta_{14|23}$ is invariant under the cyclic permutation $(1\ 2\ 3\ 4)$ of its coordinates, and under the transposition $(1\ 3)$ of its coordinates, so is μ'. On the other hand, $\mu' \subseteq \mu$ implies that μ' is the graph of a function $\mathrm{pr}_{\{1,2,3\}}\mu' \to A$, namely the graph of the restriction of M to $\delta_0 := \mathrm{pr}_{\{1,2,3\}}\mu'$. The inclusion $\Delta_{12|34} \cup \Delta_{14|23} \subseteq \mu'$ implies that $\Delta_{12|3} \cup \Delta_{1|23} \subseteq \delta_0$. Suppose that δ_0 is a nontrivial relation. It follows from Claim 3.9 (2) that either $\delta_0 \cap \iota_3 = \iota_3$, and hence δ_0 is totally reflexive, or $\delta_0 \cap \iota_3 = \Delta_{12|3} \cup \Delta_{1|23}$. The first case is impossible by Claim clm-m-less-k (1), since $k > 4$. Therefore the second case holds for δ_0, that is, up to a permutation of coordinates, δ_0 satisfies condition (III). But then by Claim 3.16, $\mathcal{R}$ contains a quaternary relation λ satisfying (3.6). This contradicts our assumption that (†) fails. Hence δ_0 is a trivial relation. Since $\Delta_{12|3} \cup \Delta_{1|23} \subseteq \delta_0$, it follows that $\delta_0 = A^3$. This proves that $\mu = \mu'$. Hence μ satisfies the assumptions of Claim 3.19, and therefore μ is a prime affine relation.

This completes the proof of Claim 3.20. ◇

Theorem 3.6 now follows from Claims 3.10, 3.12, and 3.20. □

thm-trts

Theorem 3.21. *Let A be a k-element set ($k \geq 3$), and let $\mathcal{R}$ be a relational clone on A that contains a nontrivial totally reflexive, totally symmetric relations of arity $< k$. Let m ($1 \leq m \leq k-1$) be the largest integer such that $\mathcal{R}$ contains a nontrivial m-ary totally reflexive, totally symmetric relation ρ, and let ρ be maximal among the nontrivial m-ary totally reflexive, totally symmetric relations in $\mathcal{R}$. Then one of the following conditions holds:*

1. *ρ is a central relation ($1 \leq m \leq k-1$),*
2. *ρ is an almost central relation ($2 \leq m \leq k-2$),*
3. *ρ is a nontrivial equivalence relation ($m = 2$),*
4. *ρ is an m-regular relation ($3 \leq m \leq k-1$),*
5. *ρ is an almost m-regular relation ($3 \leq m \leq k-1$).*

Proof. If $m = 1$, then ρ is a nontrivial unary relation, so ρ is a central relation, and (1) holds. From now on we will assume that $m \geq 2$.

For $m \leq t \leq k-1$ let

$$\sigma_t := \{(a_1, \dots, a_t) \in A^t : \text{there exists } c \in A \text{ such that } (a_{i_1}, \dots, a_{i_{m-1}}, c) \in \rho \text{ for all } 1 \leq i_1 < \dots < i_{m-1} \leq t\}.$$

clm-trts1

Claim 3.22. *Assume $m \geq 2$. For all t ($m \leq t \leq k-1$), σ_t is a totally symmetric relation in $\mathcal{R}$. Moreover, for $t = m$ we have that either $\sigma_m = \rho$ or $\sigma_m = A^m$.*

The first statement of the claim is clear from the definition of σ_t. To verify the second statement we will first argue that $\rho \subseteq \sigma_m$. Indeed, if $(a_1, \dots, a_m) \in \rho$, then the defining condition of σ_m holds for this tuple with the choice of $a = a_1$, because ρ is totally reflexive and totally symmetric. The maximality of ρ therefore implies that $\sigma_m = \rho$ or $\sigma_m = A^m$. This completes the proof of Claim 3.22. ◇

We will first consider the case when $\sigma_m = A^m$.

clm-central

Claim 3.23. *If $m \geq 2$ and $\sigma_m = A^m$, then ρ is either a central relation or an almost central relation.*

Assume that $\sigma_m = A^m$. First we will argue that $\sigma_{k-1} = A^{k-1}$. Suppose not, and let t be the smallest integer such that $m \leq t \leq k-1$ and $\sigma_t \neq A^t$. Then $t > m$ and $\sigma_{t-1} = A^{t-1}$. The latter equality and the definition of σ_t immediately imply that σ_t is totally reflexive. By Claim 3.22, σ_t is also totally symmetric. Furthermore, by the choice of t we have that $t > m \geq 2$ and $\sigma_t \neq A^t$, so σ_t is nontrivial. This contradicts the choice of m and ρ, and therefore proves that $\sigma_{k-1} = A^{k-1}$.

Let $D = \{d_1, d_2, \dots, d_{k-1}\}$ be an arbitrary $(k-1)$-element subset of A. Since $(d_1, d_2, \dots, d_{k-1}) \in A^{k-1} = \sigma_{k-1}$, the definition of σ_{k-1} yields the existence of an element $c_D \in A$ such that $(d_{i_1}, \dots, d_{i_{m-1}}, c_D) \in \rho$ for all $1 \leq i_1 < \dots < i_{m-1} \leq k-1$. Since ρ is totally reflexive and totally symmetric, this shows that $D^{m-1} \times \{c_D\} \subseteq \rho$.

If, for some D, c_D can chosen to be the unique element of A outside D, then by the total reflexivity of ρ, $A^{m-1} \times \{c_D\} \subseteq \rho$, so c_D is a central element of ρ and $\rho\,(\neq A^m)$ is a central relation. Otherwise ρ is not a central relation, but $c_D \in D$ and $D^{m-1} \times \{c_D\} \subseteq \rho$ hold for all $(k-1)$-element subsets D of A. Hence each c_D is a central element for the relation $\rho|_D$, which implies that $\rho|_D$ is either central or equal to D^m. It cannot be the case that $\rho|_D = D^m$ for all $(k-1)$-element subsets of A, because, in view of $m \leq k-1$, that would imply $\rho = A^m$, contradicting the nontriviality of ρ. Thus in this case ρ is an almost central relation. ◇

Claim 3.23 shows that if $m \geq 2$, then either ρ satisfies one of conditions (1)–(2) of the theorem, or $\sigma_m = \rho$. Hence it remains to prove that if $\sigma_m = \rho$, then ρ satisfies one of conditions (3)–(5). Therefore we will assume from now on that $\sigma_m = \rho$.

We define a binary relation ε as follows:

$$\varepsilon := \{(a, b) \in A^2 : (a_1, \dots, a_{m-2}, a, b) \in \rho \text{ for all } a_1, \dots, a_{m-1} \in A\}.$$

Notice that for $m = 2$ we have $\varepsilon = \rho$.

clm-eqrel

Claim 3.24. *Assume that $m \geq 2$ and $\sigma_m = \rho$. Then*

1. *ε is an equivalence relation on A, and*
2. *ρ is ε-saturated, i.e., whenever $(a_1, \dots, a_m) \in \rho$ and $(a_i, b_i) \in \varepsilon$ for all i ($1 \leq i \leq m$), then $(b_1, \dots, b_m) \in \rho$.*

In particular, if $m = 2$ and $\sigma_2 = \rho$, then ρ is a nontrivial equivalence relation.

Assume that $m \geq 2$ and $\sigma_m = \rho$. The relation ε is clearly reflexive and symmetric, since ρ is totally reflexive and totally symmetric. To see that ε is transitive, let $(a,c),(c,b) \in \varepsilon$. Since ε is symmetric, $(a,c),(b,c) \in \varepsilon$. This means that for arbitrary tuple $(a_1,\ldots,a_{m-2}) \in A^{m-2}$, both $(a_1,\ldots,a_{m-2},a,c) \in \rho$ and $(a_1,\ldots,a_{m-2},b,c) \in \rho$, and hence, by the definition of σ_m, $(a_1,\ldots,a_{m-2},a,b) \in \sigma_m = \rho$. This proves that $(a,b) \in \varepsilon$, and concludes the argument for (1).

To prove (2) we will first establish the following special case:

$$(a_1,\ldots,a_{m-1},a) \in \rho \text{ and } (a,b) \in \varepsilon \implies (a_1,\ldots,a_{m-1},b) \in \rho.$$

Indeed, let $(a_1,\ldots,a_{m-1},a) \in \rho$ and $(a,b) \in \varepsilon$. Writing a_m for b, the latter condition implies that $(a_m,a) \in \varepsilon$, and hence $(a_{i_1},\ldots,a_{i_{m-1}},a) \in \rho$ for all $1 \leq i_1 < \cdots < i_{m-2} < i_{m-1} = m$. Combining this with the assumption $(a_1,\ldots,a_{m-1},a) \in \rho$ we obtain that $(a_1,\ldots,a_{m-1},b) = (a_1,\ldots,a_{m-1},a_m) \in \sigma_m = \rho$. This proves the displayed propery of ρ. Since ρ is totally symmetric, ρ has analogous properties in all coordinates. Hence, appying these properties in each coordinate one-by-one, we get the stronger property stated in (2).

Finally, as we observed after the definition of ε, in the case when $m = 2$, ε is equal to ρ. Hence in this case ρ is an equivalence relation, which is nontrivial by assumption. ◇

The last statement of Claim 3.24 proves that if $\sigma_m = \rho$ holds with $m = 2$, then ρ satisfies condition (3) of the theorem. Therefore we are left with proving that if $m \geq 3$ and $\sigma_m = \rho$, then ρ satisfies one of conditions (4)–(5) of the theorem. Hence we will assume from now on that $m \geq 3$ and $\sigma_m = \rho$.

For $m \leq t \leq k-1$ let ρ_t denote the set of all t-tuples $\mathbf{b} = (b_1,\ldots,b_t) \in A^r$ such that $\mathrm{pr}_I(\mathbf{b}) \in \rho$ for all t-element subsets I of $\{1,\ldots,t\}$. Furthermore, for $m \leq t \leq k-1$ let

$$\begin{aligned}\tau_t := \{(a_1,\ldots,a_t) \in A^t : {} & \text{there exists } (c_1,\ldots,c_t) \in \rho_t \text{ such that} \\ & (a_{i_1},\ldots,a_{i_{m-1}},c_{i_j}) \in \rho \\ & \text{for all } 1 \leq i_1 < \cdots < i_{m-1} \leq t,\ 1 \leq j \leq m-1\}.\end{aligned}$$

We start with an analog of Claim 3.22.

clm-trts2

Claim 3.25. *Assume $m \geq 3$. For all t $(m \leq t \leq k-1)$, ρ_t and τ_t are totally symmetric relations in $\mathcal{R}$. Moreover, for $t = m$, either $\tau_m = \rho$ or $\tau_m = A^m$.*

The first statement of the claim is clear from the definitions of ρ_t and τ_t. To prove the second statement we will first show that $\rho \subseteq \tau_m$. Let $(a_1,\ldots,a_m) \in \rho$. Clearly, $\rho_m = \rho$, so we can choose $(c_1,\ldots,c_m)$ to be $(a_1,\ldots,a_m)$. With this choice the condition for $(a_1,\ldots,a_m)$ to belong to τ_m clearly holds, since ρ is totally reflexive. This proves that $\rho \subseteq \tau_m$. Now the maximality of ρ implies that $\tau_m = \rho$ or $\tau_m = A^m$, completing the proof of Claim 3.25. ◇

Definition 3.26. For $m \geq 3$ an m-ary relation δ on A is called *strongly homogeneous* if for all $(a_1,\ldots,a_m) \in A^m$ and $(c_1,\ldots,c_m) \in \delta$ such that

$$(a_1,\ldots,a_{i-1},c_j,a_{i+1},\ldots,a_m) \in \delta \quad \text{whenever } i \neq j \ (1 \leq i,j \leq m),$$

it is the case that $(a_1,\ldots,a_m) \in \delta$.

Using the assumption that ρ is totally symmetric, one can easily see that that ρ is strongly homogeneous if and only if $\tau_m \subseteq \rho$. As we have seen in the proof of

Claim 3.25, $\rho \subseteq \tau_m$ also holds. Thus

$$\tau_m = \rho \iff \rho \text{ is strongly homogeneous.}$$

clm-strhomog

Claim 3.27. *If $m \geq 3$ and $\sigma_m = \rho$, then ρ is strongly homogeneous.*

Assume that $m \geq 3$ and $\sigma_m = \rho$. By the previous claim we have either $\tau_m = \rho$ or $\tau_m = A^m$. In the first case we are done. To prove that the second case is impossible, we assume for the rest of the proof that $\tau_m = A^m$, and work towards a contradiction.

First we will argue that $\tau_{k-1} = A^{k-1}$. Suppose not, and let t be the smallest integer such that $m \leq t \leq k-1$ and $\tau_t \neq A^t$. Then $t > m$ and $\tau_{t-1} = A^{t-1}$. We want to show that τ_t is totally reflexive. Since τ_t is totally symmetric (see Claim 3.25), it suffices to show that $(a_1, a_1, a_2, \dots, a_{t-1}) \in \tau_t$ for all $(a_1, \dots, a_{t-1}) \in A^{t-1}$. If $(a_1, \dots, a_{t-1}) \in A^{t-1}$, then also $(a_1, \dots, a_{t-1}) \in \tau_{t-1}$, as $\tau_{t-1} = A^{t-1}$. Thus there exists $(c_1, \dots, c_{t-1}) \in \rho_{t-1}$ such that $(a_{i_1}, \dots, a_{i_{m-1}}, c_{i_j}) \in \rho$ for all $1 \leq i_1 < \cdots < i_{m-1} \leq t-1$ and $1 \leq j \leq m-1$. But then $(b_1, b_1, b_2, \dots, b_{t-1}) \in \rho_t$ and this tuple witnesses that $(a_1, a_1, a_2, \dots, a_{t-1}) \in \tau_t$. Thus τ_t is totally reflexive and totally symmetric. Furthermore, by the choice of t we have that $t > m \geq 2$ and $\tau_t \neq A^t$, so τ_t is nontrivial. This contradicts the choice of m and ρ, and therefore proves that $\tau_{k-1} = A^{k-1}$.

Now let $D = \{d_1, d_2, \dots, d_{k-1}\}$ be an arbitrary $(k-1)$-element subset of A. Since $(d_1, d_2, \dots, d_{k-1}) \in A^{k-1} = \tau_{k-1}$, the definition of τ_{k-1} yields the existence of a $(k-1)$-tuple $(c_1, \dots, c_{k-1}) \in A^{k-1}$ such that

(i) $(c_{i_1}, \dots, c_{i_m}) \in \rho$ for all $1 \leq i_1 < \cdots < i_m \leq k-1$, and
(ii) $(d_{i_1}, \dots, d_{i_{m-1}}, c_{i_j}) \in \rho$ for all $1 \leq i_1 < \cdots < i_{m-1} \leq k-1$ and $1 \leq j \leq m-1$.

Our goal is to show that $D^m \subseteq \rho$. This will yield the desired contradiction, because $m \leq k-1$, and therefore the inclusion $D^m \subseteq \rho$ for all $(k-1)$-element subsets D of A implies that $\rho = A^m$, which contradicts our assumption that ρ is nontrivial. To show that $D^m \subseteq \rho$ we will consider two cases.

Case 1: $c_1, \dots, c_{k-1} \in D$.

Since ρ is totally reflexive, $D^m \subseteq \rho$ will follow if we show that every m-tuple of distinct elements of D belongs to ρ. Without loss of generality, it suffices to do the proof for the m-tuple $(d_1, \dots, d_m)$. We will prove $(d_1, \dots, d_m) \in \rho$ by showing that

$(*)$ $(d_1, \dots, d_l, c_{l+1}, \dots, c_m) \in \rho$ for all $l = 0, 1, \dots, m$.

We proceed by induction on l. For $l = 0$ statement $(*)$ is $(c_1, \dots, c_m) \in \rho$, which is true by (i). Now assume that $(*)$ holds for l (≥ 0), and prove it for $l+1$; that is, we assume that $(d_1, \dots, d_l, c_{l+1}, c_{l+2} \dots, c_m) \in \rho$, and want to prove that $(d_1, \dots, d_l, d_{l+1}, c_{l+2} \dots, c_m) \in \rho$. Since $\sigma_m = \rho$, it suffices to prove that $(d_1, \dots, d_l, d_{l+1}, c_{l+2} \dots, c_m) \in \sigma_m$. By the definition of σ_m and the total symmetry of ρ, this will follow if we check that every m-tuple obtained from $(d_1, \dots, d_l, d_{l+1}, c_{l+2} \dots, c_m)$ by replacing one of the coordinated by c_{l+1} belongs to ρ. The m-tuple obtained by replacing d_{l+1} by c_{l+1} is $(d_1, \dots, d_l, c_{l+1}, c_{l+2} \dots, c_m)$, which belongs to ρ by assumption. The m-tuples obtained by replacing some c_i $(l+1 < i \leq m)$ with hace two occurences of c_{l+1}, and hence will belong to ρ since ρ is totelly reflexive. The remaining m-tuples, those obtained by replacing some d_i $(1 \leq i < l+1)$ by c_{l+1}, will all contain both c_{l+1} and d_{l+1} among their coordinates, and will have all unchanged coordinates d_j $(1 \leq j \leq l+1,\ j \neq i)$, c_j $(l+2 \leq j \leq m)$

in D (by the assumption of Case 1). Therefore these m-tuples will belong to ρ by condition (ii) and by the total symmetry of ρ, if the unchanged coordinates are distinct, and by the total reflexivity of ρ otherwise. This completes the induction and hence the proof of $D^m \subseteq \rho$ in Case 1.

Case 2: Some $c_j \notin D$, so $A = D \cup \{c_j\}$.

We will show that in this case $(c_j, d_j) \in \varepsilon$, that is, $(a_1, \dots, a_{m-2}, c_j, d_j) \in \rho$ for all $a_1, \dots, a_{m-2}$. Since ρ is totally reflexive, $(a_1, \dots, a_{m-2}, c_j, d_j) \in \rho$ is clear if the coordinates of this m-tuple are not pairwise distinct. Therefore assume that the coordinates are distinct. Then $\{a_1, \dots, a_{m-2}, d_j\} = D$, and hence $(a_1, \dots, a_{m-2}, c_j, d_j) \in \rho$ follows from (ii) and the total symmetry of ρ. Thus $(c_j, d_j) \in \varepsilon$, as claimed. Now get from Claim 3.24 (2) that if we replace c_j by d_j in $(c_1, \dots, c_{k-1})$ we get a new tuple $(c'_1, \dots, c'_{k-1})$ which satisfies the same conditions (i)–(ii) as $(c_1, \dots, c_{k-1})$, but also satisfies the assumption of Case 1. Therefore $D^m \subseteq \rho$ follows from Case 1. This finishes the proof of Claim 3.25. $\diamond$

Definition 3.28. Let B be a finite set, let $m \geq 3$, and let $U = \{1, 2, \dots, m\}$.

- An m-ary relation δ on B is called *universal* if for some $r \geq 1$ there exists a surjective function $f\colon U^r \to B$ such that $f[(\iota_m^U)^r] \subseteq \delta$.
- An m-ary relation δ on B will be called *almost universal* if it is not universal, but for every subset D of B of size $|D| = |B| - 1$ there exist $r \geq 1$ and a surjective function $f\colon U^r \to D$ such that $f[(\iota_m^U)^r] \subseteq \delta$.

clm-universal

Claim 3.29. *If $m \geq 3$, then ρ is either universal or almost universal.*

Let $U = \{1, 2, \dots, m\}$. We define relations χ_t for $m \leq t \leq k$ as follows:

$$\chi_t := \bigl\{(a_1, \dots, a_t) \in A^t : \{a_1, \dots, a_t\} \text{ is contained in the range of some } f\colon U^r \to A \text{ such that } f[(\iota_m^U)^r] \subseteq \rho\bigr\}.$$

Clearly, ρ is universal if and only if $\chi_k = A^k$ and ρ is almost universal if and only if $\chi_k \neq A^k$, but $\chi_{k-1} = A^{k-1}$. Since $\chi_k = A^k$ implies that $\chi_{k-1} = A^{k-1}$, we will be done if we prove that $\chi_{k-1} = A^{k-1}$.

Suppose $\chi_{k-1} \neq A^{k-1}$, and let t be the smallest integer such that $m \leq t \leq k-1$ and $\chi_t \neq A^t$. It cannot be the case that $t = m$, because $\chi_m = A^m$, as we will argue now. Indeed, for arbitrary tuple $(a_1, \dots, a_m) \in A^m$ consider the unary function $f\colon U \to A$, $i \mapsto a_i$ $(1 \leq i \leq m)$. It is clear that $f[\iota_m^U] \subseteq \rho$, since ρ is totally reflexive. Thus f witnesses that $(a_1, \dots, a_m) \in \chi_m$, and hence proves that $\chi_m = A^m$. Thus $t > m$, and hence by the minimality of t, $\chi_{t-1} = A^{t-1}$. This implies that every m-tuple with fewer than m distinct coordinates belongs to χ_m, so χ_m is totally reflexive. It is clear from the definition that χ_m is also totally symmetric.

Now we want to argue that $\chi_t \in \langle \rho \rangle$, and hence $\chi_t \in \mathcal{R}$. To this end it suffices to show that χ_t is preserved by all operations preserving ρ. So, let g be an arbitrary operation preserving ρ, say g is n-ary, and let $\mathbf{a}^{(i)} = (a_1^{(i)}, \dots, a_t^{(i)}) \in \chi_t$ $(1 \leq i \leq n)$. Then, there exist functions $f_i\colon U^{r_i} \to A$ $(1 \leq i \leq n)$ such that $\{a_1^{(i)}, \dots, a_t^{(i)}\}$ is contained in the range of f_i and $f_i[(\iota_m^U)^{r_i}] \subseteq \rho$. Choose $\mathbf{u}_j^{(i)} \in U^{r_i}$ such that $a_j^{(i)} = f_i(\mathbf{u}_j^{(i)})$ $(1 \leq i \leq n,\ 1 \leq j \leq t)$. Let $r = r_1 + \cdots r_n$ and let $\widehat{f}\colon U^r \to A$ be the function defined by

$$\widehat{f}(x_1, \dots, x_r) := g(f_1(x_1, \dots, x_{r_1}), f_2(x_{r_1+1}, \dots, x_{r_1+r_2}), \dots, f_n(x_{r_1+\dots+r_{n-1}+1}, \dots, x_r)).$$

Then $\widehat{f}$ witnesses that $g(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)}) \in \chi_t$. Indeed, the j-th coordinate of the t-tuple $g(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)})$ is

$$g(a_j^{(1)}, \ldots, a_j^{(n)}) = g(f_1(\mathbf{u}_j^{(1)}), \ldots, f_n(\mathbf{u}_j^{(n)})) = \widehat{f}(\mathbf{u}_j^{(1)}, \ldots, \mathbf{u}_j^{(n)}),$$

which is in the range of $\widehat{f}$ for all j $(1 \leq j \leq n)$. Moreover,

$$\widehat{f}[(\iota_m^U)^r] = g\big[f_1[(\iota_m^U)^{r_1}], \ldots, f_n[(\iota_m^U)^{r_n}]\big] \subseteq g[\rho, \ldots, \rho] \subseteq \rho,$$

as g preserves ρ. This shows that g preserves χ_t, and hence proves that $\chi_t \in \mathcal{R}$.

Summarizing, $\chi_t \neq A^t$ is a nontrivial totally reflexive, totally symmetric relation in $\mathcal{R}$ whose arity is $t > m$. By our assumptions on ρ, such a relation does not exist in $\mathcal{R}$. This contradiction proves that $\chi_{k-1} \neq A^{k-1}$, and concludes the proof of Claim 3.29. ◇

To finish the proof of Theorem 3.21 we will use the lemma below, which is part of the proof of Rosenberg's Theorem (see, e.g., Rosenberg[], Quackenbush[], Lau[]).

lm-Ros

Lemma 3.30. ([],[],[]) *Let B be a finite set, let $m \geq 3$, and let δ be an m-ary relation on B such that $\rho \neq B^m$. If δ is totally reflexive, totally symmetric, strongly homogeneous, and universal, then δ is an m-regular relation.*

clm-reg

Claim 3.31. *If $m \geq 3$ and $\sigma_m = \rho$, then ρ is either m-regular or almost m-regular.*

By assumption, $\rho \neq A^m$ and ρ is totally reflexive, totally symmetric. By Claim 3.27, ρ is also strongly homogeneous. Finally, by Claim 3.29, ρ is either universal or almost universal. If ρ is universal, then all assumptions of Lemma 3.30 hold for ρ, so we get that ρ is an m-regular relation on A.

For the rest of the proof of the claim let us assume that ρ is almost universal. Let D be a $(k-1)$-element subset of A, and let $\rho_D = \rho \cap D^m$ be the restriction of ρ to D. Clearly, ρ_D is totally reflexive and totally symmetric. Since ρ is almost universal, there exist $r \geq 1$ and a function $f\colon U^r \to A$ such that the range of f contains D and $f[(\iota_m^U)^r] \subseteq \rho$. As ρ is not universal, f is not onto. Hence the range of f is D. So, the function f_D obtained from f by changing its codomain to D we get a surjective function $f_D\colon U^r \to D$ such that $f_D[(\iota_m^U)^r] \subseteq \rho \cap D^m = \rho_D$. This proves that ρ_D is universal. Finally, we want to show that ρ_D is strongly homogeneous. Let $(a_1, \ldots, a_m) \in D^m$ and $(c_1, \ldots, c_m) \in \rho_D$ be such that $(a_1, \ldots, a_{i-1}, c_j, a_{i+1}, \ldots, a_m) \in \rho_D$ whenever $i \neq j$ $(1 \leq i, j \leq m)$. Then $(a_1, \ldots, a_m) \in A^m$ and $(c_1, \ldots, c_m) \in \rho$ satisfy $(a_1, \ldots, a_{i-1}, c_j, a_{i+1}, \ldots, a_m) \in \rho$ whenever $i \neq j$ $(1 \leq i, j \leq m)$. Hence the strong homogeneity of ρ implies that $(a_1, \ldots, a_m) \in \rho$. But also $(a_1, \ldots, a_m) \in D^m$, so $(a_1, \ldots, a_m) \in \rho_B$, as required. This shows that ρ_B is a totally reflexive, totally symmetric, strongly homogeneous, and universal relation on D. It follows from Lemma 3.30 that either $\rho_D = D^m$ or ρ_D is m-regular. Since this conclusion holds for all $(k-1)$-element subsets D of A, and ρ is not m-regular, we conclude that ρ is almost m-regular. ◇

Claim 3.31 proves that if $m \geq 3$ and $\sigma_m = \rho$, then ρ satisfies one of conditions (4)–(5) of the theorem. This finishes the proof of Theorem 3.21. □

4. Completeness criteria for $\mathcal{S}$ and $\mathcal{S}^-$

thm-compl-slup

Theorem 4.1. *Let A be a k-element set $(k \geq 3)$. The maximal subclones of Słupecki's clone $\mathcal{S}$ are the following clones:*

(1) *$\mathcal{S}(\mathcal{T}^- \cup G)$ where G is a maximal subgroup of the symmetric group on A;*

(2) $\mathcal{S}_{k-2}$ *if* $k > 3$*;*
(3) *Burle's clone* $\mathcal{B}$ *if* $k = 3$.

thm-compl-nonsurj

Theorem 4.2. *Let A be a k-element set ($k \geq 3$). Every maximal subclone of $\mathcal{S}^-$ is of the form $\mathcal{S}^- \cap \{\rho\}^\perp$ for one of the relations ρ of types (1), (4)–(9) listed in Theorem 2.1.*

Proof. Let $\mathcal{C}$ be a proper subclone of $\mathcal{S}^-$. Then $\mathcal{S}^- \not\subseteq \mathcal{C}$, therefore by Theorem 2.1, $\mathcal{C} \subseteq \{\rho\}^\perp$ holds for one of the relations ρ listed in Theorem 2.1 (1)–(9). Hence also $\mathcal{C} \subseteq \mathcal{S}^- \cap \{\rho\}^\perp$ for one of these relations. If ρ is a prime permutation of order p (p prime), and hence $k = |A| = np$ for some integer $n \geq 1$, then every operation preserving ρ has range that is a union of some of the p-element orbits of ρ. Therefore every nonsurjective operation preserving ρ has range of size at most $r = k - p$. In particular, if $k = p$, this means that every operation in $\mathcal{S}^- \cap \{\rho\}^\perp$ has to be a projection. Similarly, if ρ is a prime affine relation associated to an elementary abelian p-group $\mathbf{A} = (A; +, -, 0)$, and hence $k = p^n$ for some integer $n \geq 1$, then every operation preserving ρ has range that is a coset of a subgroup of $\mathbf{A}$. Therefore every nonsurjective operation preserving ρ has range of size at most $r = k/p$. In particular, if $k = p$, this means that every operation in $\mathcal{S}^- \cap \{\rho\}^\perp$ is either a projection or constant. Thus, in both cases we have

$$\mathcal{S}^- \cap \{\rho\}^\perp \subseteq \begin{cases} \mathcal{S}^-_{k-p} \subseteq \mathcal{S}^-_{k-2} = \{\iota_{k-1}\}^\perp & \text{if } k > p, \\ \langle \mathcal{T}^- \rangle \subseteq \mathcal{S}^-_{k-2} = \{\iota_{k-1}\}^\perp & \text{if } k = p > 3, \\ \langle \mathcal{T}^- \rangle \subseteq \mathcal{B} = \{\beta\}^\perp & \text{if } k = p = 3. \end{cases}$$

Since ι_{k-1} is one of the almost m-regular relations, this shows that $\mathcal{C} \subseteq \mathcal{S}^- \cap \{\rho\}^\perp$ holds for one of the relations of types (1), (4)–(9) in Theorem 2.1. In particular, every maximal subclone of $\mathcal{S}^-$ is equal to one of these clones $\mathcal{S}^- \cap \{\rho\}^\perp$. This completes the proof. □

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