A Completeness Criterion for Semi-Affine Algebras

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Abstract

A Rosenberg-type completeness criterion is proved for a semi-affine algebra to be a simple affine algebra.

Introduction

Affine algebras, i.e. algebras polynomially equivalent to modules, and their reducts, called semi-affine algebras, are of interest in universal algebra as well as in multiple-valued logic: in universal algebra they play an important role in the study of congruence modular varieties and in the structure theory of finite algebras, while in multiple-valued logic they come up most naturally as algebras whose clones are contained in one of the maximal clones of linear type in Rosenberg’s Theorem [3].

In both of these areas it is a basic question: what operations can be constructed from a given set \( F \) of operations by composition, or, alternatively, given a clone (a composition closed set) \( C \) of operations, under what conditions a subset \( F \) of \( C \) generates \( C \) (via composition); if it does, then \( F \) is said to be complete in \( C \).

In a more algebraic setting, the question is: under what conditions a reduct \( (A; F) \) of an algebra \( (A; C) \) is term equivalent to \( (A; C) \). The most important result of this type is Rosenberg’s Theorem [3] solving the problem for primal algebras (i.e., for \( C \) the clone of all operations on a finite set \( A \)), and it is typical, too, in that completeness is characterized in terms of excluded compatible relations of \( (A; F) \). It is clear that the structure of the algebra \( (A; C) \) and its reducts \( (A; F) \) might be essential in these considerations, therefore an algebraic approach proves often useful.

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In this paper we consider a semi-affine algebra ‘complete’ if it is a simple affine algebra, and investigate the question under what conditions a semi-affine algebra is complete. We get the following Rosenberg-type completeness criterion (Theorem 2.1): a finite algebra \( A \) that is semi-affine with respect to an elementary Abelian group \( \hat{A} \) is complete if and only if \( A \) admits no nontrivial congruence of \( \hat{A} \) and no \( q \)-regular relation corresponding to a \( q \)-regular family of congruences of \( \hat{A} \), and \( A \) is not isomorphic to a matrix power of a unary semi-affine algebra.

We note that Slupecki-type completeness criteria for reducts of certain simple affine algebras were proved earlier in [8].

Preliminaries

An algebra is a pair \( A = (A; F) \) with \( A \) a nonvoid set called the universe of \( A \), and \( F \) a set of finitary operations on \( A \) called the set of fundamental operations of \( A \). An operation \( f \) on \( A \) is a term operation [polynomial operation] of \( A \) if \( f \) can be constructed from the fundamental operations of \( A \) and from projection operations [from the fundamental operations of \( A \), from projections, and from constant operations] via composition.

A set \( C \) of operations on \( A \) is called a clone if it contains the projections and is closed under composition. Obviously, the term operations [polynomial operations] of any algebra form a clone.

If not stated otherwise, algebras are denoted by boldface capitals and their universes by the corresponding letters in italics. The clone of term operations [the set of \( n \)-ary term operations] of an algebra \( A \) is denoted by \( \text{Clo} A \) [resp., \( \text{Clo}_n A \)]. Similarly, the clone of polynomial operations [the set of \( n \)-ary polynomial operations] of \( A \) is denoted by \( \text{Pol} A \) [resp., \( \text{Pol}_n A \)].
We will call an algebra $A$ surjective if every fundamental operation of $A$ is surjective. For algebras $A = (A; F)$ and $A' = (A'; F')$, we say that $A$ is a reduct [polynomial reduct] of $A'$ if $A = A'$ and $F \subseteq \text{Clo} A' [F \subseteq \text{Pol} A']$. The algebras $A = (A; F)$ and $A' = (A'; F')$ are called term equivalent [polynomially equivalent] if $A = A'$ and $\text{Clo} A = \text{Clo} A'$ [Pol $A = \text{Pol} A'$].

For a set $N$, let $T_N$, $S_N$, and $C_N$ denote the full transformation monoid on $N$, the full symmetric group on $N$ and the set of (unary) constant operations on $N$, respectively. The identity mapping and the equality relation on $N$ are denoted by $\text{id}$ and $\Delta$, respectively ($N$ will be clear from the context). For convenience we identify every natural number $n$ with the set $n = \{0, 1, \ldots, n - 1\}$.

For a set $\mathcal{A}$ and for $k \geq 1$, the nonvoid subsets of $\mathcal{A}^k$ will also be called $k$-ary relations (on $\mathcal{A}$), and for an algebra $A$ the universes of subalgebras of $\mathcal{A}^k$ will be called compatible relations of $A$. An operation $f$ on $A$ is said to preserve a relation $\rho$ if $\rho$ is a compatible relation of the algebra $(A; f)$.

We say that an algebra $A$ is semi-affine with respect to an Abelian group $\hat{A} = (A; +)$ if $A$ and $\hat{A}$ have the same universe and

$$Q_{\hat{A}} = \{(a, b, c, d) \in A^4: a - b + c = d\}$$

is a compatible relation of $A$ (or equivalently, the operations of $A$ commute with $x - y + z$). Furthermore, $A$ is said to be affine with respect to $\hat{A}$ if it is semi-affine with respect to $\hat{A}$ and, in addition, $x - y + z$ is a term operation of $A$. It is well known (cf. [10; 2.1, 2.7–2.8]) that

— an algebra $A$ is semi-affine with respect to an Abelian group $\hat{A}$ if and only if $A$ is a polynomial reduct of the module $(\text{End} \hat{A}) \hat{A}$ (i.e. $\hat{A}$ considered as a module over its endomorphism ring $\text{End} \hat{A}$), and

— $A$ is affine with respect to $\hat{A}$ if and only if $A$ is polynomially equivalent to a module $\mu \hat{A}$ for some subring $R$ of $\text{End} \hat{A}$.

Let $q \geq 3$. A family $T = \{\Theta_0, \ldots, \Theta_{m-1}\}$ ($m \geq 1$) of equivalence relations on $A$ is called $q$-regular if each $\Theta_i$ ($0 \leq i \leq m - 1$) has exactly $q$ blocks and $\Theta_T = \Theta_0 \cap \ldots \cap \Theta_{m-1}$ has exactly $q^m$ blocks. A relation on $A$ is called $q$-regular if it is of the form

$$\lambda_T = \{(a_0, \ldots, a_{q-1}) \in A^q: \text{for all } i (0 \leq i \leq m - 1), \text{ } a_0, \ldots, a_{q-1} \text{ are not pairwise incongruent modulo } \Theta_i\}$$

for a $q$-regular family $T$ of equivalence relations on $A$. 
Let \( U \) be a \( q \)-element set and \( m \geq 1 \). The kernels of the \( m \) distinct projections \( U^m \to U \) form a \( q \)-regular family of equivalences on \( U^m \), which will be called the standard \( q \)-regular family of equivalences on \( U^m \); the corresponding \( q \)-regular relation is called the standard \( q \)-regular relation on \( U^m \). It is well known that the \( m \)th matrix power \( U^{[m]} \) of any unary algebra \( U = (U; F) \) admits the standard \( q \)-regular relation as a compatible relation. We recall that the universe of \( U^{[m]} \) is \( U^m \), and its operations are exactly all operations \( h^\mu_{\sigma} [g_0, \ldots, g_{m-1}] \) defined for arbitrary mappings \( \sigma : m \to m, \mu : m \to n \) and \( g_0, \ldots, g_{m-1} \in \text{Clo}_U U \) as follows: for \( x_i = (x_0^i, \ldots, x_{m-1}^i) \in U^m (0 \leq i \leq n-1), \)

\[
h^\mu_{\sigma} [g_0, \ldots, g_{m-1}](x_0, \ldots, x_{n-1}) = (g_0(x_0^\sigma), \ldots, g_{m-1}(x_{(m-1)^\sigma})^\mu).
\]

The mappings \( \sigma, \mu \) will be called the component mapping and the variable mapping of \( h^\mu_{\sigma} [g_0, \ldots, g_{m-1}] \), respectively. For unary operations the subscript indicating the variable mapping \( m \to 1 \) will be omitted.

In the lemma below we collect some well-known facts on finite algebras admitting \( q \)-regular compatible relations.

**Lemma 1.1.** Let \( A = (A; F) \) be a finite algebra, and let \( T = \{\Theta_0, \ldots, \Theta_{m-1}\} \) be a \( q \)-regular family of equivalence relations on \( A \) such that \( \lambda_T \) is a compatible relation of \( A \).

1. (1.1.i) \( T/\Theta_T = \{\Theta_0/\Theta_T, \ldots, \Theta_{m-1}/\Theta_T\} \) is a \( q \)-regular family of equivalences on \( A/\Theta_T \), and there exists a bijection \( \varphi : A/\Theta_T \to q^m \) carrying \( T/\Theta_T \) into the standard \( q \)-regular family of equivalences on \( q^m \).

2. (1.1.ii) If \( f \in F \) is an \( n \)-ary operation whose range meets each block of some \( \Theta_i \), then there exist \( j, l \) \( (0 \leq j \leq m - 1, 0 \leq l \leq n - 1) \) such that for \( x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1} \in A \) we have

\[
f(x_0, \ldots, x_{m-1}) \Theta_i f(y_0, \ldots, y_{m-1}) \text{ whenever } x_i \Theta_j y_i.
\]

3. (1.1.iii) If \( A \) is a surjective algebra, then

   (1) \( \Theta_T \) is a congruence of \( A \),

   (2) the relation \( \lambda_T/\Theta_T \) is a compatible relation of \( A/\Theta_T \), and

   (3) the bijection \( \varphi \) yields an isomorphism between \( A/\Theta_T \) and a reduct of the matrix power \( (q; S_q)^{[m]} \).

The proof of (1.1.ii) can be found, e.g. in \cite[Lemma 7.3]{5}. The claims in (1.1.iii) are well-known consequences of (1.1.i) and (1.1.ii); see \cite{6}, \cite{4}. We note that Rousseau \cite{6} (cf. \cite{4}) proved (1.1.iii)(3) for the case \( \Theta_T = \Delta \), however, in view of (1.1.iii)(1)–(2) the more general claim follows immediately from this special case.
Our basic tool in proving the main result of this paper is a strong version of Rosenberg’s primal algebra characterization theorem [3]. Recall that a finite algebra \( A \) is called \textit{quasiprimal} ([1], [2]) if every operation on \( A \) preserving the internal isomorphisms (i.e., isomorphisms between subalgebras) of \( A \) is a term operation of \( A \). Further, a \( k \)-ary relation \( \rho \) on \( A \) is said to be \textit{central} if \( \rho \neq A^k \), \( \rho \) is totally reflexive, totally symmetric, and there exists a \( c \in A \) such that \((c,a_1,\ldots,a_{k-1}) \in \rho \) for all \( a_1,\ldots,a_{k-1} \in A \).

\textbf{Theorem 1.2.} [11] Let \( A \) be a finite simple algebra having no proper subalgebra. Then one of the following conditions holds:

1. \( A \) is quasiprimal;
2. \( A \) is affine with respect to an elementary Abelian \( p \)-group \( (p \text{ prime});
3. \( A \) is isomorphic to a reduct of \((2;T_2)^{[m]} \) for some integer \( m \geq 1 \);
4. \( A \) has a compatible \( q \)-regular relation for some \( q \geq 3 \);
5. \( A \) has a compatible \( k \)-ary central relation for some \( k \geq 2 \);
6. \( A \) has a compatible bounded partial order.

\textbf{Main results}

On the base set \( p \) (\( p \text{ prime} \), \( + \) and \( \cdot \) will always denote addition, resp. multiplication modulo \( p \). Further, we let \( L_p \) denote the set of all unary linear operations on \( p \), i.e.

\[ L_p = \{cx + a: 0 \leq c, a \leq p - 1 \}. \]

Our main result is

\textbf{Theorem 2.1.} For arbitrary finite algebra \( A \) that is semi-affine with respect to an elementary Abelian \( p \)-group \( \hat{A} = (A;+) \) (\( p \text{ prime} \), one of the following conditions holds:

1. \( A \) is affine with respect to \( \hat{A} \);
2. \( A \) has a nontrivial congruence which is a congruence of \( \hat{A} \);
3. there is a group isomorphism \( \hat{A} \to (p;+)^{[m]} \) which is simultaneously an isomorphism between \( A \) and a reduct of \((p;L_p)^{[m]} \);
4. \( A \) has a compatible relation \( \lambda_T \) for some \( q \)-regular family \( T \) of congruences of \( \hat{A} \) with \( q > p \).

Clearly, if for an algebra \( A \) as in Theorem 2.1 condition (2.1.c) or (2.1.d) holds, then \( A \) cannot be affine. Thus Theorem 2.1 yields a necessary and sufficient condition for simple semi-affine algebras to be affine.
Corollary 2.2. Let $\mathbf{A}$ be a finite simple algebra that is semi-affine with respect to an elementary Abelian $p$-group $\hat{\mathbf{A}} = (\mathbf{A}; +)$ ($p$ prime). Then $\mathbf{A}$ is affine with respect to $\hat{\mathbf{A}}$ if and only if both of conditions (2.1.c) and (2.1.d) fail for $\mathbf{A}$.

The rest of this section is devoted to the proof of Theorem 2.1.

Let $\hat{\mathbf{A}} = (\mathbf{A}; +)$ be an Abelian group. The group \{x + a: a ∈ A\} of all translations of $\hat{\mathbf{A}}$ will be denoted by $T(\hat{\mathbf{A}})$. For an algebra $\mathbf{A} = (\mathbf{A}; F)$ that is semi-affine with respect to $\hat{\mathbf{A}}$, $\mathbf{A}^\ast$ will stand for the algebra $(\mathbf{A}; F, T(\hat{\mathbf{A}}))$ arising from $\mathbf{A}$ by adding all translations of $\hat{\mathbf{A}}$ as unary operations.

Lemma 2.3. For an algebra $\mathbf{A}$ that is semi-affine with respect to an Abelian group $\hat{\mathbf{A}} = (\mathbf{A}; +)$, $\mathbf{A}^\ast$ is affine with respect to $\hat{\mathbf{A}}$ if and only if $\mathbf{A}$ is such.

Proof. It is straightforward to check that the clone of $\mathbf{A}^\ast$ is

$$\text{Clo}\mathbf{A}^\ast = \{ \sum_{i=0}^{n-1} r_i x_i + a: n \geq 1, a \in A, \text{and} \sum_{i=0}^{n-1} r_i x_i + a_0 \in \text{Clo}\mathbf{A} \text{ for some } a_0 \in A \}.$$ 

This implies the claim of the lemma.

In view of this lemma, when we want to prove Theorem 2.1 via applying Theorem 1.2 for semi-affine algebras $\mathbf{A}$, we can always replace $\mathbf{A}$ by $\mathbf{A}^\ast$, i.e. we may assume that the translations in $T(\hat{\mathbf{A}})$ are operations of $\mathbf{A}$. Thus, in what follows, we look more closely at the relations preserved by all translations of an Abelian group.

For equivalence relations the following fact is easy and well-known.

Lemma 2.4. For an Abelian group $\hat{\mathbf{A}} = (\mathbf{A}; +)$, if $\Theta$ is an equivalence relation on $\mathbf{A}$ such that $\Theta$ is preserved by all translations in $T(\hat{\mathbf{A}})$, then $\Theta$ is a congruence of $\hat{\mathbf{A}}$.

For studying $q$-regular relations we shall need a group theoretical result. First we recall some notions and notation. Let $G \subseteq S_N$ be a permutation group acting on a set $N$. The orbits of $G$ are the minimal nonvoid subsets of $N$ that are closed under all permutations in $G$. Clearly, the orbits of $G$ yield a partition of $N$. We say that $G$ is transitive on $N$ if $N$ is an orbit of $G$, and $G$ acts regularly on $N$ if it is transitive and no non-identity permutation in $G$ has fixed points.

Let $k$ and $m$ be arbitrary positive integers, and let $P$ be a subgroup of $S_m$. Clearly, the unary term
operations \( h^\sigma [g_0, \ldots, g_{m-1}] \) of \((k; S_k)^m\) with \( \sigma \in P \) form a permutation group acting on the set \( k^m \). In group theory this group is called the general wreath product of \( S_k \) and \( P \), and is denoted by \( S_k \Wr P \) (cf. [7; p. 272]). In \( S_k \Wr P \) the elements \( h^{id}[g_0, \ldots, g_{m-1}] \) form a normal subgroup (isomorphic to the \( m \)th direct power of \( S_k \)), which will be denoted by \((S_k)^m\), while the elements \( h^\sigma[id, \ldots, id] \) form a subgroup (isomorphic to \( P \)), which will be denoted by \( P \). Obviously, \( P \) is a complement of \((S_k)^m\) in \( S_k \Wr P \) in the sense that \((S_k)^m \cap P = \{id\} \) and \((S_k)^m P = S_k \Wr P \).

If \( P \) is a regular permutation group on \( m \), then \( S_k \Wr P \) essentially coincides with the so-called complete wreath product of \( S_k \) and \( P \) (cf. [7; pp. 270, 272]).

Lemma 2.5. Let \( G \) be a subgroup of the permutation group \( S_q \Wr S_m \) where \( q \) is a power of a prime number \( p \) and \( m \) is an arbitrary positive integer. If \( G \) is an elementary Abelian \( p \)-group which acts regularly on \( q^m \), then \( G \) is a subgroup of \((S_q)^m\).

Proof. Let \( G \) be a subgroup of \( S_q \Wr S_m \) satisfying the assumptions of the lemma, and let \( P \) denote the group of component mappings of permutations in \( G \). Thus \( G \) is an elementary Abelian \( p \)-subgroup of \( S_q \Wr P \) acting regularly on \( q^m \). Let \( I_0, \ldots, I_{t-1} \) denote the orbits of \( P \). Then each member \( h^\sigma[g_0, \ldots, g_{m-1}] \) of \( G \) acts componentwise, via \( h^\sigma[I_i] = [g_i; i \in I_i] \) \((l = 0, \ldots, t - 1) \) on the set \( q^m = q^{i_0} \times \cdots \times q^{i_{t-1}} \). By the well-known fact that every commutative, transitive permutation group is regular, it follows that in each component we have a regular permutation group. Consequently, for cardinality reasons, \( G \) splits into a direct product of \( t \) regular, elementary Abelian \( p \)-subgroups of \( S_q \Wr I_i \) \((l = 0, \ldots, t - 1) \), respectively. Hence it suffices to prove that if \( P \) is transitive, then \( m = 1 \).

Assume that \( P \) is transitive. Since \( P \) is a homomorphic image of \( G \), therefore \( P \) is an elementary Abelian \( p \)-group. From the transitivity and commutativity of \( P \) it follows that \( P \) is regular as well.

Consider the subgroup \( G_0 = G \cap (S_q)^m \) of \( G \).

Since \( G \) is finite and Abelian, it has a subgroup \( P_0 \) that is a complement of \( G_0 \) in \( G \) (that is, \( G_0 \cap P_0 = \{id\} \) and \( G_0 P_0 = G \)). Clearly, for each \( \sigma \in P \), \( P_0 \) contains exactly one permutation with component mapping \( \sigma \). Thus \( P_0 \) is a complement of \((S_q)^m\) in the complete wreath product \( S_q \Wr P \). It is known (cf. [7; 10.7 in Chapter 2]) that any two complements of \((S_q)^m\) in \( S_q \Wr P \) — specifically \( P \) and \( P_0 \) — are conjugate. Since all assumptions on \( G \) and the required conclusion as well are invariant under conjugation, we may assume without loss of generality that \( P \subseteq G \). How-
ever, as \( G \) is Abelian, \( G \) is contained in the centralizer of \( \overline{P} \) in \( S_q \wr P \), which is easily seen to be equal to

\[
\{h^\sigma [g, \ldots, g] : g \in S_q, \sigma \in P\}
\]

(cf. [7; Exercise 2 on p. 277]). Obviously, this group is transitive only if \( m = 1 \), completing the proof. \( \diamond \)

**Lemma 2.6.** Let \( \hat{A} \) be a finite elementary Abelian \( p \)-group \((p \text{ prime})\), and let \( T = \{\Theta_0, \ldots, \Theta_{m-1}\} \) be a q-regular family of equivalences on \( A \) such that \( \lambda_T \) is preserved by all translations in \( T(\hat{A}) \). Then

(2.6.i) \( \Theta_0, \ldots, \Theta_{m-1} \), and hence their intersection \( \Theta_T \) as well, are congruences of \( \hat{A} \), and

(2.6.ii) for any elementary Abelian \( p \)-group \((q;+)\), there exists an isomorphism \( \hat{A}/\Theta_T \to (q;+)^m \)
carrying \( T/\Theta_T \) into the standard q-regular family of equivalences on \( q^m \).

**Proof.** Consider the unary algebra \( A = (A; T(\hat{A})) \). By our assumption \( \lambda_T \) is a compatible relation of \( A \). Since \( A \) is surjective, we get from Lemma 1.1 (1.1.iii)(1) that \( \Theta_T \) is a congruence of \( A \).

So by Lemma 2.4 \( \Theta_T \) is a congruence of \( \hat{A} \). Applying Lemma 1.1 (1.1.i) and (1.1.iii)(3) we get also that there exists an isomorphism \( \varphi \) between the algebra \( \mathcal{A}/\Theta_T = (\hat{A}/\Theta; T(\hat{A}/\Theta)) \) and a reduct of the matrix power \((q;S_q)^{[m]} \) such that \( \varphi \) carries \( T/\Theta_T \) into the standard q-regular family \( \{\Theta_0, \ldots, \Theta_{m-1}\} \) of equivalences on \( q^m \). Let \( G \) denote the subgroup of \( S_q^m \) corresponding to the group \( T(\hat{A}/\Theta) \) under \( \varphi \). Clearly, \( G \) is a subgroup of \( S_q \wr S_m \). Furthermore, by construction, \( G \) is an elementary Abelian \( p \)-group, which acts transitively on \( q^m \). Now Lemma 2.5 states that \( G \subset (S_q^m) \), whence it follows that \( \Phi_0, \ldots, \Phi_{m-1} \) are congruences of \((q^m;G)\). Via the isomorphism \( \varphi \) we get that \( \Theta_0/\Theta_T, \ldots, \Theta_{m-1}/\Theta_T \) are congruences of \( \mathcal{A}/\Theta_T \), and hence \( \Theta_0, \ldots, \Theta_{m-1} \) are congruences of \( A \). Now by Lemma 2.4 we conclude that (2.6.i) holds.

Since the family \( T \) of congruences of \( \hat{A} \) is q-regular, the natural embedding

\[
\hat{A}/\Theta_T \to \hat{A}/\Theta_0 \times \cdots \times \hat{A}/\Theta_{m-1}
\]

is an isomorphism, and all quotient groups on the right are elementary Abelian \( p \)-groups with \( q \) elements. Up to isomorphism, we can replace them with the given group \((q;+)\), and the requirements in (2.6.ii) obviously hold. \( \diamond \)

**Lemma 2.7.** Let \( A \) be a finite algebra that is semi-affine with respect to an elementary Abelian \( p \)-group \( \hat{A} = (A;+) \) \((p \text{ prime})\), and let \( T \) be a q-regular family of congruences of \( \hat{A} \) such that \( \lambda_T \) is a compatible relation of \( A^* \). Then
(2.7.i) \(\Theta_T\) is a congruence of \(A\), and
(2.7.ii) if \(\Theta_T = \Delta\), then there is a group isomorphism \(\hat{A} \rightarrow (p;+)^m\) which is simultaneously an isomorphism between \(A\) and a reduct of \((p; L_p)^m\).

Proof. Let \(T = \{\Theta_0, \ldots, \Theta_{m-1}\}\). By the previous lemma these equivalences are congruences of \(\hat{A}\), and so is their intersection \(\Theta_T\).

To prove (2.7.i) let \(f\) be an \(n\)-ary operation of \(A\), and let \(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1} \in A\) be arbitrary elements of \(A\) such that \(x_k \Theta_T y_k\) for all \(0 \leq k \leq n - 1\). Let \(0 \leq i \leq m - 1\). Assume first that the range of \(f\) meets at least two blocks of \(\Theta_i\). Since \(\hat{A}/\Theta_i\) is a \(p\)-element cyclic group and \(A\) is semi-affine with respect to \(\hat{A}\), it is clear that the range of \(f\) meets each block of \(\Theta_i\). Thus we get from Lemma 1.1 (1.1.ii) that \(f(x_0, \ldots, x_{n-1}) \Theta_i f(y_0, \ldots, y_{n-1})\). The same conclusion is obvious, if the range of \(f\) meets only one block of \(\Theta_i\). Since \(i\) was arbitrary, we conclude that \(f(x_0, \ldots, x_{n-1}) \Theta_T f(y_0, \ldots, y_{n-1})\), as required.

Now let \(\Theta_T = \Delta\). By Lemma 2.6 (2.6.ii) there exists an isomorphism \(\hat{A} \rightarrow (p;+)^m\) carrying \(T\) into the standard \(p\)-regular family of equivalences on \(p^m\). Let \(B = (p^m; F)\) be the algebra corresponding to \(A\) under this isomorphism. Notice that the standard \(p\)-regular relation on \(p^m\) is a compatible relation of \(B\), and apply Lemma 1.1 (1.1.ii) to each operation \(f\) of \(B\). Let, say, \(f\) be \(n\)-ary. For \(b \in p^m\) the components of \(b\) will be denoted by \(b^0, \ldots, b^{m-1}\). Let \(0 \leq i \leq m - 1\) be arbitrary. As in the previous paragraph, we see that the set of \(i\)th components of \(f(b_0, \ldots, b_{n-1})\) as the arguments run over all elements of \(p^m\) is either \(p\) or a one-element set. In the first case we get from (1.1.ii) that there exist indices \(j_i, l_i\), \(0 \leq j_i \leq m - 1\), \(0 \leq l_i \leq n - 1\) and a permutation \(g_i \in S_p\) such that the \(i\)th component of \(f(b_0, \ldots, b_{n-1})\) equals \(g_i(h_i^{j_i}(b_0, \ldots, b_{n-1}))\) for all \(b_0, \ldots, b_{n-1} \in p^m\). In the second case the same holds with \(g_i\) constant (and \(j_i, l_i\) arbitrary). Thus \(f = h_i^o[g_0, \ldots, g_{m-1}]\) where \(\sigma\) and \(\mu\) are the mappings \(\sigma: m \rightarrow m, i \mapsto j_i\) and \(\mu: m \rightarrow n, i \mapsto l_i\). Hence \(B\) is a reduct of \((p; S_p \cup C_p)^m\). Taking into consideration that \(B\) is semi-affine with respect to \((p;+)^m,\) one can easily derive that \(B\) is a reduct of \((p; L_p)^m\), completing the proof.

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Let \(A\) be a finite algebra that is semi-affine with respect to an elementary Abelian \(p\)-group \(\hat{A} = (A;+)(p\ prime)\), and consider the algebra \(A^*\). Because of the translations, \(A^*\) has no proper subalgebra, no compatible bounded partial order and no compatible central relation. If \(A^*\) is not simple, then by Lemma 2.4 (2.1.b) trivially holds, so assume \(A^*\) is simple. Now we can apply Theorem 1.2
for $A^*$. Since a semi-affine algebra cannot be quasiprimal, condition (1.2.b), (1.2.c) or (1.2.d) in Theorem 1.2 holds for $A^*$.

Assume first that (1.2.b) holds for $A^*$. It is well known that if an algebra is affine with respect to an Abelian group, then (because of the term operation $x - y + z$) this group is uniquely determined up to the choice of the element $0$. Thus (2.1.a) holds for $A^*$ and hence for $A$ as well.

Now let us consider the case when (1.2.c) holds for $A^*$, that is, there exists an isomorphism $\varphi$ between $A^*$ and a reduct of the matrix power $(2; T_2)^m$ (hence $p = 2$). Let $G$ denote the subgroup of $S_{2m}$ corresponding to the group $T(\hat{A})$ under $\varphi$. Clearly, $G$ is a subgroup of $S_2$ wr $S_m$, and $G$ is an elementary Abelian 2-group acting transitively on $2^m$. By Lemma 2.5 we have $G \subseteq (S_2)^m$, so for cardinality reasons $G = (S_2)^m$.

Let $\omega$ be the image of $0 \in A$ under $\varphi$, and let $\tau$ be the translation $x + \omega$ of the Abelian group $(2; +)^m$. It is straightforward to check that the mapping $\varphi_\tau$ is a group isomorphism $\hat{A} \to (2; +)^m$ which is simultaneously an isomorphism between $A$ and a reduct of $(2; T_2)^m$. Obviously, $T_2 = L_2$, hence (2.1.c) holds with $p = 2$.

Finally, suppose condition (1.2.d) holds for $A^*$, and let $T$ be a $q$-regular family of equivalences on $A$ such that $\lambda_T$ is a compatible relation of $A^*$. Obviously, $\lambda_T$ is preserved by all translations in $T(\hat{A})$, so by Lemma 2.6 $T$ consists of congruences of $\hat{A}$. It follows now that $q$ is a power of $p$. If $q > p$, then (2.1.d) trivially holds, while if $q = p$, then by Lemma 2.7 and by the simplicity of $A$ we have $\Theta_T = \Delta$ and condition (2.1.c) holds for $A$.

Concluding remarks

1. For an elementary Abelian $p$-group $\hat{A} = (A; +)$ ($p$ prime) let $Q(\hat{A})$ denote the clone consisting of all operations on $A$ preserving the relation $Q_\hat{A}$; in other words, $Q(\hat{A})$ is the largest one among the clones of those algebras on $A$ that are semi-affine with respect to $A$. These clones constitute one of the six classes of maximal clones in Rosenberg’s theorem [3]. Making use of Theorem 2.1 one can easily determine the maximal subclones of $Q(\hat{A})$. There are three types:

— those containing the operation $x - y + z$; to find them explicitly one can apply the description of the clones of affine algebras (cf. [9], [10; 2.6]);

— the inverse images of $\operatorname{Clo}(p; L_p)^m$ under all isomorphisms $\hat{A} \to (p; +)^m$; and
— for each $q$-regular family $T$ of congruences of $\hat{A}$ with $q > p$, the clone of all operations in $Q(\hat{A})$ preserving $\lambda_T$.

2. Let $A$ be a surjective, finite, simple algebra that is semi-affine with respect to an elementary Abelian $p$-group $A = (A; +)$ ($p$ prime). Combining Corollary 2.2 and the claims in Lemma 1.1 (1.1.iii) we get that either $A$ is affine with respect to $\hat{A}$, or it is isomorphic to a reduct of $(q; S_p)^{\lfloor m \rfloor}$ for some power $q$ of $p$. An application of this observation yields an alternative proof for the result shown in [12] stating that all surjective, finite, simple algebras of type 2 are affine.


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