ALGEBRAS FROM CONGRUENCES

PETER MAYR AND ÁGNES SZENDREI

ABSTRACT. We present a functorial construction which, starting from a congruence α of finite index in an algebra $\mathbf{A}$, yields a new algebra $\mathbf{C}$ with the following properties: the congruence lattice of $\mathbf{C}$ is isomorphic to the interval of congruences between 0 and α on $\mathbf{A}$, this isomorphism preserves higher commutators and TCT types, and $\mathbf{C}$ inherits all idempotent Maltsev conditions from $\mathbf{A}$.

As applications of this construction, we first show that supernilpotence is decidable for congruences of finite algebras in varieties that omit type $\mathbf{1}$. Secondly, we prove that the subpower membership problem for finite algebras with a cube term can be effectively reduced to membership questions in subdirect products of subdirectly irreducible algebras with central monoliths. As a consequence, we obtain a polynomial time algorithm for the subpower membership problem for finite algebras with a cube term in which the monolith of every subdirectly irreducible section has a supernilpotent centralizer.

1. INTRODUCTION

This paper was motivated by Problems 1.1–1.2 below. In each problem the question is whether or not a result that is known for certain algebras can be lifted to congruences.

Problem 1.1. Let $\mathbf{A}$ be a finite algebra (in a finite language). Given a congruence α of $\mathbf{A}$, is it decidable if α is supernilpotent?

This question arises from the result that finite nilpotent Maltsev algebras are non-dualizable if they have some non-abelian supernilpotent congruence [3].

In the special case when α is the total congruence of $\mathbf{A}$, i.e., when the question is whether it is decidable if $\mathbf{A}$ itself is supernilpotent, the answer to Problem 1.1 has been known to be YES by a combination of results from [12, 13, 1] (see Section 4), provided the variety generated by $\mathbf{A}$ is assumed to omit type $\mathbf{1}$.

Problem 1.2. Let **A** be a finite algebra (in a finite language) such that **A** has a cube term. If for every subdirectly irreducible section **S** of **A** the centralizer of the monolith of **S** is supernilpotent, does there exist a polynomial time algorithm for solving the Subpower Membership Problem for **A**?

In the special case when **A** itself is supernilpotent (and, without loss of generality, has prime power order), the answer has been known to be YES by a result of [17].

Our goal in this paper is to use the composition of two known functors to create "algebras from congruences", to study the resulting (functorial) construction, and show that it preserves many algebraic properties. Then we apply these results to prove that the answer to Problem 1.2 is YES, and the answer to Problem 1.1 is also YES provided the variety generated by **A** omits type **1**. For finite algebras **A** not satisfying this assumption, Problem 1.1 remains open even for the special case when α is the total congruence of **A**.

To describe the idea of the construction we use, let us look at a single algebra **A** and a congruence α of **A** with finitely many blocks. The first step of the construction yields a multisorted algebra in which the sorts are the α-blocks, and the multisorted operations are the restrictions of the operations of **A** to the sorts in all possible ways. The second step of the construction takes this multisorted algebra as its input, and yields a single-sorted algebra on the product of the sorts, where the operations are the diagonal operation of the product of the sorts and some totally defined operations that faithfully represent the operations of the multisorted algebra.

By restricting to pairs $(\mathbf{A}, \alpha)$ where α is the kernel of a homomorphism $\mathbf{A} \to \mathbf{I}$ of **A** into a fixed finite algebra **I**, the first step of the construction can be made into a functor $\mathfrak{M}$ from a category of algebras over **I** (cf. [15, Ch. II, Sec. 6]) into a category of multisorted algebras, while the second step of the construction can be made into a functor $\mathfrak{P}$ from a category of multisorted algebras to a category of (single-sorted) algebras. We will introduce these functors in Section 2, and will prove that their composition $\mathfrak{C} := \mathfrak{P} \circ \mathfrak{M}$ — which is our construction of "algebras from congruences" — is a categorical equivalence, provided we restrict to the cases where the homomorphism $\mathbf{A} \to \mathbf{I}$ is onto, that is, the sorts of the corresponding multisorted algebras are nonempty.

The functors $\mathfrak{M}$ and $\mathfrak{P}$ were introduced by Novotný [20] and Gardner [10], respectively. A precursors of $\mathfrak{P}$ appears in [2] (see also the last section of [11]). Recently, the functor $\mathfrak{P}$ was applied by Mučka, Romanowska, and Smith [19] to find good sets of explicit defining identities and quasi-identities for the variety of single-sorted algebras equivalent to the class of all multisorted algebras of a given language where either all sorts are nonempty or all sorts are empty. Freese and McMenzie [9] used a variant of the construction provided by the functor $\mathfrak{C}$ to associate modules (over appropriate rings) to abelian congruences of algebras in a congruence modular variety, which has played a prominent role in commutator theory for over three decades.

In Section 3 we will study the algebraic properties of the categorical equivalence $\mathfrak{C}$. To describe some of the properties we prove, it will be convenient to introduce a notation for the $\mathfrak{C}$-image of an algebra $\mathbf{A}$ (over $\mathbf{I}$). In this section, we will use the informal notation $\mathbf{A}^{\mathfrak{C}}$. (The precise definition of $\mathfrak{C}$ and the notation that goes along with it, and is used outside this section, can be found in Corollary 2.8.)

For congruences, we prove in Section 3 that for every algebra $\mathbf{A}$ and surjective homomorphism $\mathbf{A} \to \mathbf{I}$ with kernel α, the functor $\mathfrak{C}$ yields an isomorphism $I_{\mathbf{A}}(0, \alpha) \to \mathrm{Con}(\mathbf{A}^{\mathfrak{C}})$ between the interval $I_{\mathbf{A}}(0, \alpha) = \{\beta \in \mathrm{Con}(\mathbf{A}) : 0 \le \beta \le \alpha\}$ of the congruence lattice of $\mathbf{A}$ and the whole congruence lattice $\mathrm{Con}(\mathbf{A}^{\mathfrak{C}})$ of $\mathbf{A}^{\mathfrak{C}}$ (see Corollary 3.4). Moreover, we show that this isomorphism preserves commutators and higher arity commutators (see Theorem 3.6). In particular, α is a supernilpotent congruence of $\mathbf{A}$ iff $\mathbf{A}^{\mathfrak{C}}$ is a supernilpotent algebra, and the degrees of supernilpotence are the same. We also clarify the relationship between the clones of $\mathbf{A}$ and $\mathbf{A}^{\mathfrak{C}}$, by explicitly describing how one can obtain the clone of term operations and the clone of polynomial operations of $\mathbf{A}^{\mathfrak{C}}$ from the corresponding clone of $\mathbf{A}$ (see Theorem 3.9 and Corollary 3.12). An important consequence of this relationship between the clones of term operations of $\mathbf{A}$ and $\mathbf{A}^{\mathfrak{C}}$ is that the variety generated by $\mathbf{A}^{\mathfrak{C}}$ satisfies all idempotent Maltsev conditions that hold in the variety generated by $\mathbf{A}$ (cf. Corollary 3.14). Finally, we use the relationship between the clones of polynomial operations of $\mathbf{A}$ and $\mathbf{A}^{\mathfrak{C}}$ to prove that for finite $\mathbf{A}$, the tame congruence theoretic types of covering pairs are preserved by the isomorphism $I_{\mathbf{A}}(0, \alpha) \to \mathrm{Con}(\mathbf{A}^{\mathfrak{C}})$ mentioned above (see Theorem 3.16).

Sections 4 and 5 answer Problem 1.1 (for varieties omitting type $\mathbf{1}$) and Problem 1.2, respectively, in the affirmative. In Section 4 (Theorem 4.2), for finite algebras in a variety omitting type $\mathbf{1}$, we give a characterization for a congruence α to be supernilpotent, which immediately proves that supernilpotence is decidable. In Section 5 the theorems we prove apply to finite sets of finite algebras — not just single finite algebras. First we show that the subpower membership problem for a finite set of finite algebras with a cube term is polynomial time reducible to its subproblem where the algebras are subdirectly irreducible and have central monoliths (see Theorem 5.7(3)). From this we derive a common generalization of the main results of [7] and [17] (see Theorem 5.9), which answers Problem 1.2.

2. Three equivalent categories

Throughout this paper $\mathcal{F}$ will denote an algebraic language. So, for each symbol $f \in \mathcal{F}$ there is an associated natural number, $\mathrm{ar}(f) \ge 0$, the *arity* of f, which indicates that in every $\mathcal{F}$-algebra $\mathbf{A} = (A; \mathcal{F})$ each symbol f is interpreted as an $\mathrm{ar}(f)$-ary operation $f^{\mathbf{A}} \colon A^{\mathrm{ar}(f)} \to A$. To simplify notation, we will usually drop the superscript $\mathbf{A}$ from the operations $f^{\mathbf{A}}$. If $\mathcal{F}$ contains no nullary symbols, we will allow the universe A of an $\mathcal{F}$-algebra to be empty.

Analogously, $\mathcal{G}$ will always denote a multisorted algebraic language. There is an associated nonempty set, J, indexing the sorts, and for each symbol $g \in \mathcal{G}$ there is an associated natural number $\mathrm{ar}(g) \geq 0$, the *arity* of g, and an associated pair $(\mathbf{j}, j')$, where $\mathbf{j} = (j_1, \ldots, j_{\mathrm{ar}(g)}) \in J^{\mathrm{ar}(g)}$ is the *sequence of input sorts* of g and $j' \in J$ is the *output sort* of g. These data indicate that in every multisorted $\mathcal{G}$-algebra $\mathbf{M} = \big((M^{(j)})_{j \in J}; \mathcal{G}\big)$ each symbol g is interpreted as a multisorted operation $g^{\mathbf{M}} \colon M^{(j_1)} \times \cdots \times M^{(j_{\mathrm{ar}(g)})} \to M^{(j')}$. To simplify notation, we will usually drop the superscript $\mathbf{M}$ from the multisorted operations $g^{\mathbf{M}}$. For each $j' \in J$, if $\mathcal{G}$ contains no nullary symbol with output sort j', we will allow the universe $M^{(j')}$ of a multisorted $\mathcal{G}$-algebra to be empty.

Next we introduce several categories.

Definition 2.1.   (1) $\mathsf{Alg}(\mathcal{F})$ will denote the category of all $\mathcal{F}$-algebras; that is, the objects are all $\mathcal{F}$-algebras, and the morphisms are all homomorphisms between $\mathcal{F}$-algebras.
  (2) Given a nonempty $\mathcal{F}$-algebra $\mathbf{I} = (I; \mathcal{F})$, the category $\big(\mathsf{Alg}(\mathcal{F}) \!\downarrow\! \mathbf{I}\big)$ of all $\mathcal{F}$-algebras over $\mathbf{I}$ is defined (cf. [15, Ch. II, Sec. 6]) to be the category in which
   • the objects are all pairs $(\mathbf{A}, \chi)$ where $\mathbf{A}$ is an $\mathcal{F}$-algebra and χ is a homomorphism $\mathbf{A} \to \mathbf{I}$ of $\mathcal{F}$-algebras, while
   • a morphism between two such objects $(\mathbf{A}, \chi)$ and $(\mathbf{B}, \xi)$ is a homomorphism $\psi \colon \mathbf{A} \to \mathbf{B}$ of $\mathcal{F}$-algebras such that $\chi = \xi \circ \psi$.
  (3) $\mathsf{MSAlg}(\mathcal{G})$ will denote the category of all multisorted $\mathcal{G}$-algebras; that is, the objects are all multisorted $\mathcal{G}$-algebras, and the morphisms are all homomorphisms between multisorted $\mathcal{G}$-algebras. Recall that a homomorphism $\zeta \colon \mathbf{M} \to \mathbf{N}$ between multisorted $\mathcal{G}$-algebras $\mathbf{M} = (M^{(j)})_{j \in J}; \mathcal{G})$ and $\mathbf{N} = (N^{(j)})_{j \in J}; \mathcal{G})$ is a J-tuple $\zeta = (\zeta^{(j)})_{j \in J}$ of functions $\zeta^{(j)} \colon M^{(j)} \to N^{(j)}$ such that ζ preserves all multisorted operations $g \in \mathcal{G}$.

The next statement introduces a functor $\mathfrak{M} \colon \big(\mathsf{Alg}(\mathcal{F}) \!\downarrow\! \mathbf{I}\big) \to \mathsf{MSAlg}(\mathcal{G})$ for an appropriately defined multisorted language $\mathcal{G}$, which is analogous to a functor considered in [20].

Proposition 2.2. *Given an algebraic language $\mathcal{F}$ and a nonempty $\mathcal{F}$-algebra $\mathbf{I} = (I, \mathcal{F})$, let $\mathcal{F}_{\mathbf{I}}$ be the multisorted language $\mathcal{F}_{\mathbf{I}}$ defined as follows:*
   • *the sorts are indexed by the set I, and*
   • *the operation symbols of the multisorted language $\mathcal{F}_{\mathbf{I}}$ are all symbols*

$$f_{\mathbf{i}} \quad \text{with } f \in \mathcal{F} \text{ and } \mathbf{i} \in I^{\mathrm{ar}(f)},$$

where $\mathrm{ar}(f_{\mathbf{i}}) = \mathrm{ar}(f)$, $\mathbf{i}$ is the sequence of input sorts,

and $f(\mathbf{i})$ (computed in $\mathbf{I}$) is the output sort.

The following functions define a functor $\mathfrak{M} \colon \big(\mathsf{Alg}(\mathcal{F}) \!\downarrow\! \mathbf{I}\big) \to \mathsf{MSAlg}(\mathcal{F}_{\mathbf{I}})$:

- $\mathfrak{M}$ *assigns to every object* $(\mathbf{A}, \chi)$ *of* $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$ *the multisorted algebra*

$$\mathfrak{M}(\mathbf{A}, \chi) := \big((\chi^{-1}(i))_{i \in I}; \mathcal{F}_\mathbf{I}\big)$$

  where the sorts are the congruence classes of the kernel of χ *(indexed by* χ*), and for each* $f \in \mathcal{F}$ *and for every tuple* $\mathbf{i} = (i_1, \ldots, i_{\mathrm{ar}(f)}) \in I^{\mathrm{ar}(f)}$, *the operation* $f_\mathbf{i}$ *is the restriction of the operation* f *of* $\mathbf{A}$ *to the set* $\chi^{-1}(i_1) \times \cdots \times \chi^{-1}(i_{\mathrm{ar}(f)})$.

- $\mathfrak{M}$ *assigns to every morphism* $\psi \colon (\mathbf{A}, \chi) \to (\mathbf{B}, \xi)$ *of* $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$ *the multisorted homomorphism*

$$\mathfrak{M}(\psi) := \big(\psi^{(i)}\big)_{i \in I} \colon \mathfrak{M}(\mathbf{A}, \chi) \to \mathfrak{M}(\mathbf{B}, \xi)$$

  where for each $i \in I$, $\psi^{(i)}$ *is the restriction of* ψ *to* $\chi^{-1}(i)$, *which is a function* $\chi^{-1}(i) \to \xi^{-1}(i)$.

Definition 2.3. (1) Given a nonempty $\mathcal{F}$-algebra $\mathbf{I} = (I; \mathcal{F})$, $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$ will be our notation for the full subcategory of $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$, which consists of all objects $(\mathbf{A}, \chi)$ such that χ is onto.
  (2) $\mathsf{MSAlg}^{\square}(\mathcal{G})$ will denote the full subcategory of $\mathsf{MSAlg}(\mathcal{G})$, which consists of all multisorted $\mathcal{G}$-algebras $\mathbf{M} = \big((M^{(j)})_{j \in J}; \mathcal{G}\big)$ whose sorts $M^{(j)}$ $(j \in J)$ are pairwise disjoint.
  (3) $\mathsf{MSAlg}^{+}(\mathcal{G})$ will stand for the full subcategory of $\mathsf{MSAlg}(\mathcal{G})$, which consists of all multisorted $\mathcal{G}$-algebras $\mathbf{M} = \big((M^{(j)})_{j \in J}; \mathcal{G}\big)$ whose sorts $M^{(j)}$ $(j \in J)$ are all nonempty.
  (4) Finally, $\mathsf{MSAlg}^{\boxplus}(\mathcal{G})$ will be our notation for the full subcategory of $\mathsf{MSAlg}(\mathcal{G})$ whose objects are the multisorted $\mathcal{G}$-algebras that belong to both $\mathsf{MSAlg}^{\square}(\mathcal{G})$ and $\mathsf{MSAlg}^{+}(\mathcal{G})$.

Theorem 2.4. *For any algebraic language* $\mathcal{F}$ *and any nonempty* $\mathcal{F}$*-algebra* $\mathbf{I} = (I, \mathcal{F})$, *the category* $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$ *of* $\mathcal{F}$*-algebras over* $\mathbf{I}$ *is equivalent to the category* $\mathsf{MSAlg}(\mathcal{F}_\mathbf{I})$ *of multisorted* $\mathcal{F}_\mathbf{I}$*-algebras. The full subcategories* $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$ *and* $\mathsf{MSAlg}^{+}(\mathcal{F}_\mathbf{I})$ *of these categories are also equivalent.*

  In more detail, we have the following.

  (1) *The functor* $\mathfrak{M}$ *described in Proposition 2.2 maps onto the full subcategory* $\mathsf{MSAlg}^{\square}(\mathcal{F}_\mathbf{I})$ *of* $\mathsf{MSAlg}(\mathcal{F}_\mathbf{I})$, *and yields (by changing the target category) an isomorphism*

$$\mathfrak{M} \colon \big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big) \to \mathsf{MSAlg}^{\square}(\mathcal{F}_\mathbf{I}).$$

  (2) $\mathfrak{M}$ *restricts to an isomorphism between the full subcategory* $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$ *of* $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$ *and the full subcategory* $\mathsf{MSAlg}^{\boxplus}(\mathcal{F}_\mathbf{I})$ *of* $\mathsf{MSAlg}^{\square}(\mathcal{F}_\mathbf{I})$.

Proof. First we prove statement (1) by showing that the functor

$$\mathfrak{M}^{-1} \colon \mathsf{MSAlg}^{\square}(\mathcal{F}_\mathbf{I}) \to \big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$$

defined below is an inverse of $\mathfrak{M}$:

- $\mathfrak{M}^{-1}$ assigns to every multisorted $\mathcal{F}_{\mathbf{I}}$-algebra $\mathbf{M} = \big((M^{(i)})_{i \in I}; \mathcal{F}_{\mathbf{I}}\big)$ in $\mathsf{MSAlg}^{\square}(\mathcal{F}_{\mathbf{I}})$ the pair

$$\mathfrak{M}^{-1}(\mathbf{M}) := \left(\Big(\bigcup_{i \in I} M^{(i)}; \mathcal{F}\Big), \chi \right)$$

where for each symbol $f \in \mathcal{F}$, the operation f of the $\mathcal{F}$-algebra $\mathbf{A} = \big(\bigcup_{i \in I} M^{(i)}; \mathcal{F}\big)$ is the union of the multisorted operations $f_{\mathbf{i}}$ for all $\mathbf{i} \in I^{\mathrm{ar}(f)}$, and the homomorphism $\chi \colon \mathbf{A} \to \mathbf{I}$ sends each element $a \in \bigcup_{i \in I} M^{(i)}$ to the unique $i \in I$ with $a \in M^{(i)}$.

- $\mathfrak{M}^{-1}$ assigns to every homomorphism $\zeta = (\zeta^{(i)})_{i \in I} \colon \mathbf{M} \to \mathbf{N}$ the morphism

$$\mathfrak{M}^{-1}(\zeta) := \bigcup_{i \in I} \zeta^{(i)} \colon \mathfrak{M}^{-1}(\mathbf{M}) \to \mathfrak{M}^{-1}(\mathbf{N})$$

in $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$.

Let $\mathbf{M} = \big((M^{(i)})_{i \in I}; \mathcal{F}_{\mathbf{I}}\big)$ be a multisorted $\mathcal{F}_{\mathbf{I}}$-algebra in $\mathsf{MSAlg}^{\square}(\mathcal{F}_{\mathbf{I}})$, and let $J := \{j \in J : M^{(j)} \neq \emptyset\}$. Notice first that J is a subuniverse of $\mathbf{I}$; indeed, if $f \in \mathcal{F}$ and $\mathbf{i} = (i_1, \ldots, i_{\mathrm{ar}(f)}) \in J^{\mathrm{ar}(f)}$, then $\mathbf{M}$ has an operation $f_{\mathbf{i}}$ with nonempty domain $M^{(i_1)} \times \cdots \times M^{(\mathrm{ar} f)}$ and with codomain $M^{(f(\mathbf{i}))}$, so $M^{(f(\mathbf{i}))}$ must be nonempty, i.e., $f(\mathbf{i}) \in J$. It follows also that for every tuple $\mathbf{i} \in I^{\mathrm{ar} f} \setminus J^{\mathrm{ar} f}$, the multisorted operation $f_{\mathbf{i}}$ of $\mathbf{M}$ is the empty function. Thus, the algebra $\mathbf{A} = \big(\bigcup_{i \in I} M^{(i)}; \mathcal{F}\big)$ in $\mathfrak{M}^{-1}(\mathbf{M})$ described above is indeed an $\mathcal{F}$-algebra, and $\chi \colon \mathbf{A} \to \mathbf{I}$ is indeed an $\mathcal{F}$-algebra homomorphism. Thus, the object function of $\mathfrak{M}^{-1}$ is well-defined.

To check the analogous statement for the morphisms, let $\zeta = (\zeta^{(i)})_{i \in I} \colon \mathbf{M} \to \mathbf{N}$ be a homomorphism in $\mathsf{MSAlg}(\mathcal{F}_{\mathbf{I}})$, and let $\mathfrak{M}^{-1}(\mathbf{M}) = (\mathbf{A}, \chi)$ and $\mathfrak{M}^{-1}(\mathbf{N}) = (\mathbf{B}, \xi)$. It is clear from the definitions of $\mathbf{A}$ and $\mathbf{B}$ that $\bigcup_{i \in I} \zeta^{(i)}$ is a homomorphism $\mathbf{A} \to \mathbf{B}$. We also have $\xi \circ \bigcup_{i \in I} \zeta^{(i)} = \chi$, for the following reason: for every $a \in A$ there is a unique $i \in I$ with $a \in M^{(i)}$, so $\chi(a) = i$ and $M^{(i)} \neq \emptyset$; since $\zeta^{(i)} \colon M^{(i)} \to N^{(i)}$, we also have $N^{(i)} \neq \emptyset$, whence $(\xi \circ \bigcup_{i \in I} \zeta^{(i)})(a) = \xi(\zeta^{(i)}(a)) = i$. This shows that $\mathfrak{M}^{-1}(\zeta) = \bigcup_{i \in I} \zeta^{(i)}$ is indeed a morphism $\mathfrak{M}^{-1}(\mathbf{M}) \to \mathfrak{M}^{-1}(\mathbf{N})$ in $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$.

To complete the proof of statement (1), one can easily verify from the definitions that $\mathfrak{M}^{-1}$ is a functor such that $\mathfrak{M} \circ \mathfrak{M}^{-1}$ is the identity functor on $\mathsf{MSAlg}(\mathcal{F}_{\mathbf{I}})$ and $\mathfrak{M}^{-1} \circ \mathfrak{M}$ is the identity functor on $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$.

Statement (2) follows immediately from statement (1).

Finally, the first statement of the theorem about the equivalence of the categories $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$ and $\mathsf{MSAlg}(\mathcal{F}_{\mathbf{I}})$ follows from the isomorphism of $\big(\mathsf{Alg}(\mathcal{F}) \downarrow \mathbf{I}\big)$ and $\mathsf{MSAlg}^{\square}(\mathcal{F}_{\mathbf{I}})$ established in (1), and the fact that every multisorted algebra is isomorphic to one in which the sorts are pairwise disjoint. $\qquad\square$

Next we discuss a variant of the main result of [10].

Proposition 2.5. *Given a multisorted algebraic language $\mathcal{G}$ with no nullary symbols and finitely many sorts indexed by the set $[m] := \{1, \ldots, m\}$ ($m > 0$), let $\widehat{\mathcal{G}}$ be the algebraic language whose symbols are*

- *d with $\operatorname{ar}(d) = m$, and*
- *$\widehat{g}$ with $\operatorname{ar}(\widehat{g}) = \operatorname{ar}(g)$ for all $g \in G$.*

The following functions define a functor $\mathfrak{P} \colon \mathsf{MSAlg}(\mathcal{G}) \to \mathsf{Alg}(\widehat{\mathcal{G}})$:

- *$\mathfrak{P}$ assigns to every multisorted $\mathcal{G}$-algebra $\mathbf{M} = \big((M^{(i)})_{i\in[m]}; \mathcal{G}\big)$ the $\widehat{\mathcal{G}}$-algebra*

$$\mathfrak{P}(\mathbf{M}) := (M^{(1)} \times \cdots \times M^{(m)}; \widehat{\mathcal{G}})$$

  where
  - *d is the diagonal operation on the product set $M^{(1)} \times \cdots \times M^{(m)}$; that is, for any tuples $(a_j^{(i)})_{i\in[m]} \in \prod_{i\in[m]} M^{(i)}$ ($j \in [m]$),*

(2.1) $$d\big((a_1^{(i)})_{i\in[m]}, \ldots, (a_m^{(i)})_{i\in[m]}\big) = (a_1^{(1)}, \ldots, a_m^{(m)}),$$

    and
  - *for every $g \in \mathcal{G}$, if the input and output sorts are $\mathbf{i} = (i_1, \ldots, i_k) \in [m]^k$ and $i' \in [m]$ (so, $\operatorname{ar}(g) = k$), then the k-ary operation $\widehat{g}$ is the following: for any input tuples $(a_j^{(i)})_{i\in[m]} \in \prod_{i\in[m]} M^{(i)}$ ($j \in [k]$),*

(2.2) $$\widehat{g}\big((a_1^{(i)})_{i\in[m]}, \ldots, (a_k^{(i)})_{i\in[m]}\big)$$
$$= \big(a_1^{(1)}, \ldots, a_1^{(i'-1)}, g(a_1^{(i_1)}, \ldots, a_k^{(i_k)}), a_1^{(i'+1)}, \ldots, a_1^{(m)}\big).$$

- *$\mathfrak{P}$ assigns to every morphism $\zeta = (\zeta^{(i)})_{i\in[m]} \colon \mathbf{M} \to \mathbf{N}$ in $\mathsf{MSAlg}(\mathcal{G})$, where each $\zeta^{(i)}$ ($i \in [m]$) is a function $M^{(i)} \to N^{(i)}$, the morphism*

$$\mathfrak{P}(\zeta) := \zeta^{(1)} \times \cdots \times \zeta^{(m)} \colon \mathfrak{P}(\mathbf{M}) \to \mathfrak{P}(\mathbf{N})$$

  in $\mathsf{Alg}(\widehat{\mathcal{G}})$, which is a function $M^{(1)} \times \cdots \times M^{(m)} \to N^{(1)} \times \cdots \times N^{(m)}$.

As it is remarked in [10], for a fixed $\mathcal{G}$, the class of all isomorphic copies of algebras of the form $\mathfrak{P}(\mathbf{M})$, as described in Proposition 2.5, is a variety $\mathcal{D}(\widehat{\mathcal{G}})$ (including the empty algebra); $\mathcal{D}(\widehat{\mathcal{G}})$ is defined by the following identities:

(2.3) $$d(x, \ldots, x) = x,$$

(2.4) $$d\big(d(x_{11}, \ldots, x_{1m}), d(x_{21}, \ldots, x_{2m}), \ldots, d(x_{m1}, \ldots, x_{mm})\big) = d(x_{11}, \ldots, x_{mm}),$$

and for every symbol $g \in \mathcal{G}$ with input and output sorts $\mathbf{i} = (i_1, \ldots, i_k) \in [m]^k$ and $i' \in [m]$ (so, $\operatorname{ar}(g) = \operatorname{ar}(\widehat{g}) = k$),

(2.5) $$d_{i'}\big(\widehat{g}(x_1, \ldots, x_k), y\big) = d_{i'}\big(\widehat{g}\big(d_{i_1}(x_1, v_1), \ldots, d_{i_k}(x_k, v_k)\big), y\big),$$

(2.6) $$d_j\big(\widehat{g}(x_1, \ldots, x_k), y\big) = d_j(x_1, y) \quad \text{for all } j \in [m] \setminus \{i'\},$$

where $d_\ell(x, u)$ is an abbreviation for $d(u, \ldots, u, x, u, \ldots, u)$ with x in the ℓ-th position.

Briefly, the statement that every algebra $\mathbf{A} = (A; \widehat{\mathcal{G}})$ in $\mathcal{D}(\widehat{\mathcal{G}})$ is isomorphic to an algebra of the form $\mathfrak{P}(\mathbf{M})$ can be proved as follows. The identities (2.3)–(2.4) imply that for each $\ell \in [m]$ there exists an equivalence relation $\equiv_\ell$ on A such that for any $a, b \in A$ we have

$$a \equiv_\ell b \quad \text{iff} \quad a = d_\ell(b, a) \quad \text{iff} \quad d_\ell(a, c) = d_\ell(b, c) \text{ for all } c \in A.$$

Moreover, $\equiv_1, \ldots, \equiv_m$ yield a product decomposition $A \to A/\equiv_1 \times \cdots \times A/\equiv_m$ so that for each $\ell \in [m]$ and $a \in A$ the elements of A with the same ℓ-th coordinate as a are exactly the elements of the form $d_\ell(a, c)$ ($c \in A$). Thus, the identities (2.5)–(2.6) express that for any elements $a_1, \ldots, a_k \in A$, the i'-th coordinate of $\widehat{g}(a_1, \ldots, a_k)$ depends only on the coordinates $i_1, \ldots, i_k$ of $a_1, \ldots, a_k$, respectively, while for $j \in [m] \setminus \{i'\}$, the j-th coordinate of $\widehat{g}(a_1, \ldots, a_k)$ is the j-th coordinate of a_1.

Definition 2.6.   (1) $\mathsf{DAlg}(\widehat{\mathcal{G}})$ will denote the full subcategory of $\mathsf{Alg}(\widehat{\mathcal{G}})$ whose objects are the $\widehat{\mathcal{G}}$-algebras of the form $\mathfrak{P}(\mathbf{M})$ for some multisorted $\mathcal{G}$-algebra $\mathbf{M}$.
   (2) We will use the notation $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ for the full subcategory of $\mathsf{DAlg}(\widehat{\mathcal{G}})$ obtained by omitting the empty algebra; that is, the objects of $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ are exactly the algebras $\mathfrak{P}(\mathbf{M})$ for which the sorts of $\mathbf{M}$ are all nonempty.
   (3) $\mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{G}})$ will be our notation for the full subcategory of $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ consisting of those algebras $\mathfrak{P}(\mathbf{M})$ where the sorts of $\mathbf{M}$ are pairwise disjoint.
   (4) $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{G}})$ will denote the full subcategory of $\mathsf{Alg}(\widehat{\mathcal{G}})$ whose objects are the nonempty members of the variety $\mathcal{D}(\widehat{\mathcal{G}})$.

Clearly, $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ is a full subcategory of $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{G}})$.

Theorem 2.7. *For any multisorted algebraic language $\mathcal{G}$ with no nullary symbols and finitely many sorts, the category $\mathsf{MSAlg}^+(\mathcal{G})$ of multisorted $\mathcal{G}$-algebras is equivalent to the category $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{G}})$ of $\widehat{\mathcal{G}}$-algebras.*
   In more detail, we have the following.
   (1) *The functor $\mathfrak{P}$ described in Proposition 2.5 maps the full subcategory $\mathsf{MSAlg}^+(\mathcal{G})$ of $\mathsf{MSAlg}(\mathcal{G})$ into the full subcategory $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ of $\mathsf{Alg}(\widehat{\mathcal{G}})$, and yields (by changing the domain and target categories) an isomorphism*

$$\mathfrak{P} \colon \mathsf{MSAlg}^+(\mathcal{G}) \to \mathsf{DAlg}^+(\widehat{\mathcal{G}}).$$

   (2) *$\mathfrak{P}$ restricts to an isomorphism between the full subcategory $\mathsf{MSAlg}^{\boxplus}(\mathcal{G})$ of $\mathsf{MSAlg}^+(\mathcal{G})$ and the full subcategory $\mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{G}})$ of $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$.*

Proof. As in the proof of Theorem 2.4, the crucial statement is (1), because (2) is an immediate consequence of (1), while the first statement of the theorem on the equivalence of the categories $\mathsf{MSAlg}^+(\mathcal{G})$ and $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{G}})$ follows from (1) and the fact that every algebra in $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{G}})$ is isomorphic to one in $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$.

To verify statement (1) it suffices to check that

(i) the object function of $\mathfrak{P}$ is a bijection between the class of objects of $\mathsf{MSAlg}^+(\mathcal{G})$ and the class of objects of $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$,

(ii) the morphism function of $\mathfrak{P}$ is a bijection between the set of homomorphisms $\mathbf{M} \to \mathbf{N}$ in $\mathsf{MSAlg}^+(\mathcal{G})$ and the set of homomorphisms $\mathfrak{P}(\mathbf{M}) \to \mathfrak{P}(\mathbf{N})$ in $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ for all objects $\mathbf{M}, \mathbf{N}$ in $\mathsf{MSAlg}^+(\mathcal{G})$, and

(iii) the inverses of these object and morphism functions yield a functor $\mathsf{DAlg}^+(\widehat{\mathcal{G}}) \to \mathsf{MSAlg}^+(\mathcal{G})$.

For (i), the object function of $\mathfrak{P}$ is surjective by the definition of the category $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$, and it is injective, because for any multisorted $\mathcal{G}$-algebra $\mathbf{M}$, the $\widehat{\mathcal{G}}$-algebra $\mathfrak{P}(\mathbf{M})$ uniquely determines $\mathbf{M}$. Indeed, since the underlying set $M^{(1)} \times \cdots \times M^{(m)}$ of $\mathfrak{P}(\mathbf{M})$ is nonempty, this product set determines the sorts $M^{(1)}, \ldots, M^{(m)}$ of $\mathbf{M}$, and for every $g \in \mathcal{G}$, the operation $\widehat{g}$ of $\mathfrak{P}(\mathbf{M})$ determines the operation g of $\mathbf{M}$.

For (ii), let us fix two objects $\mathbf{M}, \mathbf{N}$ in $\mathsf{MSAlg}^+(\mathcal{G})$. It follows easily from the definition $\zeta = (\zeta^{(i)})_{i \in [m]} \mapsto \mathfrak{P}(\zeta) = \zeta^{(1)} \times \cdots \times \zeta^{(m)}$ of the morphism function of $\mathfrak{P}$ that homomorphisms $\mathbf{M} \to \mathbf{N}$ are mapped into homomorphisms $\mathfrak{P}(\mathbf{M}) \to \mathfrak{P}(\mathbf{N})$ injectively. To see that the morphism function is also surjective, notice that the diagonal operation d of the algebras in $\mathsf{DAlg}^+(\widehat{\mathcal{G}})$ forces every homomorphism $\mathfrak{P}(\mathbf{M}) \to \mathfrak{P}(\mathbf{N})$ to be of the form $\sigma^{(1)} \times \cdots \times \sigma^{(m)}$ for some functions $\sigma^{(i)} \colon M^{(i)} \to N^{(i)}$. So, since such a homomorphism also preserves the operations $\widehat{g}$ ($g \in \mathcal{G}$), it follows that $(\sigma^{(i)})_{i \in [m]}$ must be a homomorphism $\mathbf{M} \to \mathbf{N}$.

The last statement, (iii), follows immediately from the definitions of the object and morphism functions of $\mathfrak{P}$, completing the proof. $\square$

Later on in the paper, when we use the definitions of the operations of the algebra $\mathfrak{P}(\mathbf{M})$ (see Proposition 2.5) it will be convenient to think of the elements of $\prod_{i \in [m]} M^{(i)}$ as column vectors of length m, and k-tuples of elements of $\prod_{i \in [m]} M^{(i)}$ as $m \times k$ matrices whose columns are in $\prod_{i \in [m]} M^{(i)}$. Accordingly, we will introduce the following convention: for an $m \times k$ matrix $\mathbf{a} = (a_j^{(i)})_{i \in [m], j \in [k]} \in \left(\prod_{i \in [m]} M^{(i)}\right)^k$, we will denote the j-th column $(a_j^{(i)})_{i \in [m]}$ of $\mathbf{a}$ by $\mathbf{a}_j$, and the i-th row $(a_j^{(i)})_{j \in [k]}$ of $\mathbf{a}$ by $\mathbf{a}^{(i)}$. Thus, the definitions of the operations of $\mathfrak{P}(\mathbf{M})$ in (2.1) and (2.2) can be rewritten as follows: for every $m \times m$ matrix $\mathbf{a} = (a_j^{(i)})_{i \in [m], j \in [m]} \in \left(\prod_{i \in [m]} M^{(i)}\right)^m$,

$$(2.1)' \qquad d(\mathbf{a}) = \text{diagonal of } \mathbf{a},$$

and for every symbol $g \in \mathcal{G}$ with $\mathrm{ar}(g) = k$ and for every $m \times k$ matrix $\mathbf{a} = (a_j^{(i)})_{i \in [m], j \in [k]} \in \left(\prod_{i \in [m]} M^{(i)}\right)^k$,

$(2.2)'$ $\widehat{g}(\mathbf{a})$ is obtained from $\mathbf{a}_1$ by replacing the i'-th entry with $g(a_1^{(i_1)}, \ldots, a_k^{(i_k)})$.

This paper will focus on the composition of the functors $\mathfrak{M}$ and $\mathfrak{P}$. For further reference we now summarize the consequences of Theorems 2.4 and 2.7 that we will need.

Corollary 2.8. *For any algebraic language $\mathcal{F}$ with no nullary symbols and for any finite $\mathcal{F}$-algebra $\mathbf{I} = ([m]; \mathcal{F})$, the category $\left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\right)$ of $\mathcal{F}$-algebras is equivalent to the category $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{F}_\mathbf{I}})$.*

In more detail, the functor $\mathfrak{P} \circ \mathfrak{M}$ yields an isomorphism

$$\mathfrak{C} \colon \left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\right) \to \mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{F}_\mathbf{I}})$$

between $\left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\right)$ and the full subcategory $\mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{F}_\mathbf{I}})$ of $\overline{\mathsf{DAlg}}^+(\widehat{\mathcal{F}_\mathbf{I}})$ with object and morphism functions defined as follows:

- $\mathfrak{C}$ *assigns to every object $(\mathbf{A}, \chi)$ in $\left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\right)$ the algebra*

$$\mathfrak{C}(\mathbf{A}, \chi) := \left(D^{(1)} \times \cdots \times D^{(m)}; \widehat{\mathcal{F}_\mathbf{I}}\right)$$

  in $\mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{F}_\mathbf{I}})$ where
  - $\prod_{i \in [m]} D^{(i)}$ *is the product of the congruence classes $D^{(i)} := \chi^{-1}(i)$ of the kernel of χ,*
  - d *is the diagonal operation on $\prod_{i \in [m]} D^{(i)}$, that is, for every $m \times m$ matrix $\mathbf{a} = (a_j^{(i)})_{i \in [m], j \in [m]} \in \left(\prod_{i \in [m]} D^{(i)}\right)^m$,*

$$d(\mathbf{a}) = \text{diagonal of } \mathbf{a},$$

  and
  - *for each $f \in \mathcal{F}$ (say k-ary), $\mathbf{i} = (i_1, \ldots, i_k) \in [m]^k$, and for every $m \times k$ matrix $\mathbf{a} = (a_j^{(i)})_{i \in [m], j \in [k]} \in \left(\prod_{i \in [m]} D^{(i)}\right)^k$,*

$$\widehat{f_\mathbf{i}}(\mathbf{a}) \text{ is obtained from } \mathbf{a}_1 \text{ by replacing the } f(\mathbf{i})\text{-th entry}$$

$$\text{with } f(a_1^{(i_1)}, \ldots, a_k^{(i_k)}), \text{ where } f(\mathbf{i}) \text{ is evaluated in } \mathbf{I}.$$

- $\mathfrak{C}$ *assigns to every morphism $\psi = (\mathbf{A}, \chi) \to (\mathbf{B}, \xi)$ in $\left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\right)$ the homomorphism*

$$\mathfrak{C}(\psi) := \psi^{(1)} \times \cdots \times \psi^{(m)} \colon \mathfrak{C}(\mathbf{A}, \chi) \to \mathfrak{C}(\mathbf{B}, \xi)$$

  of $\widehat{\mathcal{F}_\mathbf{I}}$-algebras, where for each $i \in [m]$, $\psi^{(i)} \colon \chi^{-1}(i) \to \xi^{-1}(i)$ is the restriction of ψ to $\chi^{-1}(i)$, and $\psi^{(1)} \times \cdots \times \psi^{(m)}$ is the induced function

$$\chi^{-1}(1) \times \cdots \times \chi^{-1}(m) \to \xi^{-1}(1) \times \cdots \times \xi^{-1}(m).$$

Note that it is necessary to assume that the algebra $\mathbf{I}$ is finite if we want to obtain a finitary diagonal operation d. The assumption that the language $\mathcal{F}$ has no nullary symbols is also unavoidable, unless we are willing to lose nullary symbols during the construction $(\mathbf{A}, \chi) \mapsto \mathfrak{C}(\mathbf{A}, \chi)$. Indeed, if an algebra $\mathfrak{C}(\mathbf{A}, \chi)$ had a nullary

operation symbol, and hence a corresponding unary constant term operation, then by the description of the term operations of $\mathfrak{C}(\mathbf{A}, \chi)$ in Theorem 3.9, the algebra $\mathbf{A}$ should have constant term operations $t^{(i)}$ (and hence their nullary counterparts) with values in every kernel class $\chi^{-1}(i)$ ($i \in [m]$) of the homomorphism $\chi \colon \mathbf{A} \to \mathbf{I}$.

This restriction on the language $\mathcal{F}$ does not affect the applicability of the functor $\mathfrak{C}$. Indeed, if $\mathcal{F}$ is any language with at least one nullary symbol, then letting $\mathcal{F}'$ be the algebraic language obtained from $\mathcal{F}$ by replacing each nullary symbol $c \in F$ by a unary symbol c' we get a functor

$$' \colon \mathsf{Alg}(\mathcal{F}) \to \mathsf{Alg}(\mathcal{F}')$$

- by assigning to every $\mathcal{F}$-algebra $\mathbf{A} = (A; \mathcal{F})$ (which is necessarily nonempty) the algebra $\mathbf{A}' = (A, \mathcal{F}')$ obtained from $\mathbf{A}$ by defining c' to be the unary constant operation on A with value c for every nullary symbol $c \in \mathcal{F}$, and by keeping all other operations unchanged; and
- by assigning to every homomorphism $\varphi \colon \mathbf{A} \to \mathbf{B}$ of $\mathcal{F}$-algebras the same function $\varphi' := \varphi$, which is a homomorphism $\mathbf{A}' \to \mathbf{B}'$ of $\mathcal{F}'$-algebras.

Clearly, this is an isomorphism between $\mathsf{Alg}(\mathcal{F})$ and a full subcategory of $\mathsf{Alg}(\mathcal{F}')$. Given a finite $\mathcal{F}$-algebra $\mathbf{I} = ([m]; \mathcal{F})$, this isomorphism induces a functor

$$\mathfrak{I} \colon \big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big) \to \big(\mathsf{Alg}(\mathcal{F}') \downdownarrows \mathbf{I}'\big)$$

which is an isomorphism between $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$ and a full subcategory of $\big(\mathsf{Alg}(\mathcal{F}') \downdownarrows \mathbf{I}'\big)$. This can now be composed with the isomorphism from Corollary 2.8 with domain category $\big(\mathsf{Alg}(\mathcal{F}') \downdownarrows \mathbf{I}'\big)$.

3. Algebraic properties of the functor $\mathfrak{C}$

Throughout this section, $\mathcal{F}$ will be an algebraic language with no nullary symbols, and $\mathbf{I} = ([m]; \mathcal{F})$ will be a finite $\mathcal{F}$-algebra ($m > 0$). The functor $\mathfrak{C}$ (see Corollary 2.8) describes a construction which produces from every $\mathcal{F}$-algebra $\mathbf{A}$ with a fixed homomorphism χ onto $\mathbf{I}$ a new algebra $\mathfrak{C}(\mathbf{A}, \chi)$ in the language $\widehat{\mathcal{F}}_\mathbf{I}$. Our goal in this section is to show that this construction preserves a lot of important algebraic properties. We will prove preservation theorems for subalgebras of products and compatible relations. In particular, we will discuss the preservation of congruences, commutator properties, and tame congruence theoretic types. We will also describe how the clone of term (resp., polynomial) operations changes when we pass from $\mathbf{A}$ (paired with χ) to $\mathfrak{C}(\mathbf{A}, \chi)$. This will imply, for example, that under this construction, the satisfaction of idempotent Maltsev conditions by the generated varieties is also preserved.

3.1. **Subalgebras of products and compatible relations.** In the category $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$, the subobjects of an object $(\mathbf{A}, \chi)$ are (up to isomorphism) the objects $(\mathbf{B}, \xi)$ such that $B \subseteq A$ and the inclusion map $B \to A$ is a morphism $(\mathbf{B}, \xi) \to (\mathbf{A}, \chi)$;

equivalently, the subobjects of $(\mathbf{A}, \chi)$ are (up to isomorphism) the pairs $(\mathbf{B}, \xi)$ where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and $\xi = \chi|_{\mathbf{I}} \colon \mathbf{B} \to \mathbf{I}$ is onto (i.e., B has a nonempty intersection with $\chi^{-1}(i)$ for all $i \in [m]$).

The category $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$ has products for all nonempty families of objects. Namely, if $(\mathbf{A}_j, \chi_j)$ $(j \in J,\ J \neq \emptyset)$ is an indexed family of objects in $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$, then one representative of the product object is the fibered product $\prod_{j \in J}^{\downdownarrows \mathbf{I}} \mathbf{A}_j$ together with the induced homomorphism χ defined as follows:

$$\prod_{j \in J}^{\downdownarrows \mathbf{I}} A_j = \left\{ (a_j)_{j \in J} \in \prod_{j \in J} A_j : \chi_j(a_j) = \chi_{j'}(a_{j'}) \text{ for all } j, j' \in J \right\} = \bigcup_{i \in [m]} \left(\prod_{j \in J} D_j^{(i)} \right)$$

where $D_j^{(i)} = \chi_j^{-1}(i)$ for all $i \in [m]$, $j \in J$, and

$$\chi \colon \prod_{j \in J}^{\downdownarrows \mathbf{I}} \mathbf{A}_j \to \mathbf{I}, \quad (a_j)_{j \in J} \mapsto \chi_j(a_j),$$

where the image $\chi_j(a_j)$ is independent of the choice of $j \in J$.

Theorem 3.1. *Let $(\mathbf{A}_j, \chi_j)$ $(j \in J)$ be arbitrary objects of $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$, let $\mathbf{C}_j :=$ $\mathfrak{C}(\mathbf{A}_j, \chi_j)$ $(j \in J)$, and for any $i \in [m]$ let $D^{(i)} := \prod_{l \in J} \chi_l^{-1}(i)$.*

(1) *The mapping*

$$B \mapsto B^* := \left\{ \left(\begin{bmatrix} b_j^{(1)} \\ \vdots \\ b_j^{(m)} \end{bmatrix} \right)_{j \in J} \in \prod_{j \in J} C_j : (b_j^{(i)})_{j \in J} \in B \cap D^{(i)} \text{ for all } i \in [m] \right\}$$

$$= \prod_{i \in [m]} (B \cap D^{(i)}) \quad \text{(up to a rearrangement of coordinates)}$$

is an isomorphism between
  ⋄ *the ordered set of all subuniverses B of the algebra $\prod_{j \in J}^{\downdownarrows \mathbf{I}} \mathbf{A}_j$ with the property that $B \cap D^{(i)} \neq \emptyset$ for every $i \in [m]$, and*
  ⋄ *the ordered set of all nonempty subuniverses of the algebra $\prod_{j \in J} \mathbf{C}_j$.*

(2) *This mapping * preserves all intersections of subuniverses and all coordinate manipulations on subuniverses (i..e, permutation and duplication of coordinates, and projection of subuniverses onto subsets of coordinates). In more detail, for intersection and for projection this means that if R_ℓ $(\ell \in L)$ and R are subuniverses of $\prod_{j \in J}^{\downdownarrows \mathbf{I}} \mathbf{A}_j$ in the domain of * and $K \subseteq J$, then*

$$\left(\bigcap_{\ell \in L} R_\ell \right)^* = \bigcap_{\ell \in L} R_\ell^* \quad \text{and} \quad \left(R|_K \right)^* = R^*|_K,$$

respectively, where $=$ means that if the left hand side is defined, then so is the right hand side and the equality holds.

(3) *For arbitrary choice of elements $d^{(i)} = (d_j^{(i)})_{j \in J} \in D^{(i)}$ ($i \in [m]$), the function*

$$
\sim : \prod_{j \in J}^{\Downarrow \mathbf{I}} A_j \to \prod_{j \in J} C_j, \quad x = (x_j)_{j \in J} \mapsto \widetilde{x} := \left(\begin{bmatrix} d_j^{(1)} \\ \vdots \\ d_j^{(i-1)} \\ x_j \\ d_j^{(i+1)} \\ \vdots \\ d_j^{(m)} \end{bmatrix} \right)_{j \in J} \quad \text{if } x \in D^{(i)}
$$

has the following property: If $\mathbf{B}$ is a subalgebra of $\prod_{j \in J}^{\Downarrow \mathbf{I}} \mathbf{A}_j$ such that $d^{(i)} \in B \cap D^{(i)}$ for every $i \in [m]$, then for every set $G \subseteq B$ that generates $\mathbf{B}$, the set $\widetilde{G} := \{\widetilde{b} : b \in G\}$ generates $\mathbf{B}^*$.

The elements $d^{(i)}$ ($i \in [m]$) used in the definition of the function $\sim$ will be referred to as *padding elements*.

Proof of Theorem 3.1. Statements (1)–(2) follow from Corollary 2.8, the description of subobjects and products in the category $(\mathsf{Alg}(\mathcal{F}) \Downarrow \mathbf{I})$, and the fact that the functor $\mathfrak{C}$ sends the product object $\left(\prod_{j \in J}^{\Downarrow \mathbf{I}} \mathbf{A}_j, \chi \right)$ to the product algebra $\prod_{j \in J} \mathfrak{C}(\mathbf{A}_j, \chi_j) = \prod_{j \in J} \mathbf{C}_j$, and $\mathfrak{C}$ sends the subobjects $(\mathbf{B}, \chi|_{\mathbf{B}})$ of an object $(\mathbf{A}, \chi)$ to the subalgebras $\mathbf{B}^* := \mathfrak{C}(\mathbf{B}, \chi|_{\mathbf{B}})$ of $\mathfrak{C}(\mathbf{A}, \chi)$.

For the proof of statement (3) let $\mathbf{T}$ denote the subalgebra of $\prod_{j \in J} \mathbf{C}_j$ generated by the set $\widetilde{G}$. Since $\mathbf{T}$ is nonempty (as $\mathbf{B}$, and hence G are nonempty), statement (1) implies that $\mathbf{T} = \mathbf{S}^*$ for some subalgebra $\mathbf{S}$ of $\prod_{j \in J}^{\Downarrow \mathbf{I}} \mathbf{A}_j$ such that $S \cap D^{(i)} \neq \emptyset$ for every $i \in [m]$. Our goal is to show that $\mathbf{T} = \mathbf{B}^*$, or equivalently, $\mathbf{S}^* = \mathbf{B}^*$.

Since $G \subseteq B$, it follows from the description of the elements of $\mathbf{B}^*$ that $\widetilde{G} \subseteq B^*$. Therefore, $\mathbf{S}^* = \mathbf{T}$ is a subalgebra of $\mathbf{B}^*$. To prove that $\mathbf{B}^*$ is a subalgebra of $\mathbf{S}^*$, it suffices to establish that $\mathbf{B}$ is a subalgebra of $\mathbf{S}$. Since $\widetilde{G} \subseteq \mathbf{S}^*$, the description of the elements of $\mathbf{S}^*$ implies that $G \cup \{d^{(i)} : i \in [m]\} \subseteq S$. Hence the algebra $\mathbf{B}$ generated by G is a subalgebra of $\mathbf{S}$. $\square$

An important special case of Theorem 3.1 is when all objects $(\mathbf{A}_j, \chi_j)$ ($j \in J$) are the same, say $(\mathbf{A}, \chi)$. Then the algebra $\prod_{j \in J}^{\Downarrow \mathbf{I}} \mathbf{A}$ depends only on $\mathbf{A}$ and $\alpha = \ker(\chi)$, and will be denoted by $\mathbf{A}^J[\alpha]$. The underlying set is

$$
A^J[\alpha] := \{(a_j)_{j \in J} : a_j \, \alpha \, a_\ell \text{ for all } j, \ell \in J\}.
$$

In particular, $A^2[\alpha]$ is nothing else than α.

Recall that for an algebra $\mathbf{A}$ the subuniverses of $\mathbf{A}^n$ are often called n-ary *compatible relations* of $\mathbf{A}$. An n-ary compatible relation of $\mathbf{A}$, and the corresponding algebra,

is said to be *reflexive* if it contains all constant n-tuples $(a, \ldots, a)$ $(a \in A)$. The map * described in Theorem 3.1(1) is particularly well-behaved for reflexive compatible relations of $\mathbf{A}$ that are contained in $\mathbf{A}^n[\alpha]$, as the following corollary shows.

Corollary 3.2. *Let* $\mathbf{A}$ *be an* $\mathcal{F}$*-algebra with an onto homomorphism* $\chi \colon \mathbf{A} \to \mathbf{I}$*. Let* $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$ *and* $\alpha := \ker(\chi)$*.*

(1) *For every positive integer n, the mapping*

$$B \mapsto B^*$$

$$= \left\{ \left(\begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_1^{(m)} \end{bmatrix}, \ldots, \begin{bmatrix} a_n^{(1)} \\ \vdots \\ a_n^{(m)} \end{bmatrix} \right) \in \mathbf{C}^n : (a_1^{(i)}, \ldots, a_n^{(i)}) \in B \text{ for all } i \in [m] \right\}$$

is an isomorphism between
   ⋄ *the lattice of all n-ary reflexive compatible relation of $\mathbf{A}$ that are contained in $A^n[\alpha]$, and*
   ⋄ *the lattice of all n-ary reflexive compatible relations of $\mathbf{C}$.*

(2) *This mapping * between reflexive relations (applied over all arities $n \geq 1$) also preserves the following relational clone operations: intersection of relations of the same arity, composition, and coordinate manipulations (permutation, identification, and duplication of coordinates, and projection of relations to a subset of coordinates). Moreover, * preserves product in the respective categories; that is, if for any reflexive compatible relations $R \subseteq A^k[\alpha]$ and $S \subseteq A^n[\alpha]$ we define $R \times_\alpha S := (R \times S) \cap A^{k+n}[\alpha]$, then*

$$(R \times_\alpha S)^* = R^* \times S^*.$$

Proof. Statement (1) follows immediately from Theorem 3.1(1) and the fact that for any n-ary reflexive relation B of $\mathbf{A}$ with $B \subseteq A^n[\alpha]$ the property that "$B \cap (\chi^{-1}(i))^n \neq \emptyset$ for every $i \in [m]$" holds automatically, since $B \cap (\chi^{-1}(i))^n$ contains all constant tuples $(a, \ldots, a)$ with $a \in \chi^{-1}(i)$. Statement (2) is a straightforward consequence of statement (1). $\qquad\square$

By identifying endomorphisms ψ of $\mathbf{A}$ with their graphs $\{(a, \psi(a)) : a \in A\}$, which form subuniverses of $\mathbf{A}^2$, the map * described in Theorem 3.1(1) also yields the following correspondence.

Corollary 3.3. *Let* $\mathbf{A}$ *be an* $\mathcal{F}$*-algebra with an onto homomorphism* $\chi \colon \mathbf{A} \to \mathbf{I}$*. Let* $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$ *and* $\alpha := \ker(\chi)$*. The mapping*

$$\psi \mapsto \psi^* = \left\{ \left(\begin{bmatrix} a^{(1)} \\ \vdots \\ a^{(m)} \end{bmatrix}, \begin{bmatrix} \psi(a^{(1)}) \\ \vdots \\ \psi(a^{(m)}) \end{bmatrix} \right) \in \mathbf{C}^2 : a^{(i)} \in \chi^{-1}(i) \text{ for all } i \in [m] \right\}$$

is an isomorphism between

⋄ *the monoid of all endomorphisms (the group of all automorphisms) ψ of $\mathbf{A}$ such that the graph of ψ is contained in α, and*

⋄ *the endomorphism monoid (the automorphism group) of $\mathbf{C}$.*

3.2. **Congruences and commutator properties.** Corollary 3.2 specializes to congruences as follows.

Corollary 3.4. *Let $\mathbf{A}$ be an $\mathcal{F}$-algebra with an onto homomorphism $\chi\colon \mathbf{A} \to \mathbf{I}$, and let $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$, $\alpha := \ker(\chi)$.*

(1) *The mapping*

$$\beta \mapsto \beta^* = \left\{ \left(\begin{bmatrix} a_1^{(1)} \\ \vdots \\ a_1^{(m)} \end{bmatrix}, \begin{bmatrix} a_2^{(1)} \\ \vdots \\ a_2^{(m)} \end{bmatrix} \right) \in \mathbf{C}^2 : (a_1^{(i)}, a_2^{(i)}) \in \beta \text{ for all } i \in [m] \right\}$$

is an isomorphism between the interval $I(0, \alpha)$ of the congruence lattice of $\mathbf{A}$ and the congruence lattice of $\mathbf{C}$.

(2) *The mapping * preserves meet (= intersection) and join of congruences, and k-permutability of congruences for every $k \geq 2$; that is, for arbitrary congruences $\beta_j \in I(0, \alpha)$ $(j \in J)$ and $\beta, \gamma \in I(0, \alpha)$ of $\mathbf{A}$,*

$$\left(\bigcap_{j \in J} \beta_j \right)^* = \bigcap_{j \in J} \beta_j^*, \qquad \left(\bigvee_{j \in J} \beta_j \right)^* = \bigvee_{j \in J} \beta_j^*,$$

and

$$\beta \circ_k \gamma = \gamma \circ_k \beta \quad \Longleftrightarrow \quad \beta^* \circ_k \gamma^* = \gamma^* \circ_k \beta^*.$$

The description of the functor $\mathfrak{C}$ in Corollary 2.8, together with the description of the mapping * it induces on congruences, implies that quotient algebras are also preserved by $\mathfrak{C}$ in the following sense.

Corollary 3.5. *Let $\mathbf{A}$ be an $\mathcal{F}$-algebra with an onto homomorphism $\chi\colon \mathbf{A} \to \mathbf{I}$, and let $\alpha := \ker(\chi)$. For every congruence $\beta \in I(0, \alpha)$ of $\mathbf{A}$,*

$$\mathfrak{C}(\mathbf{A}/\beta, \chi/\beta) \cong \mathfrak{C}(\mathbf{A}, \chi)/\beta^*,$$

where $\chi/\beta\colon \mathbf{A}/\beta \to \mathbf{I}$ is the unique homomorphism that is obtained by factoring $\chi\colon \mathbf{A} \to \mathbf{I}$ through the natural homomorphism $\mathbf{A} \to \mathbf{A}/\beta$.

Recall (see, e.g., [5, 1, 18]) that for any algebra $\mathbf{A}$, the k-ary commutator is a k-ary operation $[-, \dots, -]$ on the congruence lattice of $\mathbf{A}$, and for any choice of $\beta_1, \dots, \beta_k \in \mathrm{Con}(\mathbf{A})$, the definition of the commutator $[\beta_1, \dots, \beta_k]$ uses a specific subalgebra $\mathbf{M_A}(\beta_1, \dots, \beta_k)$ of $\mathbf{A}^{2^k}$, called the algebra of $(\beta_1, \dots, \beta_k)$-*matrices*. It is useful to think of the elements of A^{2^k} as functions $f\colon \mathbf{2}^k \to A$ labeling the vertices of the k-dimensional cube $\mathbf{2}^k = \{0, 1\}^k$. For each $j \in [k]$, $\mathbf{2}^k$ has two 'faces' of

codimension 1 perpendicular to the j-th direction: $F_j(0)$ and $F_j(1)$, where $F_j(u)$ is the set of all elements of $\mathbf{2}^k$ with j-th coordinate u ($u \in \mathbf{2}$). The algebra $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ of $(\beta_1, \ldots, \beta_k)$-matrices is generated by all labelings of the vertices of the k-dimensional cube that are constant on the faces $F_j(0)$ and $F_j(1)$ for some $j \in [k]$, and have the property that the two values they assume are β_j-related. We introduce some notation to describe this generating set. For any $j \in [k]$ and $a_0 \, \beta_j \, a_1$, let $g_j[a_0, a_1]$ denote the labeling $g \colon \mathbf{2}^k \to A$ such that for each $u \in \mathbf{2}$ we have that $g|_{F_j(u)}$ is constant with value a_u. So, the standard generating set for $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ is

$$(3.1) \qquad G = \bigcup_{j \in [k]} G_j \quad \text{where} \quad G_j := \{g_j[a_0, a_1] : a_0 \, \beta_j \, a_1\} \quad \text{for all } j \in [k].$$

The commutator $[\beta_1, \ldots, \beta_k]$ is defined to be the least congruence γ of $\mathbf{A}$ with the following property:

(∗) whenever f is an element of $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ such that $f(\varepsilon 0) \, \gamma \, f(\varepsilon 1)$ for all $\varepsilon \in \mathbf{2}^{k-1} \setminus \{1 \ldots 1\}$, then we also have that $f(1 \ldots 10) \, \gamma \, f(1 \ldots 11)$.

Note that condition (∗) holds for $\gamma = \beta_k$, because every labeling f in G — and hence also every labeling f in $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ — satisfies $f(\varepsilon 0) \, \beta_k \, f(\varepsilon 1)$ for all $\varepsilon \in \mathbf{2}^{k-1}$. Therefore, $[\beta_1, \ldots, \beta_k] \leq \beta_k$.

Theorem 3.6. *Let $\mathbf{A}$ be an $\mathcal{F}$-algebra with an onto homomorphism $\chi \colon \mathbf{A} \to \mathbf{I}$, and let $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$, $\alpha := \ker(\chi)$. The isomorphism * between the interval $I(0, \alpha)$ of the congruence lattice of $\mathbf{A}$ and the congruence lattice of $\mathbf{C}$ (see Corollary 3.4(1)) preserves higher commutators of congruences; that is, for any integer $k \geq 2$ and for arbitrary congruences $\beta_1, \ldots, \beta_k \in I(0, \alpha)$ of $\mathbf{A}$,*

$$(3.2) \qquad\qquad\qquad [\beta_1, \ldots, \beta_k]^* = [\beta_1^*, \ldots, \beta_k^*].$$

Proof. We start by examining the effect of the mapping * from Corollary 3.2 to the algebra of $(\beta_1, \ldots, \beta_k)$-matrices for congruences $\beta_1, \ldots, \beta_k$ below α.

Claim 3.7. *Under the same assumptions on $\mathbf{A}$, χ, and $\beta_1, \ldots, \beta_k$ as in Theorem 3.6,*

$$(3.3) \qquad\qquad \big(\mathbf{M_A}(\beta_1, \ldots, \beta_k)\big)^* = \mathbf{M_C}(\beta_1^*, \ldots, \beta_k^*).$$

Proof of Claim 3.7. The standard generating set for the algebra $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ of $(\beta_1, \ldots, \beta_k)$-matrices in $\mathbf{A}$ is the set G indicated in (3.1). Analogously, the standard generating set for the algebra $\mathbf{M_C}(\beta_1^*, \ldots, \beta_k^*)$ of $(\beta_1^*, \ldots, \beta_k^*)$-matrices in $\mathbf{C}$ is the set

$$(3.4) \qquad H = \bigcup_{j \in [k]} H_j \quad \text{where} \quad H_j := \{h_j[c_0, c_1] : c_0 \, \beta_j^* \, c_1\} \quad \text{for all } j \in [k],$$

where for any elements $c_0, c_1 \in C$ with $c_0 \, \beta_j^* \, c_1$, $h_j[c_0, c_1]$ denotes the labeling $h \colon \mathbf{2}^k \to C$ of the vertices of the k-dimensional cube with elements of $\mathbf{C}$ in such a way that for each $u \in \mathbf{2}$, $h|_{F_j(u)}$ is constant with value c_u.

To see that the left hand side of (3.3) is defined, observe that $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ is a subalgebra of $\mathbf{A^{2^k}}[\alpha] = \prod_{\varepsilon \in 2^k}^{\sqcup \mathbf{I}} \mathbf{A}$, because the assumption $\beta_1, \ldots, \beta_k \leq \alpha$ ensures that $G \subseteq A^{2^k}[\alpha]$. Moreover, $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ is reflexive, since G contains all constant labelings $2^k \to A$. Choose and fix one constant labeling $g_1[d^{(i)}, d^{(i)}]$ ($= \cdots = g_k[d^{(i)}, d^{(i)}]$) with value $d^{(i)}$ ($\in \chi^{-1}(i)$) from each set $D^{(i)}$ ($i \in [m]$). By Theorem 3.1(3), the algebra $\left(\mathbf{M_A}(\beta_1, \ldots, \beta_k)\right)^*$ is generated by the set

(3.5) $\widetilde{G} = \bigcup_{j \in [k]} \widetilde{G}_j$ where $\widetilde{G}_j := \{h_j[\widetilde{a}_0, \widetilde{a}_1] : a_0 \, \beta_j \, a_1\}$ for all $j \in [k]$,

because the image of each labeling $g_j[a_0, a_1] \in G$ under the function $\widetilde{} : \mathbf{A^{2^k}} \to \mathbf{C^{2^k}}$ with padding elements $g_1[d^{(i)}, d^{(i)}]$ ($i \in [m]$) is the labeling $h_j[\widetilde{a}_0, \widetilde{a}_1]$, where $\widetilde{a}_0, \widetilde{a}_1$ are obtained from a_0, a_1 by applying the function $\widetilde{} : \mathbf{A} \to \mathbf{C}$ with padding elements $d^{(i)}$ ($i \in [m]$). Since $\widetilde{G} \subseteq H$, the inclusion $\subseteq$ in (3.3) follows. For the reverse inclusion notice that $\mathbf{M_C}(\beta_1^*, \ldots, \beta_k^*)$ is a reflexive subalgebra of $\mathbf{C^{2^k}}$, hence by Corollary 3.2(1), it is equal to $\mathbf{B}^*$ for some reflexive subalgebra $\mathbf{B}$ of $\mathbf{A^{2^k}}[\alpha]$. Thus, $\mathbf{B}^*$ is the least reflexive subalgebra of $\mathbf{C^{2^k}}$ which contains all generators $h_j[c_0, c_1] \in H$ of $\mathbf{M_C}(\beta_1^*, \ldots, \beta_k^*)$ from (3.4); so $j \in [k]$ and

$$c_0 = \begin{bmatrix} c_0^{(1)} \\ \vdots \\ c_0^{(m)} \end{bmatrix}, \quad c_1 = \begin{bmatrix} c_1^{(1)} \\ \vdots \\ c_1^{(m)} \end{bmatrix}$$

where $c_0^{(i)} \, \beta_j \, c_1^{(i)}$ and $c_0^{(i)}, c_1^{(i)} \in \chi^{-1}(i)$ for each $i \in [m]$.

It follows from Corollary 3.2(1) that $\mathbf{B}$ is the least reflexive subalgebra of $\mathbf{A^{2^k}}[\alpha]$ which contains the projections of these generators to all coordinates $i \in [m]$, which are the functions

$$g_j[c_0^{(i)}, c_1^{(i)}] \text{ with } c_0^{(i)} \, \beta_j \, c_1^{(i)} \text{ and } c_0^{(i)}, c_1^{(i)} \in \chi^{-1}(i).$$

Since all these functions belong to G, and $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ is reflexive, we conclude that $\mathbf{B}$ is a subalgebra of $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$. By applying * we get the inclusion $\supseteq$ in (3.3). This completes the proof of (3.3). $\diamond$

Now we are ready to prove (3.2). Since our assumption $\beta_1, \ldots, \beta_k \in I(0, \alpha)$ implies that $[\beta_1, \ldots, \beta_k] \leq \beta_k \leq \alpha$, the commutator $[\beta_1, \ldots, \beta_k]$ is the least congruence γ of $\mathbf{A}$ in the interval $I(0, \alpha)$ which satisfies condition $(*)$.

So, let γ be a congruence of $\mathbf{A}$ with $\gamma \leq \alpha$. Condition $(*)$ in the definition of $[\beta_1, \ldots, \beta_k]$ can be expressed by relational clone operations as follows:

$\mathbf{M_A}(\beta_1, \ldots, \beta_k) \cap (\gamma \times_\alpha \cdots \times_\alpha \gamma \times_\alpha A^2[\alpha]) \subseteq \gamma \times_\alpha \cdots \times_\alpha \gamma \times_\alpha \gamma$ (with 2^{k-1} factors)

provided the coordinates in $\mathbf{2}^k$ are listed so that $\varepsilon 0$ and $\varepsilon 1$ are consecutive for every $\varepsilon \in \mathbf{2}^{k-1}$, and the last two coordinates are $1 \ldots 10$ and $1 \ldots 11$. All three compatible relations involved in this condition are reflexive and are contained in $A^{\mathbf{2}^k}[\alpha]$. Therefore, by Corollary 3.2(2), condition $(*)$ holds for γ and $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ if and only if the analogous condition holds for γ^* and $\left(\mathbf{M_A}(\beta_1, \ldots, \beta_k)\right)^* = \mathbf{M_C}(\beta_1^*, \ldots, \beta_k^*)$. Consequently, $\gamma \in I(0, \alpha)$ is the least congruence of $\mathbf{A}$ satisfying $(*)$ for $\mathbf{M_A}(\beta_1, \ldots, \beta_k)$ if and only if γ^* is the least congruence of $\mathbf{C}$ satisfying the analogous condition for $\mathbf{M_C}(\beta_1^*, \ldots, \beta_k^*)$. By the definition of the k-ary commutator, and by our remark in the preceding paragraph, this proves the desired equality (3.2). □

For any algebra $\mathbf{A}$ and congruence β of $\mathbf{A}$, there is a largest congruence ρ of $\mathbf{A}$ such that $[\rho, \beta] = 0$. This congruence is called the *centralizer of* β, and is denoted by $(0 : \beta)$. Theorem 3.6, combined with this definition, immediately implies the following fact.

Corollary 3.8. *Let* $\mathbf{A}$ *be an* $\mathcal{F}$-*algebra with an onto homomorphism* $\chi \colon \mathbf{A} \to \mathbf{I}$, *and let* $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$, $\alpha := \ker(\chi)$. *The isomorphism* * *between the interval* $I(0, \alpha)$ *of the congruence lattice of* $\mathbf{A}$ *and the congruence lattice of* $\mathbf{C}$ *(see Corollary 3.4(1)) preserves centralizers; that is, for any congruence* $\beta \in I(0, \alpha)$ *of* $\mathbf{A}$ *with* $(0 : \beta) \leq \alpha$ *we have that*

$$(0 : \beta)^* = (0 : \beta^*).$$

3.3. **Term operations and polynomial operations.** In the next theorem we will use the following notation: $\mathbf{e}_{m \times k}$ is the $m \times k$ matrix in which the i-th row is $[i \ i \ \ldots \ i]$ for every $i \in [m]$.

Theorem 3.9. *The following statements hold for any positive integer* k.

(1) *If* T *is a* k-*ary term in the language* $\widehat{\mathcal{F}}_\mathbf{I}$, *then there exist* mk-*ary terms* $t^{(1)}, \ldots, t^{(m)}$ *in the language* $\mathcal{F}$ *such that*
   (i) $t^{(i)}(\mathbf{e}_{m \times k}) = i$ *in* $\mathbf{I}$ *for all* $i \in [m]$, *and*
   (ii) *whenever* $(\mathbf{A}, \chi)$ *is an object of* $\left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\right)$, *and* $D^{(i)} := \chi^{-1}(i)$ *for all* $i \in [m]$, *then in the algebra* $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$ *we have that*

$$T(\mathbf{a}) = \begin{bmatrix} t^{(1)}(\mathbf{a}) \\ \vdots \\ t^{(m)}(\mathbf{a}) \end{bmatrix} \quad \text{for every } m \times k \text{ matrix } \mathbf{a} \in \left(\prod_{i \in [m]} D^{(i)} \right)^k = C^k,$$

   where T *is applied to the* k *columns of* $\mathbf{a}$, *and* $t^{(1)}, \ldots, t^{(m)}$ *are applied to the* mk *entries of* $\mathbf{a}$.

(2) *Conversely, if* $t^{(1)}, \ldots, t^{(m)}$ *are* mk-*ary terms in the language* $\mathcal{F}$ *satisfying condition* (i), *then there exists a* k-*ary term* T *in the language* $\widehat{\mathcal{F}}_\mathbf{I}$ *such that condition* (ii) *holds.*

For every $k \geq 1$ and every k-ary $\widehat{\mathcal{F}}_{\mathbf{I}}$-term T, the mk-ary $\mathcal{F}$-terms $t^{(i)}$ $(i \in [m])$ satisfying conditions (i)–(ii) above will be referred to as *the coordinate terms of T*.

For later reference, we also note that condition (i) in statement (1) of the theorem is equivalent to the following:

(i)$'$ for every object $(\mathbf{A}, \chi)$ of $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$, and for every $m \times k$ matrix $\mathbf{a} \in \big(\prod_{i \in [m]} D^{(i)}\big)^k$ where $D^{(i)} := \chi^{-1}(i)$ for all $i \in [m]$, we have that

$$t^{(i)}(\mathbf{a}) \in D^{(i)} \quad \text{for every } i \in [m].$$

Proof of Theorem 3.9. To prove statement (1), let $T(\mathbf{x}) = T(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ be a k-ary $\widehat{\mathcal{F}}_{\mathbf{I}}$-term where $\mathbf{x}_j = \begin{bmatrix} x_j^{(1)} \\ \vdots \\ x_j^{(m)} \end{bmatrix}$ for every $j \in [k]$, and $\mathbf{x}$ is the $m \times k$ matrix with columns $\mathbf{x}_1, \ldots, \mathbf{x}_k$. If T is a variable, say $T = \mathbf{x}_j$, then the choices $t^{(1)} := x_j^{(1)}, \ldots, t^{(m)} := x_j^{(m)}$ are clearly appropriate. Now suppose that $T = f_{\mathbf{i}}(T_1, \ldots, T_\ell)$ where $f \in \mathcal{F}$ and $\mathbf{i} = (i_1, \ldots, i_\ell) \in [m]^\ell$, and conditions (i)–(ii) hold for each $T_j = T_j(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ $(j \in [\ell])$ with mk-ary $\mathcal{F}$-terms $t_j^{(i)} = t_j^{(i)}(\mathbf{x})$. Then, by the definition of the basic operation $f_{\mathbf{i}}$ in any algebra $\mathfrak{C}(\mathbf{A}, \chi)$, we get that (i)–(ii) also hold for T with the following mk-ary terms $t^{(i)}$:

$$t^{(i)} = \begin{cases} f(t_1^{(i_1)}, \ldots, t_\ell^{(i_\ell)}) & \text{if } i = f(\mathbf{i}), \\ t_1^{(i)} & \text{otherwise.} \end{cases}$$

Similarly, if $T = d(T_1, \ldots, T_m)$ where (i)–(ii) hold for each $T_j = T_j(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ $(j \in [\ell])$ with mk-ary $\mathcal{F}$-terms $t_j^{(i)} = t_j^{(i)}(\mathbf{x})$, then by the definition of the basic operation d in any algebra $\mathfrak{C}(\mathbf{A}, \chi)$, we get that (i)–(ii) also hold for T with the mk-ary terms $t^{(i)} := t_i^{(i)}$. This proves statement (1) by induction on the complexity of T.

For the converse statement (2), we will need that every term s in the language $\mathcal{F}$ occurs as an appropriate coordinate term of a term T_s in the language $\widehat{\mathcal{F}}_{\mathbf{I}}$. More precisely, we need the following fact.

Claim 3.10. *For every integer $\ell \geq 1$, for every ℓ-tuple $\mathbf{i} = (i_1, \ldots, i_\ell) \in [m]^\ell$, and for every ℓ-ary term s in the language $\mathcal{F}$ there exists an ℓ-ary term $T_s = T_s(\mathbf{x}_1, \ldots, \mathbf{x}_\ell)$ in the language $\widehat{\mathcal{F}}_{\mathbf{I}}$ such that*

$$s(x_1^{(i_1)}, \ldots, x_\ell^{(i_\ell)}) \text{ is the } s(\mathbf{i})\text{-th coordinate term of } T_s.$$

Proof of Claim 3.10. We proceed by induction on the complexity of s. If s is a variable, say x_j $(j \in [\ell])$, then $s(x_1^{(i_1)}, \ldots, x_\ell^{(i_\ell)}) = x_j^{(i_j)}$ is the $s(\mathbf{i}) = i_j$-th coordinate term of the ℓ-ary $\widehat{\mathcal{F}}_{\mathbf{I}}$-term $T_s := \mathbf{x}_j$, as required. Now let us assume that $s = f(s_1, \ldots, s_p)$ for some $f \in \mathcal{F}$ with $\mathrm{ar}(f) = p$ and for some ℓ-ary $\mathcal{F}$-terms $s_1, \ldots, s_p$

such that for each s_q $(q \in [p])$, $s_q(x_1^{(i_1)}, \ldots, x_\ell^{(i_\ell)})$ is the $s_q(\mathbf{i})$-th coordinate term of an ℓ-ary $\widehat{\mathcal{F}}_\mathbf{I}$-term $T_{s_q} = T_{s_q}(\mathbf{x}_1, \ldots, \mathbf{x}_\ell)$. Then

$$s(x_1^{(i_1)}, \ldots, x_\ell^{(i_\ell)}) = f\big(s_1(x_1^{(i_1)}, \ldots, x_\ell^{(i_\ell)}), \ldots, s_p(x_1^{(i_1)}, \ldots, x_\ell^{(i_\ell)})\big)$$

is the $s(\mathbf{i}) = f\big(s_1(\mathbf{i}), \ldots, s_p(\mathbf{i})\big)$-th coordinate term of the ℓ-ary $\widehat{\mathcal{F}}_\mathbf{I}$-term

$$T_s(\mathbf{x}_1, \ldots, \mathbf{x}_\ell) := f_{(s_1(\mathbf{i}), \ldots, s_p(\mathbf{i}))}\big(T_{s_1}(\mathbf{x}_1, \ldots, \mathbf{x}_\ell), \ldots, T_{s_p}(\mathbf{x}_1, \ldots, \mathbf{x}_\ell)\big).$$

This completes the proof. $\diamond$

Now to prove statement (2), assume that $t^{(1)}, \ldots, t^{(m)}$ are mk-ary $\mathcal{F}$-terms such that $t^{(i)}(\mathbf{e}_{m \times k}) = i$ for all $i \in [m]$. By Claim 3.10, there exist mk-ary $\widehat{\mathcal{F}}_\mathbf{I}$-terms

$$T_{t^{(i)}} = T_{t^{(i)}}(\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_1^{(m)}, \ldots, \mathbf{x}_k^{(1)}, \ldots, \mathbf{x}_k^{(m)}) \quad (i \in [m]),$$

where the notation for the variables indicates how they correspond to the entries of an $m \times k$ matrix, such that the i-th coordinate term of $T_{t^{(i)}}$ is the mk-ary $\mathcal{F}$-term

$$t^{(i)}\big((\mathbf{x}_1^{(1)})^{(1)}, \ldots, (\mathbf{x}_1^{(m)})^{(m)}, \ldots, (\mathbf{x}_k^{(1)})^{(1)}, \ldots, (\mathbf{x}_k^{(m)})^{(m)}\big),$$

where $(\mathbf{x}_j^{(i)})^{(i)}$ denotes the i-th entry of $\mathbf{x}_j^{(i)}$. Thus, if for each $j \in [k]$ we replace all variables $\mathbf{x}_j^{(1)}, \ldots, \mathbf{x}_j^{(m)}$ in $T_{t^{(i)}}$ with the same variable $\mathbf{x}_j$, we get a k-ary term $\overline{T}_i = \overline{T}_i(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ in the language $\widehat{\mathcal{F}}_\mathbf{I}$ such that the i-th coordinate term of $\overline{T}_i$ is

$$t^{(i)}\big((\mathbf{x}_1)^{(1)}, \ldots, (\mathbf{x}_1)^{(m)}, \ldots, (\mathbf{x}_k)^{(1)}, \ldots, (\mathbf{x}_k)^{(m)}\big)$$
$$= t^{(i)}(x_1^{(1)}, \ldots, x_1^{(m)}, \ldots, x_k^{(1)}, \ldots, x_k^{(m)}) = t^{(i)}.$$

Now let $T := d(\overline{T}_1, \ldots, \overline{T}_m)$. For any algebra $\mathfrak{C}(\mathbf{A}, \chi)$ as in condition (ii) of part (1), the property of $\overline{T}_1, \ldots, \overline{T}_m$ just established, and the definition of the operation d implies that condition (ii) in part (1) holds for T.

This completes the proof of Theorem 3.9. $\square$

Corollary 3.11. *Let $(\mathbf{A}, \chi)$ be an arbitrary object of $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$. If the clone of the algebra $\mathbf{A}$ is finitely generated, then so is the clone of the algebra $\mathfrak{C}(\mathbf{A}, \chi)$.*

Proof. If the clone of $\mathbf{A} = (A; \mathcal{F})$ is finitely generated, then $\mathbf{A}$ is term equivalent to a reduct $\mathbf{A}^\circ = (A; \mathcal{F}^\circ)$ of $\mathbf{A}$ to a finite sublanguage $\mathcal{F}^\circ$ of $\mathcal{F}$. Clearly, χ is a homomorphism of $\mathbf{A}^\circ$ onto the reduct $\mathbf{I}^\circ = ([m]; \mathcal{F}^\circ)$ of $\mathbf{I}$, and $(\mathbf{A}^\circ, \chi)$ is an object of the category $\big(\mathsf{Alg}(\mathcal{F}^\circ) \downdownarrows \mathbf{I}^\circ\big)$. Theorem 3.9 implies that the algebras $\mathfrak{C}(\mathbf{A}, \chi)$ and $\mathfrak{C}(\mathbf{A}^\circ, \chi)$ have the same clone. Since the latter algebra has a finite language, $\widehat{\mathcal{F}^\circ}_{\mathbf{I}^\circ}$, we conclude that the clone of $\mathfrak{C}(\mathbf{A}, \chi)$ is finitely generated. $\square$

Corollary 3.12. *Let $(\mathbf{A}, \chi)$ be an object of $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big)$, let $D^{(i)} := \chi^{-1}(i)$ for each $i \in [m]$, and let $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$. A function $P \colon C^k \to C$ (where $C = \prod_{i \in [m]} D^{(i)}$) is a*

k-ary polynomial operation of $\mathbf{C}$ *if and only if* $\mathbf{A}$ *has mk-ary polynomial operations* $p^{(1)}, \ldots, p^{(m)}$ *such that for every* $m \times k$ *matrix* $\mathbf{a} \in \left(\prod_{i \in [m]} D^{(i)} \right)^k$,

$$(3.6) \qquad p^{(i)}(\mathbf{a}) \in D^{(i)} \;\; \textit{for every } i \in [m], \quad \textit{and} \quad P(\mathbf{a}) = \begin{bmatrix} p^{(1)}(\mathbf{a}) \\ \vdots \\ p^{(m)}(\mathbf{a}) \end{bmatrix}.$$

Proof. We want to deduce this statement from Theorem 3.9 by using the fact that for any algebra $\mathbf{U}$ the polynomial operations of $\mathbf{U}$ are the term operations of the constant expansion of $\mathbf{U}$. In keeping with our convention of excluding nullary symbols, we define the *constant expansion* of an algebra $\mathbf{U} = (U; \mathcal{L})$ to be the algebra $(U; \mathcal{L} \cup \{c_u : u \in U\})$ where each c_u is a unary symbol, and is interpreted as the unary constant operation on U with value u.

Now let $(\mathbf{A}, \chi)$ satisfy the assumptions of Corollary 3.12, and let $\mathbf{A}^c$ be the constant expansion of $\mathbf{A}$. The language of $\mathbf{A}^c$ is $\mathcal{F}^c = \mathcal{F} \cup \{c_a : a \in A\}$. Since χ is an onto homomorphism $\mathbf{A} \to \mathbf{I}$, we can expand $\mathbf{I}$ to get an $\mathcal{F}^c$-algebra $\mathbf{I}^c = ([m], \mathcal{F}^c)$ by interpreting each symbol c_a $(a \in A)$ as the unary constant function with value $\chi(a)$ on $[m]$. Thus, χ is an onto homomorphism $\mathbf{A}^c \to \mathbf{I}^c$, and $(\mathbf{A}^c, \chi)$ is an object of the category $\left(\mathsf{Alg}(\mathcal{F}^c) \downdownarrows \mathbf{I}^c \right)$. By applying the functors $\mathfrak{C}$ with domain categories $\left(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I} \right)$ and $\left(\mathsf{Alg}(\mathcal{F}^c) \downdownarrows \mathbf{I}^c \right)$ to the objects $(\mathbf{A}, \chi)$ and $(\mathbf{A}^c, \chi)$, respectively, we get the $\widehat{\mathcal{F}}_{\mathbf{I}}$-algebra $\mathfrak{C}(\mathbf{A}, \chi)$ and the $\widehat{\mathcal{F}^c}_{\mathbf{I}^c}$-algebra $\mathfrak{C}(\mathbf{A}^c, \chi)$.

Claim 3.13. *The clone of term operations of* $\mathfrak{C}(\mathbf{A}^c, \chi)$ *coincides with the clone of term operations of the constant expansion of* $\mathfrak{C}(\mathbf{A}, \chi)$.

Proof of Claim 3.13. Let $D^{(i)} := \chi^{-1}(i)$ for all $i \in [m]$, and let $C := \prod_{i \in [m]} D^{(i)}$, which is the universe of all three algebras that play a role in the claim: $\mathfrak{C}(\mathbf{A}, \chi)$, $\mathfrak{C}(\mathbf{A}^c, \chi)$, and the constant expansion of $\mathfrak{C}(\mathbf{A}, \chi)$. It also follows from the definition of the functor $\mathfrak{C}$ that $\mathfrak{C}(\mathbf{A}^c, \chi)$ is obtained from $\mathfrak{C}(\mathbf{A}, \chi)$ by adding the unary operations $\widehat{(c_a)}_i$ $(a \in A, i \in [m])$, while the constant expansion of $\mathfrak{C}(\mathbf{A}, \chi)$ is obtained from $\mathfrak{C}(\mathbf{A}, \chi)$ by adding the unary constant operations $c_{\mathbf{c}}$ $(\mathbf{c} \in C)$. Therefore, to prove the claim it suffices to show that

(1) every operation $\widehat{(c_a)}_i$ $(a \in A, i \in [m])$ is a term operation of the constant expansion of $\mathfrak{C}(\mathbf{A}, \chi)$, and

(2) every unary constant operation $c_{\mathbf{c}}$ $(\mathbf{c} \in C)$ is a term operation of $\mathfrak{C}(\mathbf{A}^c, \chi)$.

(1) follows by observing that if $a \in D^{(i')}$ and $\mathbf{c}$ is an element of $C = \prod_{\ell \in [m]} D^{(\ell)}$ with i'-th coordinate a, then for each $i \in [m]$ we have that

$$\widehat{(c_a)}_i(\mathbf{x}) = d\big(\mathbf{x}, \ldots, \mathbf{x}, \underbrace{c_{\mathbf{c}}(\mathbf{x})}_{i'\text{-th}} \mathbf{x}, \ldots, \mathbf{x}\big).$$

For (2), notice that if $\mathbf{c} = (c_1, \dots, c_m) \in \prod_{i \in [m]} D^{(i)} = C$, then the unary operation $\mathsf{c_c}$ is the composition (in any order) of the unary operations $\widehat{(\mathsf{c}_{c_1})}_1, \dots, \widehat{(\mathsf{c}_{c_m})}_m$.   ◇

To summarize, we have that the clone of polynomial operations of $\mathfrak{C}(\mathbf{A}, \chi)$ is the clone of term operations of $\mathfrak{C}(\mathbf{A}^{\mathsf{c}}, \chi)$, while the clone of polynomial operations of $\mathbf{A}$ is the clone of term operations of $\mathbf{A}^{\mathsf{c}}$. Therefore, if we apply Theorem 3.9 to the category $\big(\mathsf{Alg}(\mathcal{F}^{\mathsf{c}}) \downdownarrows \mathbf{I}^{\mathsf{c}}\big)$ and its object $(\mathbf{A}^{\mathsf{c}}, \chi)$, and replace condition (i) with the equivalent condition (i)$'$, we get the conclusion of Corollary 3.12, which is what we wanted to prove.   □

3.4. **Varieties and Maltsev conditions.** In this subsection let $\mathcal{V}$ be any variety of (nonempty) $\mathcal{F}$-algebras, and let $\mathbf{I} = ([m], \mathcal{F})$ ($m > 0$) be a finite algebra in $\mathcal{V}$. We will use the notation $\big(\mathcal{V} \downdownarrows \mathbf{I}\big)$ for the full subcategory of $\mathsf{Alg}(\mathcal{F})$ consisting of those objects $(\mathbf{A}, \chi)$ for which $\mathbf{A} \in \mathcal{V}$.

The discussion following Proposition 2.5 and Corollary 2.8 imply that if $\mathcal{V}$ is the variety of all (nonempty) $\mathcal{F}$-algebras, then the class of all isomorphic copies of $\widehat{\mathcal{F}}_{\mathbf{I}}$-algebras in the range $\mathfrak{C}\big(\mathcal{V} \downdownarrows \mathbf{I}\big) = \mathfrak{C}\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big) = \mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{F}}_{\mathbf{I}})$ of the functor $\mathfrak{C} \colon \big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}\big) \to \mathsf{DAlg}^{\boxplus}(\widehat{\mathcal{F}}_{\mathbf{I}})$ is a variety of $\widehat{\mathcal{F}}_{\mathbf{I}}$-algebras. We can combine earlier results of this subsection to obtain an analogous conclusion for all subvarieties $\mathcal{V}$ of the variety of all $\mathcal{F}$-algebras.

Corollary 3.14. *If $\mathcal{V}$ is a variety of $\mathcal{F}$-algebras and $\mathbf{I} = ([m], \mathcal{F})$ ($m > 0$) is a finite algebra in $\mathcal{V}$, then the class of all isomorphic copies of algebras in $\mathfrak{C}\big(\mathcal{V} \downdownarrows \mathbf{I}\big)$ is a variety, $\mathcal{V}^*$, of $\widehat{\mathcal{F}}_{\mathbf{I}}$-algebras.*

Proof. The fact that $\mathcal{V}^*$ is closed under taking products and subalgebras follows from the discussion preceding Theorem 3.1 and from the first paragraph of the proof of that theorem. To show that $\mathcal{V}^*$ is closed under taking quotients, consider an algebra $\mathbf{C}$ in $\mathcal{V}^*$ and a congruence γ of $\mathbf{C}$. Since $\mathbf{C}$ is isomorphic to an algebra of the form $\mathfrak{C}(\mathbf{A}, \chi)$ for some $\mathbf{A} \in \mathcal{V}$ and some onto homomorphism $\chi \colon \mathbf{A} \to \mathbf{I}$, we may assume without loss of generality that $\mathbf{C}$ is actually equal to $\mathfrak{C}(\mathbf{A}, \chi)$. By Corollary 3.4, $\gamma = \beta^*$ for a unique congruence β of $\mathbf{A}$ such that $\beta \subseteq \ker(\chi)$. Now Corollary 3.5 shows that $\mathbf{C}/\beta^* \cong \mathfrak{C}(\mathbf{A}/\beta, \chi/\beta) \in \mathcal{V}^*$, completing the proof.   □

In the next corollary we will apply Theorem 3.9, which describes the term operations of the algebras $\mathfrak{C}(\mathbf{A}, \chi)$ in $\mathsf{DAlg}(\widehat{\mathcal{F}}_{\mathbf{I}})$ in terms of the term operations of $\mathbf{A}$. By the preceding corollary, this description specializes to any variety $\mathcal{V}$ of $\mathcal{F}$-algebras and any finite member $\mathbf{I}$ of $\mathcal{V}$ to yield a description for the term operations of algebras in $\mathcal{V}^*$ in terms of the corresponding algebras in $\mathcal{V}$. In particular, by applying this description to the countably generated free algebras, we get a description of the clone of the variety $\mathcal{V}^*$ in terms of the clone of the variety $\mathcal{V}$.

Corollary 3.15. *Let $\mathcal{V}$ be a variety of $\mathcal{F}$-algebras, let $\mathbf{I} = ([m]; \mathcal{F})$ ($m > 0$) be a finite algebra in $\mathcal{V}$, and let $\mathcal{V}^*$ be the variety of all $\widehat{\mathcal{F}}_{\mathbf{I}}$-algebras that are isomorphic to algebras in $\mathfrak{C}(\mathcal{V} \downdownarrows \mathbf{I})$.*

(1) *The map*

$$t(x_1, \ldots, x_k) \mapsto \breve{t}(\mathbf{x}_1, \ldots, \mathbf{x}_k) := \begin{bmatrix} t(x_1^{(1)}, \ldots, x_k^{(1)}) \\ \vdots \\ t(x_1^{(m)}, \ldots, x_k^{(m)}) \end{bmatrix},$$

which assigns to every idempotent term $t(x_1, \ldots, x_k)$ of $\mathcal{V}$ the idempotent term $\breve{t}(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of $\mathcal{V}^*$ with coordinate terms $t(x_1^{(i)}, \ldots, x_k^{(i)})$ ($i \in [m]$), is a clone homomorphism of the clone of all idempotent terms of $\mathcal{V}$ into the clone of all idempotent terms of $\mathcal{V}^*$.

(2) *Hence, $\mathcal{V}^*$ satisfies every idempotent Maltsev condition that holds in $\mathcal{V}$.*

Proof. Since t is an idempotent term, i.e., the identity $t(x, \ldots, x) \approx x$ holds in $\mathcal{V}$, the coordinate terms $t(x_1^{(i)}, \ldots, x_k^{(i)})$ satisfy condition (i) in Theorem 3.9(1). Therefore, by Theorem 3.9(2), there exists an $\widehat{\mathcal{F}}_{\mathbf{I}}$-term $\breve{t}$ of the variety $\mathcal{V}^*$ with these terms as coordinate terms. Clearly, $\breve{t}$ is an idempotent term of $\mathcal{V}^*$. It is also straightforward to check that the map $t \mapsto \breve{t}$ is a clone homomorphism. This proves statement (1).

For (2), recall that a strong Maltsev condition is a primitive positive sentence in the language of clones, while a Maltsev condition is a disjunction of a weakening infinite sequence of strong Maltsev conditions. Since the satisfaction of conditions of this form is preserved under homomorphisms, we get from part (1) that $\mathcal{V}^*$ satisfies every idempotent Maltsev condition that holds in $\mathcal{V}$. $\square$

3.5. **TCT types.** For the basic concepts and notation of tame congruence theory we refer the reader to [12]. If for a covering pair $\delta \prec \theta$ of congruences in some finite algebra $\mathbf{A}$ we have that $\mathrm{typ}_{\mathbf{A}}(\delta, \theta) = \mathbf{2}$, and hence the minimal algebras $\mathbf{A}|_N / \delta|_N$ associated to all $\langle \delta, \theta \rangle$-traces N are weakly isomorphic one-dimensional vector spaces over a finite field, we will refer to the characteristic of this field as *the characteristic of $\langle \delta, \theta \rangle$*. In particular, if θ is an atom in the congruence lattice of $\mathbf{A}$, then the characteristic of the prime quotient $\langle 0, \theta \rangle$ will also be called *the characteristic of θ*.

Theorem 3.16. *Let $\mathcal{F}$ be an algebraic language, and let $\mathbf{I} = ([m]; \mathcal{F})$ and $\mathbf{A}$ be finite $\mathcal{F}$-algebras. If $(\mathbf{A}, \chi)$ is an object of $(\mathrm{Alg}(\mathcal{F}) \downdownarrows \mathbf{I})$ with $\alpha := \ker(\chi)$, then for any covering pair $\delta \prec \theta$ of congruences of $\mathbf{A}$ in the interval $I(0, \alpha)$ and for every $\langle \delta, \theta \rangle$-trace N of $\mathbf{A}$ there exist a $\langle \delta^*, \theta^* \rangle$-trace $\check{N}$ of $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$ and a weak isomorphism $\mathbf{A}|_N \to \mathbf{C}|_{\check{N}}$ between the induced algebras which maps $\delta|_N$ to $\delta^*|_{\check{N}}$. Consequently,*

$$\mathrm{typ}_{\mathbf{A}}(\delta, \theta) = \mathrm{typ}_{\mathbf{C}}(\delta^*, \theta^*);$$

moreover, if $\mathrm{typ}_{\mathbf{A}}(\delta, \theta) = \mathrm{typ}_{\mathbf{C}}(\delta^*, \theta^*) = \mathbf{2}$, then the prime quotients $\langle \delta, \theta \rangle$ and $\langle \delta^*, \theta^* \rangle$ have the same prime characteristic.

Proof. As before, we will use the notation $D^{(i)} := \chi^{-1}(i)$ $(i \in [m])$ for the equivalence classes of α. So, $C := \prod_{i \in [m]} D^{(i)}$ is the underlying set of the algebra $\mathbf{C} = \mathfrak{C}(\mathbf{A}, \chi)$.

Let $\delta \prec \theta (\leq \alpha)$ be congruences in $\mathbf{A}$, and let N be a $\langle \delta, \theta \rangle$-trace of $\mathbf{A}$. Then there exist a unary polynomial operation $e = e^2$ of $\mathbf{A}$ such that $U := e(A)$ is a $\langle \delta, \theta \rangle$-minimal set of $\mathbf{A}$, and $N = U \cap (a/\theta)$ for some $a \in U$. Furthermore, $\theta|_N$ is the full relation on N and $\delta|_N \subsetneq \theta|_N$.

Since $\theta \leq \alpha$, N lies in a single α-class; without loss of generality we may assume that $N \subseteq D^{(1)}$. Now we choose and fix elements $c^{(2)} \in D^{(2)}, \ldots, c^{(m)} \in D^{(m)}$ from the remaining α-classes. For $2 \leq i \leq m$ let $q^{(i)}$ denote the unary constant polynomial operation of $\mathbf{A}$ with value $c^{(i)}$, while for $i = 1$ let $q^{(1)} := e$. Notice that $e(D^{(1)}) \subseteq D^{(1)}$, because $D^{(1)} = a/\alpha$ and $e(a) = a$ (as $a \in N \subseteq U = e(A)$). Therefore, it follows from Corollary 3.12 that the function $\check{e} \colon C \to C$ defined by

$$\check{e}(\mathbf{a}) := \begin{bmatrix} q^{(1)}(a^{(1)}) \\ q^{(2)}(a^{(2)}) \\ \vdots \\ q^{(m)}(a^{(m)}) \end{bmatrix} = \begin{bmatrix} e(a^{(1)}) \\ c^{(2)} \\ \vdots \\ c^{(m)} \end{bmatrix} \quad \text{for all} \quad \mathbf{a} = \begin{bmatrix} a^{(1)} \\ a^{(2)} \\ \vdots \\ a^{(m)} \end{bmatrix} \in C$$

is a polynomial operation of $\mathbf{C}$. Clearly, $\check{e} = \check{e}^2$, and for the set $\check{U} := \check{e}(C)$ we have that

$$\check{U} = e(D^{(1)}) \times \{c^{(2)}\} \times \cdots \times \{c^{(m)}\} = (U \cap D^{(1)}) \times \{c^{(2)}\} \times \cdots \times \{c^{(m)}\}.$$

For each $u \in U \cap D^{(1)}$ let $\check{u} := (u, c^{(2)}, \ldots, c^{(m)})$; thus, we have a bijection $U \cap D^{(1)} \to \check{U}$, $u \mapsto \check{u}$. Restricting this mapping to the set $N (\subseteq U \cap D^{(1)})$ we obtain a bijection

$$N \to \check{N} := N \times \{c^{(2)}\} \times \cdots \times \{c^{(m)}\}, \quad u \mapsto \check{u}.$$

It follows from Corollary 3.4(1) that $\delta^*|_{\check{N}} \subsetneq \theta^*|_{\check{N}}$ and $\check{N} = \check{U} \cap (\check{a}/\theta^*)$ (so $\theta^*|_{\check{N}}$ is the full relation on $\check{N}$). To establish that $\check{N}$ is a $\langle \delta^*, \theta^* \rangle$-trace of $\mathbf{C}$ it remains to show that $\check{U}$ is a $\langle \delta^*, \theta^* \rangle$-minimal set of $\mathbf{C}$.

Assume not, and let $\check{V} := V \times \{c^{(2)}\} \times \cdots \times \{c^{(m)}\}$ be a proper subset of $\check{U}$ which is $\langle \delta^*, \theta^* \rangle$-minimal in $\mathbf{C}$. Then there exists a unary polynomial g of $\mathbf{C}$ such that

(3.7) $$\check{V} = g(C), \quad g = g^2, \quad \text{and} \quad \delta^*|_{\check{V}} \subsetneq \theta^*|_{\check{V}}.$$

Hence, by Corollary 3.4(1), $\delta|_V \subsetneq \theta|_V$ where $V \subsetneq U \cap D^{(1)}$. Since both g and $\check{e}$ fix every element of $V (\subseteq \check{U})$, so will the unary polynomial $g \circ \check{e}$. Therefore, by replacing g with $g \circ \check{e}$, we may assume from now on that, in addition to (3.7), g also satisfies $g = g \circ \check{e}$. Let $g^{(1)}$ denote the m-ary polynomial operation of $\mathbf{A}$ that is the first coordinate function of g according to Corollary 3.12, and define a unary polynomial

operation f of $\mathbf{A}$ by $f(x) := g^{(1)}\big(e(x), c^{(2)}, \ldots, c^{(m)}\big)$. Then,

$$g(\mathbf{a}) = (g \circ \check{e})(\mathbf{a}) = g\left(\begin{bmatrix} e(a^{(1)}) \\ c^{(2)} \\ \vdots \\ c^{(m)} \end{bmatrix}\right) = \begin{bmatrix} g^{(1)}\big(e(a^{(1)}), c^{(2)}, \ldots, c^{(m)}\big) \\ c^{(2)} \\ \vdots \\ c^{(m)} \end{bmatrix} = \begin{bmatrix} f(a^{(1)}) \\ c^{(2)} \\ \vdots \\ c^{(m)} \end{bmatrix}$$

$$\text{for all} \quad \mathbf{a} = \begin{bmatrix} a^{(1)} \\ a^{(2)} \\ \vdots \\ a^{(m)} \end{bmatrix} \in C.$$

The first property of g in (3.7) implies that $f(U \cap D^{(1)}) = V$, while the second one implies that $f|_{U \cap D^{(1)}} = (f|_{U \cap D^{(1)}})^2$. Hence, f fixes all elements of V. Since by the definition of f we have $f = f \circ e$, it follows that $\ker(e) \subseteq \ker(f) \subseteq \ker(e \circ f)$. The elements of $U \cap D^{(1)}$ are in distinct kernel classes of e, because e fixes all elements of U. However, some elements of $U \cap D^{(1)}$ are in the same kernel class of $e \circ f$, because $(e \circ f)(U \cap D^{(1)}) = e(V) = V \subsetneq U \cap D^{(1)}$. Hence, $\ker(e) \subsetneq \ker(e \circ f)$, and therefore $U = e(A) \supsetneq (e \circ f)(A)$. Combining the facts that $\delta|_V \subsetneq \theta|_V$ and $e \circ f$ fixes every element of V we also get that $\theta|_V \subseteq (e \circ f)(\theta)$, and hence $(e \circ f)(\theta) \not\subseteq \delta$. The existence of a unary polynomial $e \circ f$ of $\mathbf{A}$ with these properties contradicts our initial assumption that U is a $\langle \delta, \theta \rangle$-minimal set of $\mathbf{A}$. This contradiction shows that $\check{U}$ is a $\langle \delta^*, \theta^* \rangle$-minimal set of $\mathbf{C}$, and $\check{N}$ is $\langle \delta^*, \theta^* \rangle$-trace of $\mathbf{C}$.

Now we will prove that the induced algebras $\mathbf{A}|_N$ and $\mathbf{C}|_{\check{N}}$ are weakly isomorphic. Let P be a k-ary polynomial operation of $\mathbf{C}$ such that $P(\check{N}^k) \subseteq \check{N}$. If we write P in the form described in Corollary 3.12, where the coordinate polynomials of P are the mk-ary polynomial operations $p^{(1)}, \ldots, p^{(m)}$ of $\mathbf{A}$, then we get that for $2 \leq i \leq m$, $p^{(i)}$ must be constant with value $c^{(i)}$ for all allowable inputs. Thus, we may assume without loss of generality that $p^{(2)}, \ldots, p^{(m)}$ are these constant polynomials. For $i = 1$, the assumption $P(\check{N}^k) \subseteq \check{N}$ forces that the first coordinate function of P assigns to any tuple $(\check{u}_1, \ldots, \check{u}_k) \in \check{N}^k$ the element $p^{(1)}(\check{u}_1, \ldots, \check{u}_k) \in \check{N}$. By the fact that the second through m-th coordinates of $\check{u}_1, \ldots, \check{u}_k$ are $c^{(2)}, \ldots, c^{(m)}$, we have the equality $p^{(1)}(\check{u}_1, \ldots, \check{u}_k) = p(u_1, \ldots, u_k)$ for all $\check{u}_1, \ldots, \check{u}_k \in \check{N}$ if we denote by p the k-ary polynomial of $\mathbf{A}$ obtained from $p^{(1)}$ by replacing appropriate variables by $c^{(2)}, \ldots, c^{(m)}$. This shows that if P is a k-ary polynomial operation of $\mathbf{C}$ such that $P(\check{N}^k) \subseteq \check{N}$, then $\mathbf{A}$ has a k-ary polynomial operation p such that $p(N^k) \subseteq N$ and

$$P|_{\check{N}}(\check{u}_1, \ldots, \check{u}_k) = P|_{\check{N}}\left(\begin{bmatrix} u_1 \\ c^{(2)} \\ \vdots \\ c^{(m)} \end{bmatrix}, \ldots, \begin{bmatrix} u_k \\ c^{(2)} \\ \vdots \\ c^{(m)} \end{bmatrix}\right) = \begin{bmatrix} p|_N(u_1, \ldots, u_k) \\ c^{(2)} \\ \vdots \\ c^{(m)}. \end{bmatrix}$$

or equivalently,

$$(3.8) \qquad P|_{\check{N}}(\check{u}_1, \ldots, \check{u}_k) = \big(p|_N(u_1, \ldots, u_k)\big)^{\check{}} \qquad \text{for all } u_1, \ldots u_k \in N.$$

Conversely. if p is a k-ary polynomial operation of $\mathbf{A}$ satisfying $p(N^k) \subseteq N$, then the polynomial operation P of $\mathbf{C}$ whose coordinate functions are p and the unary constant polynomials with values $c^{(2)}, \ldots, c^{(m)}$ satisfies these equalities. Clearly, each one of the induced operations $p|_N$ and $P|_{\check{N}}$ uniquely determines the another. This proves that if we make a correspondence between the operations of the non-indexed algebras $\mathbf{A}|_N$ and $\mathbf{C}|_{\check{N}}$ via the assignment $p|_N \mapsto P|_{\check{N}}$ indicated in (3.8), then the bijection $N \to \check{N}$, $u \mapsto \check{u}$ is an isomorphism. Hence, this bijection is a weak isomorphism $\mathbf{A}|_N \to \mathbf{C}|_{\check{N}}$. Moreover, by Corollary 3.4(1), it maps $\delta|_N$ to $\delta^*|_{\check{N}}$, which completes the proof of the first statement of Theorem 3.16.

The second statement now follows easily from the facts that $\mathrm{typ}(\delta, \theta)$ is the type of the minimal algebra $\mathbf{A}|_N/\delta|_N$, $\mathrm{typ}(\delta^*, \theta^*)$ is the type of the minimal algebra $\mathbf{C}|_{\check{N}}/\delta^*|_{\check{N}}$, we have a weak isomorphism $\mathbf{A}|_N/\delta|_N \to \mathbf{C}|_{\check{N}}/\delta^*|_{\check{N}}$ induced by the weak isomorphism $\mathbf{A}|_N \to \mathbf{C}|_{\check{N}}$ mapping $\delta|_N$ to $\delta^*|_{\check{N}}$ obtained above, and weakly isomorphic minimal algebras have the same type; moreover, if that type is $\mathbf{2}$, then they also have the same prime characteristic. $\qquad \square$

4. APPLICATION TO SUPERNILPOTENT CONGRUENCES

A congruence α of an algebra $\mathbf{A}$ is called k-*supernilpotent* if

$$[\underbrace{\alpha, \ldots, \alpha}_{k+1 \ \alpha\text{'s}}] = 0,$$

and α is called *supernilpotent* if it is k-supernilpotent for some $k \geq 1$. An algebra $\mathbf{A}$ is called k-supernilpotent or supernilpotent if its congruence 1 has the property.

It is well known (see e.g. [1, p. 370]) and easy to check that for every $k \geq 1$ the 4-element algebra $(\mathbb{Z}_4; +, 2x_1 \ldots x_k)$ $(k \geq 2)$ is k-supernilpotent, but not $(k-1)$-supernilpotent. Therefore, even for finite algebras of a fixed size, there is no a priori bound on the arity of the higher commutator $[1, \ldots, 1]$ to be checked if one wants to determine whether the algebra is supernilpotent. Hence, it is not clear from the definition whether supernilpotence is a decidable property for congruences of finite algebras.

Under mild assumptions on a finite algebra in a finite language, a combination of basic facts from tame congruence theory (see [12]) and results from [13] and [1] yields the following characterization of supernilpotence, which implies that supernilpotence for these algebras is decidable.

Theorem 4.1. [12, 13, 1] *Let $\mathbf{A}$ be a finite algebra in a finite language such that the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$ omits type $\mathbf{1}$. Then $\mathbf{A}$ is supernilpotent if and only if $\mathbf{A}$ factors as a direct product of nilpotent algebras of prime power order.*

In more detail, the three theorems on finite algebras $\mathbf{A}$, which combine to yield the characterization in Theorem 4.1, are as follows:

(I) If $\mathcal{V}(\mathbf{A})$ omits type $\mathbf{1}$ and $\mathbf{A}$ is supernilpotent or nilpotent, then $\mathbf{A}$ is solvable, and hence $\mathcal{V}(\mathbf{A})$ is congruence permutable (i.e., $\mathbf{A}$ has a Maltsev term). [12, Thm. 7.2, Cor. 7.6, Thm. 7.11]

(II) If $\mathbf{A}$ is nilpotent in a finite language and $\mathcal{V}(\mathbf{A})$ is congruence modular, then
   - $\mathbf{A}$ factors as a direct product of nilpotent algebras of prime power order if and only if $\mathbf{A}$ has a finite bound on the arities of nontrivial commutator terms. [13, Thm. 3.14]

(III) Assuming $\mathcal{V}(\mathbf{A})$ is congruence permutable,
   - if $\mathbf{A}$ is supernilpotent, then $\mathbf{A}$ is nilpotent; moreover,
   - $\mathbf{A}$ is supernilpotent if and only if $\mathbf{A}$ has a finite bound on the arities of nontrivial commutator terms. [1, Cor. 6.15, Lm. 7.5]

Our goal in this section is to use the techniques developed in Sections 2–3 to lift the characterization of supernilpotence in Theorem 4.1 from algebras to congruences as follows.

Theorem 4.2. *Let $\mathbf{A}$ be a finite algebra in a finite language such that $\mathcal{V}(\mathbf{A})$ omits type $\mathbf{1}$. For any congruence α of $\mathbf{A}$ the following conditions are equivalent:*

(a) *α is supernilpotent.*
(b) *either $\alpha = 0$, or else α is nilpotent, and $\mathbf{A}$ has congruences $\beta_1, \dots, \beta_\ell \leq \alpha$ (for some $\ell > 0$) such that*
   (1) *$\beta_1 \wedge \cdots \wedge \beta_\ell = 0$,*
   (2) *$(\beta_1 \wedge \cdots \wedge \beta_{i-1}) \circ \beta_i = \beta_i \circ (\beta_1 \wedge \cdots \wedge \beta_{i-1}) = \alpha$ for every $i \in [\ell]$, $i > 1$, and*
   (3) *for each $i \in [\ell]$ there exists a prime p_i such that every block of α/β_i in $\mathbf{A}/\beta_i$ has size a power of p_i.*

Proof. Suppose $\mathbf{A}$ satisfies the assumptions of the theorem, let $\mathcal{F}$ denote the (finite) language of $\mathbf{A}$, and let $\mathcal{V} := \mathcal{V}(\mathbf{A})$. Let us consider any congruence α of $\mathbf{A}$. The statement of the theorem is trivial if $\alpha = 0$, therefore we will assume from now on that $\alpha > 0$. Let us choose and fix an algebra $\mathbf{I} = ([m]; \mathcal{F})$ isomorphic to the (finite) algebra $\mathbf{A}/\alpha$, and let us fix an onto homomorphism $\chi \colon \mathbf{A} \to \mathbf{I}$. Thus, $(\mathbf{A}, \chi)$ is an object of the category $(\mathcal{V} \downdownarrows \mathbf{I})$.

Now let $\mathbf{C} := \mathfrak{C}(\mathbf{A}, \chi)$. Clearly, $\mathbf{C}$ is a finite algebra in a finite language, $\widehat{\mathcal{F}}_{\mathbf{I}}$, which belongs to the variety $\mathcal{V}^*$ consisting of all isomorphic copies of algebras in the category $\mathfrak{C}(\mathcal{V} \downdownarrows \mathbf{I})$ (cf. Corollary 3.14). Clearly, $\mathcal{V}(\mathbf{C})$ is a subvariety of $\mathcal{V}^*$. Since $\mathcal{V}$ omits type $\mathbf{1}$, and omitting type $\mathbf{1}$ is characterized, for locally finite varieties, by an idempotent Maltsev condition (see [12, Thm. 9.6]), Corollary 3.15(2) implies that $\mathcal{V}(\mathbf{C})$ also satisfies this Maltsev condition, and therefore $\mathcal{V}(\mathbf{C})$ omits type $\mathbf{1}$. In

summary, our discussion in this paragraph shows that $\mathbf{C}$ satisfies the assumptions of Theorem 4.1.

Let us return to the congruence α of $\mathbf{A}$. By Corollary 3.4(1), its image under $\mathfrak{C}$ is the congruence $\alpha^* = 1$ of $\mathbf{C}$, and by Theorem 3.6, α is a supernilpotent congruence of $\mathbf{A}$ if and only if 1 is a supernilpotent congruence of $\mathbf{C}$, that is, if and only if the algebra $\mathbf{C}$ is supernilpotent. By Theorem 4.1, the latter condition holds if and only if $\mathbf{C}$ factors as a direct product of nilpotent algebras of prime power order; that is, $\mathbf{C}$ is nilpotent and factors as a direct product of algebras of prime power order. Using congruences of $\mathbf{C}$, namely the projection kernels of the direct factorization, this condition can be expressed as follows:

(b)* the congruence 1 of $\mathbf{C}$ is nilpotent, and $\mathbf{C}$ has congruences $\gamma_1, \ldots, \gamma_\ell$ (for some $\ell > 0$) such that
   (1) $\gamma_1 \wedge \cdots \wedge \gamma_\ell = 0$,
   (2) $(\gamma_1 \wedge \cdots \wedge \gamma_{i-1}) \circ \gamma_i = \gamma_i \circ (\gamma_1 \wedge \cdots \wedge \gamma_{i-1}) = 1$ for every $i \in [\ell]$, $i > 1$, and
   (3) for each $i \in [\ell]$ there exists a prime p_i such that the algebra $\mathbf{C}/\gamma_i$ has size a power of p_i.

By Theorem 3.6, $1 = \alpha^*$ is a nilpotent congruence of $\mathbf{C}$ if and only if α is a nilpotent congruence of $\mathbf{A}$. Furthermore, using the bijection * from Corollary 3.4 between the interval $I(0, \alpha)$ of the congruence lattice of $\mathbf{A}$ and the congruence lattice of $\mathbf{C}$, which also preserves $\wedge$ and $\circ$, we can write each γ_i as $\gamma_i = \beta_i^*$ for a unique $\beta_i \in I(0, \alpha)$, and we see that the existence of $\gamma_1, \ldots, \gamma_\ell$ with properties (1)–(3) is equivalent to the existence of $\beta_1, \ldots, \beta_\ell \in I(0, \alpha)$ such that conditions (1)–(3) in (b) hold. For translating condition (3) in (b)* to condition (3) in (b) we also use the fact that the underlying set of $\mathbf{C}/\beta_i^*$ is the product of the blocks of the congruence α/β_i of $\mathbf{A}/\beta_i$. $\qquad\square$

5. Application to the Subpower Membership Problem

Let $\mathcal{K}$ be a finite set of finite algebras in a finite language. The *Subpower Membership Problem* for $\mathcal{K}$ is the following combinatorial decision problem:

> SMP($\mathcal{K}$)
> Input: $a_1, \ldots, a_k, b \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ with $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K}$.
> Question: Is b a member of the subalgebra of $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ generated by the set $\{a_1, \ldots, a_k\}$?

For background and recent results on the subpower membership problem, see [17, 6, 22, 23, 21, 7].

In this section we will assume that the set $\mathcal{K}$ of algebras fixed for the Subpower Membership Problem satisfies the following condition for some integer $\mathsf{d} \geq 2$:

(5.1) $\mathcal{V}$ is a variety in a finite language $\mathcal{F}$ with a d-cube term, and
$$\mathcal{K} \text{ is a finite set of finite algebras in } \mathcal{V}.$$

For the definition of a d-cube term, the reader is referred to [4, 14]. Note also that every variety with a cube term is congruence modular by [4, Thm. 4.2] (for an easy proof, see [8]), therefore we may use concepts and results from the theory of the modular commutator (see [9]).

It was proved in [7, Thm. 6.4] that under assumption (5.1) on $\mathcal{K}$, $\mathrm{SMP}(\mathcal{K}) \in \mathsf{P}$ provided $\mathcal{K}$ generates a residually small variety. This was done by reducing the general problem $\mathrm{SMP}(\mathcal{K})$ to a 'well-structured' subproblem.

In this section we will employ the techniques developed in Sections 2–3 to further reduce this subproblem of $\mathrm{SMP}(\mathcal{K})$, and use this reduction to extend the result of [7, Thm. 6.4] on $\mathrm{SMP}(\mathcal{K}) \in \mathsf{P}$ mentioned above to a wider family of sets $\mathcal{K}$.

Our starting point for the new reduction will be the reduction proved in [7], therefore we need to recall the relevant concepts and results from [7].

Definition 5.1 ([7, Def. 6.2]). Let $a_1, \ldots, a_k, b \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ $(\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K})$ be an input for $\mathrm{SMP}(\mathcal{K})$ where $a_r = (a_{r1}, \ldots, a_{rn})$ $(r \in [k])$ and $b = (b_1, \ldots, b_n)$. We call this input d-*coherent* if the following conditions are satisfied:

(i) $n \geq \max\{\mathsf{d}, 3\}$;

(ii) $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are similar subdirectly irreducible algebras, and each $\mathbf{A}_\ell$ has abelian monolith μ_ℓ; let ρ_ℓ denote the centralizer of μ_ℓ;

(iii) for all $I \subseteq [n]$ with $|I| < \max\{\mathsf{d}, 3\}$, the subalgebra of $\prod_{i \in I} \mathbf{A}_i$ generated by $\{a_1|_I, \ldots, a_k|_I\}$ is a subdirect subalgebra of $\prod_{i \in I} \mathbf{A}_i$, and $b|_I$ is a member of this subalgebra;

(iv) for all $i, j \in [n]$, the subalgebra of $\mathbf{A}_i/\rho_i \times \mathbf{A}_j/\rho_j$ generated by

$$\{(a_{1i}/\rho_i, a_{1j}/\rho_j), \ldots, (a_{ki}/\rho_i, a_{kj}/\rho_j)\}$$

is the graph of an isomorphism $\mathbf{A}_i/\rho_i \to \mathbf{A}_j/\rho_j$.

It is easy to see that d-coherence for inputs of $\mathrm{SMP}(\mathcal{K})$ can be checked in polynomial time.

Definition 5.2 ([7, Def. 6.3]). We define $\mathrm{SMP}_{\mathsf{d\text{-}coh}}(\mathcal{K})$ to be the restriction of $\mathrm{SMP}(\mathcal{K})$ to d-coherent inputs.

Theorem 5.3 ([7, Thm. 6.4]). *If $\mathcal{V}$ is a variety in a finite language with a d-cube term, then the decision problems $\mathrm{SMP}(\mathcal{K})$ and $\mathrm{SMP}_{\mathsf{d\text{-}coh}}(\mathbb{HSK})$ are polynomial time equivalent for every finite family $\mathcal{K}$ of finite algebras in $\mathcal{V}$.*

Now we are ready to define our new reduction for $\mathrm{SMP}(\mathcal{K})$.

Definition 5.4. An input $a_1, \ldots, a_k, b \in \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ $(\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathcal{K})$ for $\mathrm{SMP}(\mathcal{K})$ will be called d-*central* if it satisfies conditions (i) and (iii) in Definition 5.1 and the following new condition:

(ii)′ $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are subdirectly irreducible algebras such that the monolith μ_ℓ of each $\mathbf{A}_\ell$ is a central congruence of $\mathbf{A}_\ell$ (i.e., $[1, \mu_\ell] = 0$, hence in particular, μ_ℓ is abelian), and the monoliths $\mu_1, \ldots, \mu_n$ have the same prime characteristic.

As before, it is clear that d-centrality for inputs of $\mathrm{SMP}(\mathcal{K})$ can be checked in polynomial time.

Definition 5.5. We define $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K})$ to be the restriction of $\mathrm{SMP}(\mathcal{K})$ to d-central inputs.

Our reduction theorem will reduce the solution of $\mathrm{SMP}(\mathcal{K})$ to the solution of $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}^\star)$ for several new sets $\mathcal{K}^\star$ of algebras constructed from $\mathcal{K}$, which are defined as follows.

Definition 5.6. Given $\mathcal{K}$ as in (5.1), let $\overline{\overline{\mathcal{K}}}$ denote the set of all subdirectly irreducible algebras $\mathbf{S}$ in $\mathbb{HS}\mathcal{K}$ whose monolith $\mu_{\mathbf{S}}$ is abelian. Recall that similarity of subdirectly irreducible algebras is an equivalence relation on $\overline{\overline{\mathcal{K}}}$, so let $\overline{\mathcal{K}}_1, \ldots, \overline{\mathcal{K}}_q$ denote its equivalence classes. For each $\ell \in [q]$, let $\mathbf{I}_\ell = ([m_\ell], \mathcal{F})$ be a fixed algebra that is isomorphic to $\mathbf{S}/(0 : \mu_{\mathbf{S}})$ for all $\mathbf{S} \in \overline{\mathcal{K}}_\ell$, and let

$$\mathcal{K}_\ell^\star := \{\mathfrak{C}(\mathbf{S}, \chi) : \mathbf{S} \in \overline{\mathcal{K}}_\ell, \ \chi \text{ is an onto homomorphism } \mathbf{S} \to \mathbf{I}_\ell \text{ with kernel } (0 : \mu_{\mathbf{S}})\},$$

where $\mathfrak{C}$ is the functor with domain category $\big(\mathsf{Alg}(\mathcal{F}) \downdownarrows \mathbf{I}_\ell\big)$.

Theorem 5.7. *Let $\mathcal{V}$ be a variety in a finite language $\mathcal{F}$ with a d-cube term. For any finite set $\mathcal{K}$ of finite algebras in $\mathcal{V}$, the sets $\mathcal{K}_1^\star, \ldots, \mathcal{K}_q^\star$ of algebras have the following properties:*

 (1) *Each $\mathcal{K}_\ell^\star$ $(\ell \in [q])$ is a finite set of finite algebras in a variety in a finite language with a d-cube term.*
 (2) *Each $\mathcal{K}_\ell^\star$ $(\ell \in [q])$ is a set of subdirectly irreducible algebras whose monoliths are central and have the same prime characteristic.*
 (3) $\mathrm{SMP}(\mathcal{K})$ *is polynomial time reducible to the problems $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star)$ $(\ell \in [q])$, and the sets $\mathcal{K}_\ell^\star$ $(\ell \in [q])$ of algebras can be computed from $\mathcal{K}$ in constant time.*

Proof. We will use all notation introduced in Definition 5.6. For the proofs of statements (1) and (2) we fix an $\ell \in [q]$.

To prove (1), let $\mathcal{W}_\ell$ denote the subvariety of $\mathcal{V}$ generated by $\overline{\mathcal{K}}_\ell$. Since

$$(5.2) \quad \{(\mathbf{S}, \chi) : \mathbf{S} \in \overline{\mathcal{K}}_\ell, \ \chi \text{ is an onto homomorphism } \mathbf{S} \to \mathbf{I}_\ell \text{ with kernel } (0 : \mu_{\mathbf{S}})\}$$

is a finite set of objects in $\big(\mathcal{W}_\ell \downdownarrows \mathbf{I}_\ell\big)$ with all $\mathbf{S} \in \overline{\mathcal{K}}_\ell$ finite, it follows from Corollary 3.14 that $\mathcal{K}_\ell^\star$ is a finite set of finite algebras in the variety $\mathcal{W}_\ell^*$ (whose language $\widehat{\mathcal{F}}_{\mathbf{I}_\ell}$ is finite). Since $\mathcal{V}$ has a d-cube term, so does its subvariety $\mathcal{W}_\ell$. The existence of a d-cube term is a strong idempotent Maltsev condition, therefore the variety $\mathcal{W}_\ell^*$ also has a d-cube term by Corollary 3.15.

For statement (2), let $\mathbf{C} := \mathfrak{C}(\mathbf{S}, \chi)$ be any algebra in $\mathcal{K}_\ell^\star$. Here $(\mathbf{S}, \chi)$ is a member of the set in (5.2), so $\mathbf{S}$ is subdirectly irreducible with abelian monolith $\mu_{\mathbf{S}}$, and for the centralizer $\alpha_{\mathbf{S}} = (0 : \mu_{\mathbf{S}}) \, (\geq \mu_{\mathbf{S}})$ we have that $\alpha_{\mathbf{S}} = \ker(\chi)$. The map * described in Corollary 3.4 is an isomorphism between the interval $I(0, \alpha_{\mathbf{S}})$ in the congruence lattice

of $\mathbf{S}$ and the congruence lattice of $\mathbf{C}$. Therefore, $\mathbf{C}$ is subdirectly irreducible with monolith $\mu_{\mathbf{S}}^*$. Moreover, by Corollary 3.8, we have that $(0 : \mu_{\mathbf{S}}^*) = (0 : \mu_{\mathbf{S}})^* = \alpha^* = 1$, so the monolith $\mu_{\mathbf{S}}^*$ of $\mathbf{C}$ is central. Since $\mathbf{S} \in \mathcal{V}$ and $\mathcal{V}$ is congruence modular, Theorem 3.16 and the fact that $\mu_{\mathbf{S}}$ is abelian imply that $\mathrm{typ}_{\mathbf{C}}(0, \mu_{\mathbf{S}}^*) = \mathrm{typ}_{\mathbf{S}}(0, \mu_{\mathbf{S}}) = \mathbf{2}$, and the congruences $\mu_{\mathbf{S}}^*$ and $\mu_{\mathbf{S}}$ — i.e., the prime quotients $\langle 0, \mu_{\mathbf{S}}^* \rangle$ and $\langle 0, \mu_{\mathbf{S}} \rangle$ — have the same prime characteristic. Since all algebras $\mathbf{S}$ in (5.2) are similar, and hence by the definition or by the characterization of similarity in [9, Def. 10.6, Thm. 10.8] they have the same characteristic, we conclude that the monoliths of all members of $\mathcal{K}_\ell^\star$ have the same prime characteristic.

In statement (3) it is clear that the sets $\mathcal{K}_\ell^\star$ ($\ell \in [q]$) of algebras can be computed from $\mathcal{K}$ in constant time. We need to argue that $\mathrm{SMP}(\mathcal{K})$ is polynomial time reducible to the problems $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star)$ ($\ell \in [q]$). In view of Theorem 5.3, it suffices to show that $\mathrm{SMP}_{\mathsf{d\text{-}coh}}(\mathbb{HSK})$ is polynomial time reducible to the problems $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star)$ ($\ell \in [q]$). We will prove this by describing how to assign to every input $a_1, \ldots, a_k, b$ for $\mathrm{SMP}_{\mathsf{d\text{-}coh}}(\mathbb{HSK})$ a number $\ell \in [q]$ and an input $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ for $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star)$ in such a way that

- the answer to the input $a_1, \ldots, a_k, b$ is 'yes' if and only if the answer to the input $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ is 'yes', and
- $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ can be computed from $a_1, \ldots, a_k, b$ in polynomial time.

Let $a_1, \ldots, a_k, b \in \prod_{j \in [n]} \mathbf{A}_j$ ($\mathbf{A}_1, \ldots, \mathbf{A}_n \in \mathbb{HSK}$) be an input for $\mathrm{SMP}_{\mathsf{d\text{-}coh}}(\mathbb{HSK})$. Then this is a d-coherent input for $\mathrm{SMP}(\mathcal{K})$, that is, conditions (i)–(iv) from Definition 5.1 hold. In particular, (ii) implies that there is a unique $\ell \in [q]$ such that $\mathbf{A}_1, \ldots, \mathbf{A}_n \in \overline{\mathcal{K}}_\ell$. This ℓ can be found in constant time. Now let us fix $\mathbf{I} := \mathbf{I}_\ell$ as described in Definition 5.6, choose an isomorphism $\varphi_1 \colon \mathbf{A}_1/\rho_1 \to \mathbf{I}$, and for $2 \leq j \leq n$ define the isomorphisms $\varphi_j \colon \mathbf{A}_j/\rho_j \to \mathbf{I}$ by $\varphi_j := \varphi_1 \circ \iota_{j,1}$ where $\iota_{j,1}$ is the isomorphism $\mathbf{A}_j/\rho_j \to \mathbf{A}_1/\rho_1$ from condition (iv) in Definition 5.1. For each $j \in [n]$ let $\chi_j \colon \mathbf{A}_j \to \mathbf{I}$ be defined by $\chi_j := \varphi_j \circ \nu_j$ where $\nu_j \colon \mathbf{A}_j \to \mathbf{A}_j/\rho_j$ is the natural homomorphism. So, $\ker(\chi_j) = \rho_j = (0 : \mu_j)$ for every $j \in [n]$. Clearly, the algebra $\mathbf{I}$ and each isomorphism $\iota_{j,1}$ can be computed in constant time, so the homomorphisms χ_j ($j \in [n]$) can be computed in $O(n)$ time.

Let $\mathbf{B}$ and $\mathbf{B}^+$ denote the subalgebras of $\prod_{j \in [n]} \mathbf{A}_j$ generated by the sets $\{a_1, \ldots, a_k\}$ and $\{a_1, \ldots, a_k, b\}$, respectively. Using the homomorphisms χ_j ($j \in [n]$) we can express condition (iv) as follows: $\mathbf{B}$ is a subalgebra of the algebra $\prod_{j \in [n]}^{\sqcup\mathbf{I}} \mathbf{A}_j$, where $(\prod_{j \in [n]}^{\sqcup\mathbf{I}} \mathbf{A}_j; \chi)$ with χ defined by $(x_1, \ldots, x_n) \mapsto \chi_1(x_1) = \cdots = \chi_n(x_n)$, is the product of the objects $(\mathbf{A}_j, \chi_j)$ ($j \in [n]$) in the category $(\mathsf{Alg}(\mathcal{F}) \sqcup \mathbf{I})$. Condition (iii) of Definition 5.1 implies that $\mathbf{B}$ and $\mathbf{B}^+$ are both subdirect subalgebras of $\prod_{j \in [n]} \mathbf{A}_j$, moreover, the requirements on b in this condition ensure that $\mathbf{B}^+$ is a subalgebra of $\prod_{j \in [n]}^{\sqcup\mathbf{I}} \mathbf{A}_j$ as well. Therefore, the map χ restricts to $\mathbf{B}$ and $\mathbf{B}^+$ as onto homomorphisms $\chi|_{\mathbf{B}} \colon \mathbf{B} \to \mathbf{I}$ and $\chi|_{\mathbf{B}^+} \colon \mathbf{B}^+ \to \mathbf{I}$. The kernel classes of these homomorphisms

are the sets $B \cap D^{(i)}$ and $B^+ \cap D^{(i)}$ ($i \in [m]$), respectively, where $D^{(i)} = \prod_{j \in [n]} \chi_j^{-1}(i)$. Hence, $B^+ \cap D^{(i)} \supseteq B \cap D^{(i)} \neq \emptyset$ for every $i \in [m]$.

We can compute representatives $d^{(i)} \in B \cap D^{(i)}$ for each $i \in [m]$ by generating all elements of $\mathbf{B}/\ker(\chi|_{\mathbf{B}}) \cong \mathbf{I}$ using $\chi(a_1), \ldots, \chi(a_k)$ (at most m distinct elements), and replicating the same computation using at most m of the generators $a_1, \ldots, a_k$ instead, one from each set $\{a_1, \ldots, a_k\} \cap \chi^{-1}\big(\chi(a_r)\big)$ ($r \in [k]$). This requires $O(kn)$ time.

Now let $\mathbf{C}_j := \mathfrak{C}(\mathbf{A}_j, \chi_j)$ for every $j \in [n]$, and let $\mathbf{B}^*$ and $(\mathbf{B}^+)^*$ be the subalgebras of $\prod_{j \in [n]} \mathbf{C}_j$ obtained from $\mathbf{B}$ and $\mathbf{B}^+$ by applying the map * described in Theorem 3.1(1). Furthermore, let $\widetilde{} : \prod_{j \in [n]}^{\sqcup \mathbf{I}} A_j \to \prod_{j \in [n]} C_j$ be the function defined in Theorem 3.1(3), using the elements $d^{(i)}$ ($i \in [m]$) from the preceding paragraph as padding elements. Let $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ be the elements of $(B^+)^*$ obtained from the given input elements $a_1, \ldots, a_k, b$ by applying this function. Clearly, $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ can be computed from $a_1, \ldots, a_k, b$ and $d^{(1)}, \ldots, d^{(m)}$ in $O\big(kn\big)$ time.

Since $\{a_1, \ldots, a_k\}$ generates $\mathbf{B}$, and $\{a_1, \ldots, a_k, b\}$ generates $\mathbf{B}^+$, Theorem 3.1(3), together with the fact $d^{(1)}, \ldots, d^{(m)} \in B \subseteq B^+$, implies that

(5.3) $\quad\quad\quad \{\widetilde{a}_1, \ldots, \widetilde{a}_k\}$ is a generating set for $\mathbf{B}^*$, and
$\quad\quad\quad\quad\quad\quad \{\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}\}$ is a generating set for $(\mathbf{B}^+)^*$.

Now we want to show that

(5.4) $\quad\quad\quad \widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b} \in \mathbf{C}_1 \times \cdots \times \mathbf{C}_n \ (\mathbf{C}_1, \ldots, \mathbf{C}_n \in \mathcal{K}_\ell^\star)$

is a correct input for $\mathrm{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star)$, that is, $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ is a d-central input for $\mathrm{SMP}(\mathcal{K}_\ell^\star)$. We need to check that conditions (i), (ii)$'$, and (iii) from Definition 5.4 hold for the new input (5.4), that is, they hold for $\widetilde{a}_r$ ($r \in [k]$), $\widetilde{b}$, $\mathbf{C}_j$ ($j \in [n]$), and $\mathcal{K}_\ell^\star$ in place of a_r ($r \in [k]$), b, $\mathbf{A}_j$ ($j \in [n]$), and $\mathcal{K}$. Note first that our general assumption (5.1) (suppressed in Definition 5.4) holds for $\mathcal{K}_\ell^\star$ by Theorem 5.7(1). Condition (i) from Definition 5.4 holds for the new input (5.4), because it is identical to condition (i) of d-coherence for the original input $a_1, \ldots, a_k, b$. Condition (ii)$'$ from Definition 5.4 holds for (5.4) by Theorem 5.7(2), since $\mathbf{C}_1, \ldots, \mathbf{C}_n \in \mathcal{K}_\ell^\star$. To check condition (iii) from Definition 5.4 for the new input (5.4), let $I \subseteq [n]$ be such that $|I| < \max\{\mathsf{d}, 3\}$. In view of (5.3), the subalgebra of $\prod_{j \in I} \mathbf{C}_j$ generated by $\{\widetilde{a}_1|_I, \ldots, \widetilde{a}_k|_I\}$ is $\mathbf{B}^*|_I$, while the subalgebra of $\prod_{j \in I} \mathbf{C}_j$ generated by $\{\widetilde{a}_1|_I, \ldots, \widetilde{a}_k|_I, \widetilde{b}|_I\}$ is $(\mathbf{B}^+)^*|_I$. Therefore, condition (iii) from Definition 5.4 for (5.4) and for the chosen I is equivalent to saying that $\mathbf{B}^*|_I$ projects to each coordinate $j \in I$ to be $\mathbf{C}_j$, and $(\mathbf{B}^+)^*|_I = \mathbf{B}^*|_I$. To prove that $\mathbf{B}^*|_I$ and $(\mathbf{B}^+)^*|_I$ meet these conditions, notice that our assumption that the original input $a_1, \ldots, a_k, b$ is d-coherent, implies that the analogous conditions hold for $\mathbf{B}|_I$, $\mathbf{B}^+|_I$, and $\mathbf{A}_j$ ($j \in I$). Consequently, they also hold if we pass to their images under the map * described in Theorem 3.1(1). Thus, $(\mathbf{B}|_I)^*$ projects to each

coordinate $j \in I$ to be $\mathbf{C}_j$, and $(\mathbf{B}^+|_I)^* = (\mathbf{B}|_I)^*$. According to Theorem 3.1(2), projection onto a set of coordinates is preserved by *, therefore these conditions hold with the order of $|_I$ and * switched. This proves (iii) from Definition 5.4 for the new input (5.4), and hence finishes the proof that the input (5.4) is d-central.

Finally, using again (5.3), we get that

the answer of $\text{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star)$ to the input $\widetilde{a}_1, \ldots, \widetilde{a}_k, \widetilde{b}$ is 'yes'

$$\Leftrightarrow \quad \widetilde{b} \in \mathbf{B}^* \quad \Leftrightarrow \quad (\mathbf{B}^+)^* = \mathbf{B}^* \quad \overset{\text{Thm 3.1}}{\Leftrightarrow} \quad (\mathbf{B}^+) = \mathbf{B} \quad \Leftrightarrow \quad b \in \mathbf{B}$$

$$\Leftrightarrow \quad \text{the answer of } \text{SMP}_{\mathsf{d\text{-}coh}}(\mathbb{HSK}) \text{ to the input } a_1, \ldots, a_k, b \text{ is 'yes'}.$$

The crucial step is $\overset{\text{Thm 3.1}}{\Leftrightarrow}$, where we use the bijective property of the map * in Theorem 3.1(1). The proof of Theorem 5.7 is complete. □

Corollary 5.8. *The following conditions are equivalent:*

(a) $\text{SMP}(\mathcal{K}) \in \mathsf{P}$ *for every finite set $\mathcal{K}$ of finite algebras in a variety in a finite language with a d-cube term.*

(b) $\text{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}) \in \mathsf{P}$ *for every finite set $\mathcal{K}$ of finite algebras in a variety in a finite language with a d-cube term.*

Theorem 5.9. *Let $\mathcal{V}$ be a variety in a finite language $\mathcal{F}$ with a d-cube term. If $\mathcal{K}$ is a finite set of finite algebras in $\mathcal{V}$ such that*

(‡) *for every subdirectly irreducible algebra $\mathbf{S} \in \mathbb{HSK}$ with abelian monolith $\mu_{\mathbf{S}}$ the centralizer of $\mu_{\mathbf{S}}$ is supernilpotent,*

then $\text{SMP}(\mathcal{K}) \in \mathsf{P}$.

The result of [7, Thm. 6.4] mentioned earlier is a special case of Theorem 5.9; it is obtained from this theorem by replacing the word 'supernilpotent' in condition (‡) by the word 'abelian'. Indeed, the strengthening of condition (‡) where 'supernilpotent' is replaced by 'abelian' is equivalent to the condition that $\mathcal{K}$ generates a residually small variety, and hence $\mathcal{V}$ can be chosen to be residually small (see. e.g., [7, Cor. 2.4] and the discussion preceding it).

Finite algebras with a Maltsev term that satisfy a slightly weaker condition than (‡), namely that in every subdirectly irreducible homomorphic image the monolith has supernilpotent centralizer, already appeared in [16] where a relational description of their polynomial clones was given.

Proof of Theorem 5.9. We will use the notation introduced in Definition 5.6. By Theorem 5.7(3), it suffices to show that under the assumptions of Theorem 5.9 we have $\text{SMP}_{\mathsf{d\text{-}centr}}(\mathcal{K}_\ell^\star) \in \mathsf{P}$ for every $\ell \in [q]$. Let $\ell \in [q]$ be fixed for the rest of the proof. For notational convenience, we will drop the subscript ℓ; that is, we will write $\overline{\mathcal{K}}$ for $\overline{\mathcal{K}}_\ell$, $\mathbf{I} = ([m]; \mathcal{F})$ for $\mathbf{I}_\ell = ([m_\ell]; \mathcal{F})$, and $\mathcal{K}^\star$ for $\mathcal{K}_\ell^\star$. Furthermore, let $\mathbf{K} := \prod \mathcal{K}^\star$

be the product of all members of $\mathcal{K}^\star$. Clearly, the variety $\mathcal{V}(\mathbf{K})$ it generates coincides with the variety generated by $\mathcal{K}^\star$.

We know from Theorem 5.7(1) that $\mathcal{K}^\star$ is a finite set of finite algebras in a variety in a finite language (namely, $\widehat{\mathcal{F}}_\mathbf{I}$) with a d-cube term. Therefore, $\mathbf{K}$ is a finite algebra, the variety $\mathcal{V}(\mathbf{K})$ has a d-cube term, and hence is congruence modular.

In the proof of Theorem 5.7(2) we saw that for every algebra $\mathbf{C} = \mathfrak{C}(\mathbf{S}, \chi)$ in $\mathcal{K}^\star$,

- $\mathbf{C}$ is subdirectly irreducible with monolith $\mu_\mathbf{S}^*$, where $\mu_\mathbf{S}$ is the monolith of $\mathbf{S}$, and
- $1 = \alpha_\mathbf{S}^*$ centralizes $\mu_\mathbf{S}^*$, where $\alpha_\mathbf{S} = (0 : \mu_\mathbf{S})$.

Now, our additional assumption (‡) implies that $\alpha_\mathbf{S}$ is a supernilpotent congruence of $\mathbf{S}$. Hence, by Theorem 3.6, $1 = \alpha_\mathbf{S}^*$ is a supernilpotent congruence of $\mathbf{C}$. Thus, all algebras in $\mathcal{K}^\star$ are supernilpotent. Since they are also subdirectly irreducible, Theorem 4.1 implies that they are nilpotent algebras of prime power order. By Theorem 5.7(2) we also have that the algebras in $\mathcal{K}^\star$ have the same prime characteristic. It follows that the cardinality of every algebra in $\mathcal{K}^\star$ is a power of the same prime. Hence, the product of these algebras, $\mathbf{K}$, is also nilpotent of prime power order. Statement (I) following Theorem 4.1 also implies that $\mathbf{K}$ has a Maltsev term.

Now we can use [17, Thm. 1.2] to conclude that $\mathrm{SMP}(\{\mathbf{K}\}) \in \mathsf{P}$. Clearly, $\mathrm{SMP}(\{\mathbf{K}\})$ is polynomial time reducible to $\mathrm{SMP}(\mathcal{K}^\star)$, since every computation for $\mathrm{SMP}(\{\mathbf{K}\})$ can be viewed as a computation for $\mathrm{SMP}(\mathcal{K}^\star)$. Conversely, $\mathrm{SMP}(\mathcal{K}^\star)$ is also polynomial time reducible to $\mathrm{SMP}(\{\mathbf{K}\})$ for the following reason: by [7, Thm. 4.11], if $\mathcal{L}$ is a finite set of finite algebras in a variety in a finite language with a cube term, then $\mathrm{SMP}(\mathcal{L})$ and $\mathrm{SMP}(\mathbb{HS}\mathcal{L})$ are polynomial time equivalent. This theorem applies to $\mathcal{L} = \{\mathbf{K}\}$, so since $\mathcal{K}^\star \subseteq \mathbb{HS}\{\mathbf{K}\}$, we get that $\mathrm{SMP}(\mathcal{K}^\star)$ is a subproblem of $\mathrm{SMP}(\mathbb{HS}\{\mathbf{K}\})$, which is polynomial time equivalent to $\mathrm{SMP}(\{\mathbf{K}\})$. This proves that $\mathrm{SMP}(\mathcal{K}^\star) \in \mathsf{P}$. It follows that for its subproblem we also have $\mathrm{SMP}_{\text{d-centr}}(\mathcal{K}^\star) \in \mathsf{P}$, which completes the proof of Theorem 5.9. $\qquad\square$

References

[1] Aichinger, E.; Mudrinski, N. *Some applications of higher commutators in Mal'cev algebras.* Algebra Universalis, **63** (2010), 367–403.

[2] Barr, M. *The point of the empty set.* Cahiers Topologie Géom. Différentielle **13** (1972), 357–368, 442.

[3] Bentz, W.; Mayr, P. *Supernilpotence prevents dualizability.* J. Aust. Math. Soc., **96** (2014), no. 1, 1–24.

[4] Berman, J.; Idziak, P.; Marković, P.; McKenzie, R.; Valeriote, M.; Willard, R. *Varieties with few subalgebras of powers.* Trans. Amer. Math. Soc. **362** (2010), no. 3, 1445–1473.

[5] Bulatov, A. *On the number of finite Maltsev algebras.* Contributions to General Algebra, vol. 13 (Velké Karlovice, 1999/Dresden, 2000), pp. 41–54, Heyn, Klagenfurt, 2001.

[6] Bulatov, A.; Kozik, M.; Mayr, P.; Steindl, M. *The subpower membership problem for semigroups.* Internat. J. Algebra Comput. **26** (2016), 1435–1451.

[7] Bulatov, A.; Mayr, P.; Szendrei, Á. *The subpower membership problem for finite algebras with cube term.* Log. Methods Comput. Sci. **15** (2019), no. 1, Paper No. 11, 48.

[8] Dent, T.; Kearnes, K. A.; Szendrei, Á. *An easy test for congruence modularity.* Algebra Universalis **67** (2012), no. 4, 375–392.

[9] Freese, R.; McKenzie, R. *Commutator theory for congruence modular varieties.* London Mathematical Society Lecture Note Series, **125**. Cambridge University Press, Cambridge, 1987.

[10] Gardner, B. J. *How to make many-sorted algebras one-sorted.* Comment. Math. Univ. Carolin. **30** (1989), no. 4, 627–635.

[11] Goguen, J. A.; Meseguer, J. *Completeness of many-sorted equational logic.* Houston J. Math. **11** (1985), no. 3, 307–334.

[12] Hobby, D.; McKenzie, R. *The structure of finite algebras.* Contemporary Mathematics, **76**. American Mathematical Society, Providence, RI, 1988.

[13] Kearnes, K. A. *Congruence modular varieties with small free spectra.* Algebra Universalis, **42** (1999), 165–181.

[14] Kearnes, K. A.; Szendrei, Á. *Clones of algebras with parallelogram terms.* Internat. J. Algebra Comput. **22** (2012), no. 1, 1250005, 30 pp.

[15] MacLane, S. *Categories for the working mathematician.* Graduate Texts in Mathematics, **5**. Springer-Verlag, New York-Berlin, 1971.

[16] Mayr, P. *Mal'cev algebras with supernilpotent centralizers.* Algebra Universalis **65** (2011), 193–211.

[17] Mayr, P. *The subpower membership problem for Mal'cev algebras.* Internat. J. Algebra and Comput. **22** (2012), 1250075, 23 pp.

[18] Moorhead, A. *Higher commutator theory for congruence modular varieties.* J. Algebra **513** (2018), 133–158.

[19] Mučka, A.; Romanowska, A. B.; Smith, J. D. H. *Many-sorted and single-sorted algebras.* Algebra Universalis **69** (2013), no. 2, 171–190.

[20] Novotný, M. *Remarks on heterogeneous algebras.* Časopis Pěst. Mat. **107** (1982), no. 4, 397–406.

[21] Shriner, J. *Hardness Results for the Subpower Membership Problem.* Internat. J. Algebra Comput. **28** (2018), 719–732.

[22] Steindl, M. *On semigroups with PSPACE-complete subpower membership problem.* J. Aust. Math. Soc. **106** (2019), no. 1, 127–142.

[23] Steindl, M. *The subpower membership problem for bands.* J. Algebra **489** (2017), 529–551.

(Peter Mayr) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER CO, USA, 80309-0395

E-mail address: Peter.Mayr@Colorado.EDU

(Ágnes Szendrei) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER CO, USA, 80309-0395

E-mail address: Agnes.Szendrei@Colorado.EDU