Clones of algebras with parallelogram terms

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Joint work with K. Kearnes

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Algebraic Relations

A, an algebra

An ℓ-ary relation $R \subseteq A^\ell$ is an *algebraic relation* of **A**

$$\stackrel{\text{Def}}{\Longleftrightarrow} \mathbf{R} = (R; \dots) \leq \mathbf{A}^\ell$$

Central in

- clone theory and
- duality theory,

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Fact. *f is locally in* $\mathrm{Clo}(\mathbf{A}) \iff$ *f preserves all alg. rel's of* **A**

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Critical Relations

R, an ℓ-ary algebraic relation of $\mathbf{A}$

R or the corresponding subalgebra $\mathbf{R} \leq \mathbf{A}^\ell$ is

- *critical* $\overset{\text{Def}}{\iff}$ completely $\bigcap$-irreducible in $\mathrm{Sub}(\mathbf{A}^\ell)$
  & directly indecomposable: $R \neq \mathrm{pr}_U(R) \times \mathrm{pr}_V(R)$

Examples. Critical relations of a

- primal algebra: equality relation
- 1-dim. vector space: solution set of $\sum_{i=1}^{\ell} c_i x_i = 0$, $c_i \neq 0$

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NU Algebras

Theorem. *For a variety* $\mathcal{V}$ *and* $k > 1$*:*

- $\forall \mathbf{A} \in \mathcal{V} \quad \forall \ell \geq k$ *there is no critical* $\mathbf{R} \leq \mathbf{A}^\ell$ $\iff$ $\mathcal{V}$ *has a* k*-NU term* U*:*

$$\mathcal{V} \models \mathrm{U}\begin{pmatrix} x & y & \cdots & y \\ y & x & & y \\ \vdots & & \ddots & \vdots \\ y & y & \cdots & x \end{pmatrix} = \begin{pmatrix} y \\ y \\ \vdots \\ y \end{pmatrix}$$

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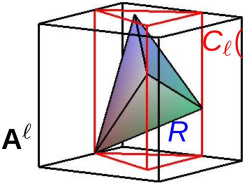
($\Longleftarrow$)



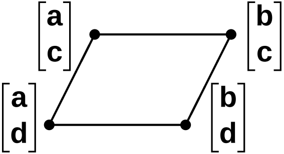
$C_\ell(R) = \mathrm{pr}_{1,\ldots,\ell-1}(R) \times A$

Baker–Pixley Theorem:

$R = \bigcap_i C_i(R)$ if $\ell \geq k$

Maltsev Algebras and the Parallelogram Property

(m,n)-▱ in $\mathbf{A}^\ell$:



$$|\mathbf{a}| = |\mathbf{b}| = m$$
$$|\mathbf{c}| = |\mathbf{d}| = n$$
$$m + n = \ell$$

$\mathbf{R} \le \mathbf{A}^\ell$ has the

- (m,n)-▱ *property* $\overset{\text{Def}}{\iff} \left(\begin{bmatrix}\mathbf{a}\\ \mathbf{c}\end{bmatrix}, \begin{bmatrix}\mathbf{a}\\ \mathbf{d}\end{bmatrix}, \begin{bmatrix}\mathbf{b}\\ \mathbf{d}\end{bmatrix} \in \mathbf{R} \Rightarrow \begin{bmatrix}\mathbf{b}\\ \mathbf{c}\end{bmatrix} \in \mathbf{R} \right)$
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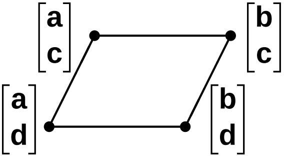

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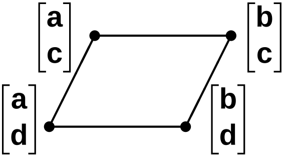

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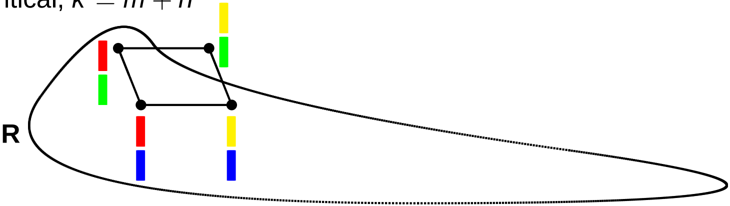
Parallelogram Terms

To prevent failure of (m, n)-▱ property

$\mathbf{R} \leq \mathbf{A}^k$ critical, $k = m + n$



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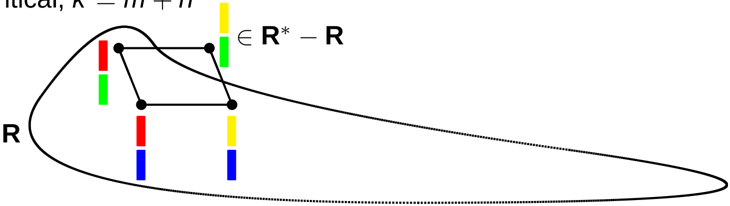

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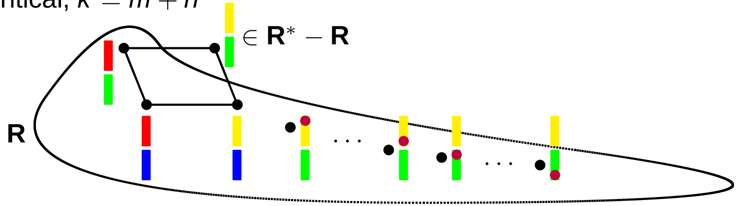

Prevented by (m,n)-▱ *term*:

$$\mathcal{V} \models \mathrm{P}_{m,n} \quad \left(\begin{array}{ccc|cccccc} x & x & y & z & \cdots & y & y & \cdots & y \\ & \vdots & & \vdots & \ddots & & & & \vdots \\ x & x & y & y & & z & y & & y \\ \hline y & x & x & y & & y & z & & y \\ & \vdots & & \vdots & & & & \ddots & \vdots \\ y & x & x & y & \cdots & y & y & \cdots & z \end{array}\right) = \left(\begin{array}{c} y \\ \vdots \\ y \\ y \\ \vdots \\ y \end{array}\right)$$

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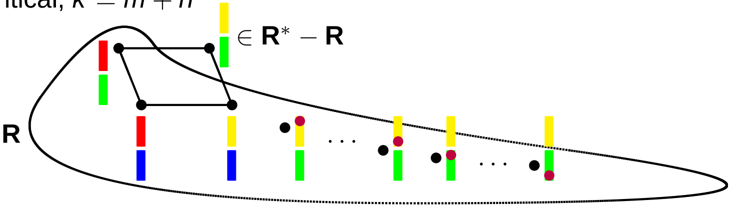

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Parallelogram Terms
Characterization Theorem

Theorem. *TFAE for $\mathcal{V}$ and $k > 1$:*

(i) *$\mathcal{V}$ has an (m,n)-▱ term for some $m+n=k$ $(m,n \geq 1)$.*

(ii) *$\mathcal{V}$ has (m,n)-▱ terms for all $m+n \geq k$ $(m,n \geq 1)$.*

k-▱ term = any (m,n)-▱ term with $m+n=k$

Theorem. *For a variety $\mathcal{V}$ and $k > 1$:*

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Edge Terms

Berman, Idziak, Marković, McKenzie, Valeriote, Willard (2006)

Theorem. (BIMMVW) *For finite* $\mathbf{A}$

(1) $\log_2 |\mathrm{Sub}(\mathbf{A}^n)| \leq p(n)$ *for some polynomial* $p(x)$
$\iff$ $\mathbf{A}$ *has a* k*-edge-term for some* k

For arbitrary $\mathcal{V}$

(2) $\mathcal{V}$ *has a* k*-edge term* $\iff$ $\mathcal{V}$ *has a* k*-cube term*
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Structure Theorem

Theorem.
Assume

- $\mathbf{A} \in \mathcal{V}$, $\mathcal{V}$ *has a* k-▱ *term, and*
- $\mathbf{R} \leq \mathbf{A}^{\ell}$ *is critical,* $\ell \geq k$.

Let

- $\mathbf{R} \leq_{\mathrm{sd}} \mathbf{R}_1 \times \cdots \times \mathbf{R}_\ell$ $(\mathbf{R}_i \leq \mathbf{A})$,
- $\theta = \theta_1 \times \cdots \times \theta_\ell$ $(\theta_i \in \mathrm{Con}(\mathbf{R}_i))$ **largest** *with* $\mathbf{R}$ θ*-saturated.*

Then

(1) *the algebras* $\mathbf{R}_i/\theta_i$ *are SI;*

(2) *they have abelian monoliths if* $\ell \geq 3$;

(3) $\mathbf{R}/\theta$ *is a joint similarity between them.*

Applications

Corollary. (BIMMVW, MM) *For $\mathcal{V}$ and $k \geq 3$*

$\mathcal{V}$ CD and has a k-edge(▱) term $\Longrightarrow$ $\mathcal{V}$ has a k-NU term

$\Downarrow$ $\Downarrow$ Structure Thm $\Uparrow$

no abelian congr. | for critical $\mathbf{R} \leq \mathbf{A}^{\ell}$, $\mathbf{A} \in \mathcal{V}$ $\ell \geq k \Rightarrow \mathbf{R}_i/\theta_i$ has abelian congr. | $\Uparrow$ Charact. Thm

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Applications

Theorem. *For a finite set A and fixed $k \geq 2$ there are only finitely many clones $\mathcal{C}$ on A such that*

- *$\mathcal{C}$ contains a k-▱ operation, and*
- *$(A; \mathcal{C})$ generates a residually small variety.*

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