C_0-semigroups of holomorphic Carathéodory isometries

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BASIC CONCEPTS, NOTATIONS

$\mathbf{E},\ \|\cdot\|$ complex Banach space

$\mathbf{D}$ bounded domain (=open connected set) in $\mathbf{E}$

$\mathrm{Hol}(\mathbf{D}) = \{\text{holomorphic } \mathbf{D} \to \mathbf{D} \text{ maps}\}$

$\mathrm{Aut}(\mathbf{D}) = \{\Phi \in \mathrm{Hol}(\mathbf{D}) : \ \Phi : \mathbf{D} \leftrightarrow \mathbf{D},\ \Phi^{-1} \in \mathrm{Hol}(\mathbf{D})\}$

Taylor series with Fréchet derivatives:

$\Phi(\mathbf{x} + \mathbf{v}) = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^{[n]}(\mathbf{x}) \mathbf{v}^n$

$\Phi^{[n]}(\mathbf{x}) \mathbf{v}_1 \cdots \mathbf{v}_n = \frac{\partial^n}{\partial \zeta_1 \cdots \partial \zeta_n}\Big|_0 \Phi(\mathbf{x} + \zeta_1 \mathbf{v}_1 + \cdots \zeta_n \mathbf{v}_n)$

$\left[\Phi^t : t \in \mathbb{R}_+\right]$ $\mathcal{C}_0$-semigroup of **D**-endomorphisms

$$\Phi^t \in \mathrm{Hol}(\mathbf{D}), \quad \Phi^0 = \mathrm{Id}_{\mathbf{D}}$$
$$\Phi^{t+s} = \Phi^t \circ \Phi^s \quad (t, s \in \mathbb{R}_+)$$
$$t \mapsto \Phi^t(\mathbf{x}) \text{ continuous} \quad (\mathbf{x} \in \mathbf{D})$$

Infinitesimal generator:

$\Phi' : \mathbf{x} \mapsto \lim_{t \searrow 0} t^{-1}\left[\Phi^t(\mathbf{x}) - \mathbf{x}\right]$

Remark. In real-analytic context (population dynamics), there are $\mathcal{C}_0$-semigroups with $\mathrm{dom}(\Phi') = \emptyset$ [Webb85].

Poincaré distance on $\Delta = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$:

$$\omega(\zeta,\alpha) = \operatorname{artanh}\big(M_{-\alpha}(\zeta)\big), \quad M_{-\alpha}(\zeta) = \frac{\zeta-\alpha}{1-\overline{\alpha}\zeta}$$

Carathéodory distance:

$$d_{\mathbf{D}}(x,y) = \sup\big\{\omega\big(\varphi(x),\varphi(y)\big) : \varphi \in \mathrm{Hol}(\mathbf{D},\Delta)\big\}$$

Remark. (1) Up to const. factor: ω unique $\mathrm{Aut}(\Delta)$-invariant metric;

(2) $\Psi \in \mathrm{Hol}(\mathbf{D}_1,\mathbf{D}_2)$ $\quad d_{\mathbf{D}_1} \to d_{\mathbf{D}_2}$ contraction;

(3) $\widetilde{d}_{\mathbf{D}} = d_{\mathbf{D}}^{Kobayashi}$ $\mathrm{Aut}(\mathbf{D})$-inv. metric:

$\widetilde{d}_{\mathbf{D}} \ge d \ge d_{\mathbf{D}}$ if d $\mathrm{Aut}(\Delta)$-inv metric with

Hol. $\Delta \xrightarrow{\psi} \mathbf{D} \xrightarrow{\varphi} \Delta$ maps are $\widetilde{d}_{\mathbf{D}} \to d \to d_{\mathbf{D}}$ contractions

(4) $d_{\mathbf{D}}, \widetilde{d}_{\mathbf{D}}$ locally equiv metrics to $d_{\|\cdot\|}$ on $\mathbf{D}$

Vesentini 1980: $\quad d_{B(\mathbf{D})}(0,x) = \widetilde{d}_{B(\mathbf{D})}(0,x) = \operatorname{artanh}\big(\|x\|\big)$

Lempert (finite dim), **Dineen 1982**: $\quad d_{\mathbf{D}} = \widetilde{d}_{\mathbf{D}}$ for convex $\mathbf{D}$

EARLY HISTORY

Physical motivation $\rightarrow$ symmetric **D**

$$\forall\, \mathbf{x} \in \mathbf{D} \;\; \exists\, S_{\mathbf{x}} \in \mathrm{Aut}(\mathbf{D}) \quad S_{\mathbf{x}}(\mathbf{x}) = \mathbf{x}, \;\; S_{\mathbf{x}}^{[1]}(\mathbf{x}) = -\mathrm{Id}_{\mathbf{E}}$$

E. Cartan 1933: Classification of finite dim. bded. symm. dom.-s
Biholomorphic equivalence ($\simeq$) with direct sums
of unit balls of certain matrix spaces (6 types).

L. Harris 1973: JC*-algebras with Hilbert space op.-s
$\mathcal{L}(\mathbf{H},\mathbf{K})$, $\mathcal{L}_{sym}(\mathbf{H},\bar{\cdot})$, $\mathcal{L}_{antisym}(\mathbf{H},\bar{\cdot})$, [Spin factor]$\hookrightarrow \mathcal{L}(\mathbf{H})$
triple product $\{AB^*C\} = (AB^*C + CB^*A)/2$

J.-P. Vigué 1976: In ∞-dim-s $\mathbf{D} \simeq \mathbf{D}_c$ homog. circ. bded dom.
$\mathbf{D}_c$ circular: $e^{i\tau}\mathbf{D}_c = \mathbf{D}_c \;\; (\tau \in \mathbb{R})$
$\mathbf{D}_c$ homogeneous: $\forall\, x, y \in \mathbf{D} \;\; \exists\, T \in \mathrm{Aut}(\mathbf{D}) \quad T(x) = y$.

W. Kaup 1970: **D** finite dim. circular $\rightarrow \mathrm{Aut}(\mathbf{D})$ partial J*-triple
Loc.unif.1-par.group $\equiv$ unif.cont. 1-prg
$\equiv$flow of a pol. v.-field of 2-nd order

H. Upmeier 1975: *M* Banach manifold, $\mathrm{Aut}(\mathbf{M})$-invariant metric $\rightarrow$
Banach-Lie group structure for $\mathrm{Aut}(\mathbf{M})$ with loc. unif. top. ,
Lie-algebra $\equiv$ { complete hol. vector fields} with a special top.

Kaup 1977: Symm. Hermitian Banach man. $\leftrightarrow$ Banach-Jordan-triples

JB*-TRIPLES

JB*-triple: Banach space with holomorphically symm. unit ball

$\{\mathbf{x}\mathbf{y}^*\mathbf{z}\}$ lin. in $\mathbf{x}, \mathbf{z}$, conj.-lin. in $\mathbf{y}$
$\delta_{\mathbf{a}} = [\mathbf{x} \mapsto i\{\mathbf{a}\mathbf{a}^*\mathbf{x}]\}$ derivation of $\{\ldots\}$ with $\mathrm{Sp}(\frac{1}{i}\delta_{\mathbf{a}}) \geq 0$
$\|\{\mathbf{a}\mathbf{a}^*\mathbf{a}\}\| = \|a\|^3$

Kaup 1983: Bded symm. dom-s $\longleftrightarrow$ unit balls of JB*-triples

1985: Bidualization, von Neumann type Jordan theory
Tools: projection principle [Stachó 82, Kaup 84],
ultrapower embedding of bidual [Dineen 84]

Y. Friedman - B. Russo 1985: Gelfand-Neumark repr. of JB*-triples
Closed subtriples of $\oplus_{\ell\infty}\{\text{Cartan factors}\}$.

Theory with: Loc unif. cont. 1-par. groups

VESENTINI 1980-2001

Aim: $\mathcal{C}_0$-sgr of isometries wrt. to $\mathrm{Aut}(\mathbf{D})$-inv. distances, $\mathrm{Iso}(d_{\mathbf{D}})$

Franzoni-Vesentini 1980: Holomorphic Maps and Invariant Distances
Last chapter: Hilbert ball

Visits of R. Nagel in Pisa $\rightarrow$ linear models, Hille-Yosida theory ?

$\Phi \mapsto \left[f(\in \mathrm{Hol}(\mathbf{D},\mathbf{E})) \mapsto f \circ \Phi\right]$ goes beyond H-Y,
Unsuitable estimates in Fréchet setting as far (future hopes?)

Alternative approach in ∞-dim. reflexive Cartan factors
$\mathbf{E}$ refl, $\mathbf{D} = B(\mathbf{E})$ symm $\rightarrow$ $\mathbf{E} = \left[\text{finite } \ell^\infty\text{-sum of refl. C-factors}\right]$
$\mathrm{Aut}(\mathbf{D})$ reduced by the above decomposition
∞-dim parts $\simeq$ $\mathcal{L}(\mathbf{H},\mathbf{K}), \mathrm{Spin}(\mathbf{H},\bar{\cdot})$, $\mathbf{H},\mathbf{K}$ Hilbert, $\dim(\mathbf{K}) < \infty$.

Remark. $\mathcal{L}(\mathbf{H},\mathbf{K})$ and $\mathrm{Spin}(\mathbf{H})$ are motived by Physics, $\mathbf{H} \simeq \mathcal{L}(\mathbf{H},\mathbb{C})$
In general, $\mathrm{Iso}(d_{\mathbf{B}})$ is also reduced by the factor decomposition [Apazoglou-Peralta, Stachó 2016]

Steps:

1) Generalize Hierzbruch's finite dim. matrix representations
for $\mathrm{Aut}(B(\mathbf{E}))$, $\mathbf{E} = \mathcal{L}(\mathbf{H}, \mathbf{K}), \mathrm{Spin}(\mathbf{H})$

Tools: Carathéodory dist., gen. Möbius trf.,
direct descr. of lin. isometries (with Franzoni),
Cartan uniqueness thm

Required features: lin. C_0 repr. $\longrightarrow$ C_0 sgr.

2) Characterize non-cont. lin. operators corresponsponding to
infinitesimal generators of a C_0-sgr. in Hierzbruch type repr.

Tools: lin. Hille-Yosida theory (before [Engel-Nagel])

3) Describe an integration process for the calculated inf. gen.-s.

Remark. Missing for a "perfect closed" theory:
explicit final formulas, non-continous lin. repr. may represent C_0 sgr-s

ADJUSTED CONTINUITY

Conjecture. (Botelho-Jamison 2008)
If $\mathcal{B} := \{\text{bded } N\text{-lin maps } \mathbf{X}_1 \ldots, \mathbf{X}_N \to \mathbb{C}\}$ with
$[\Lambda^t : t \in \mathbb{R}]$ $\mathcal{C}_0$-group of surj lin isometries $\mathcal{B} \to \mathcal{B}$
then

$$\Lambda^t = \underbrace{U_1^t \otimes \cdots \otimes U_N^t}_{\varphi \mapsto \varphi(U_1^t x_1, \ldots, U_N^t x_N)} \qquad t \mapsto U_k^t \quad \textit{str.cont. 1-par.grp. of surj.lin.isom. } X_k \to X_k$$

Problem. $U_1 \otimes \cdots \otimes U_N = [{\color{red}\kappa_1} U_1] \otimes \cdots \otimes [{\color{red}\kappa_N} V_N]$ if $\prod_j \kappa_j = 1, \kappa_j \in \mathbb{T}$

Stachó [JMAA 2010]: Proof for Hilbert sp $\mathbf{X}_k$, probabilistic argument

$U_1^t \otimes \cdots \otimes U_N^t = [\kappa_{1,t} U_1^t] \otimes \cdots \otimes [\kappa_{N,t} U_N^t]$ with $[\kappa_{j,t} U_j^t : t \in \mathbb{R}]$ $\mathcal{C}_0$-group

HILBERT BALL BY VESENTINI

$\mathbf{H}, \langle\cdot|\cdot\rangle$ Hilbert space, $\quad a^* := [x \mapsto \langle x|a\rangle]$

$x \oplus \xi \equiv \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$, $\quad \mathcal{L}(\mathbf{H} \oplus \mathbb{C}) \equiv \left\{ \left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] : \ A \in \mathcal{L}(\mathbf{H}),\ b, c \in \mathbf{H},\ d \in \mathbb{C} \right\}$

$\mathbf{B} = B(\mathbf{H}),\ \ I = \mathrm{Id}_{\mathbf{H}} \quad \mathcal{J} = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}$

Fractional linear trf.:

$$F \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} : x \mapsto \frac{Ax + b}{c^*x + d} \ \ (c^*x + d \neq 0), \quad \begin{matrix} F(\mathcal{G}) = F(\lambda\mathcal{G}), \\ \lambda \in \mathbb{C} \setminus \{0\} \end{matrix}$$

Matrix repr: $\quad \mathfrak{G} = \{ \mathcal{G} : \mathcal{G}^* \mathcal{J} \mathcal{G} = \mathcal{J} \}$

$\Phi \in \mathrm{Iso}(d_{\mathbf{B}}) \iff \Phi = F(\mathcal{G})$ with $\mathcal{G} \in \mathfrak{G}$

$F(\mathcal{G}_1\mathcal{G}_2) = F(\mathcal{G}_1) \circ F(\mathcal{G}_2)$ on a neighborhood of $\overline{\mathbf{B}}$ if $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{G}$

Ambiguity: $F(\mathcal{G}) = F(\kappa\mathcal{G}),\ \kappa\mathcal{G} \in \mathfrak{G} \quad$ if $\quad |\kappa| = 1,\ \mathcal{G} \in \mathfrak{G}$

Prop. The Möbius decomposition

$$\Phi^t = \Theta_{a_t} \circ U_t,\ \ a_t = \Phi^t(0),\ \ U_t\ \mathbf{H}\text{-isom},\ \ \Theta_{a_t} = F(\mathcal{M}_{a_t}),\ \ U_t = F(\mathcal{U}_t)$$

$$\mathcal{M}_a = \begin{bmatrix} \mathrm{Id}_{\mathbf{H}} - aa^* & 0 \\ 0 & 1 - a^*a \end{bmatrix}^{-1/2} \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix},\ \ \mathcal{U}_t = \begin{bmatrix} U_t & 0 \\ 0 & 1 \end{bmatrix}$$

is norm resp. str. cont. in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$ iff $t \mapsto \Phi^t$ is str. cont.

Remark. Even in 1 dim, $[\mathcal{M}_{a_t}\mathcal{U}_t : t \in \mathbb{R}_+]$ no sgr. in general.
Example: $\Phi^t(\zeta) = \frac{1-it\zeta}{1+it\zeta} \cdot M_{it/(1-it)}(\zeta)$

Theorem. [Ves87, Section 2].

Let $\mathcal{G}^t \in \mathfrak{G}\ (t \geq 0)$ and define $\Phi^t = F(\mathcal{G}^t)\big|_{\mathbf{H}}$.

If $\left[\mathcal{G}^t : t \in \mathbb{R}_+\right]$ is a $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$

then $[\Phi^t : t \in \mathbb{R}_+]$ is a $\mathcal{C}_0$-sgr of holomorphic $d_{\mathbf{B}}$-isometries whose generator Φ' is densely defined (and Φ^t-invariant) in $\mathbf{B}$

Remark. $\Phi^t = F\big(\kappa_t \mathcal{G}^t\big)\big|_{\mathbf{H}}$ with any function $t \mapsto \kappa_t \in \mathbb{T} = \{\text{unit circle}\}$

Theorem. [Ves87, Thms. V+VI with Prop.5.3; corrected in Ves94].

A possibly unbded. lin. op. $\mathcal{A} : \mathrm{dom}(\mathcal{A}) \to \mathbf{H} \oplus \mathbb{C}$
is the infinitesimal generator of a $\mathcal{C}_0$-sgr $[\mathcal{G}^t : t \in \mathbb{R}_+]$ in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$
giving rise to the $\mathcal{C}_0$-sgr $[\Phi^t : t \in \mathbb{R}_+]$, $\Phi^t = F(\mathcal{G}^t)$ of $d_{\mathbf{B}}$-isometries
such that $0 \in \mathrm{dom}(\Phi')$ if and only if

$$\mathcal{A} = \begin{bmatrix} iA + \nu I & b \\ b^* & \nu \end{bmatrix} \qquad \begin{array}{c} \nu \in \mathbb{C}, \quad b \in \mathbf{H} \\ iA = \big[\text{gen. of a } \mathcal{C}_0\text{-sgr of } \mathbf{H}\text{-isometries}\,\big] \end{array}$$

In this case, $\mathrm{dom}(\Phi') = \big\{x \in \mathbf{B} : \ x \oplus 1 \in \mathrm{dom}(\mathcal{A})\big\}$,

$$\Phi'(\mathbf{x}) = \frac{d}{dt}\Big|_{t=0+} F(\mathcal{G}^t)(x) = \frac{d}{dt}\Big|_{t=0+} \frac{\big[\mathcal{G}^t x \oplus 1\big]_{\mathbf{H}}}{\big[\mathcal{G}^t x \oplus 1\big]_{\mathbb{C}}} =$$

$$= b - \langle x|b\rangle x + iAx = b - \{xb^*x\} + iAx$$

Remark. 1) The restriction $0 \in \mathrm{dom}(\Phi')$ is harmless:

Taking any $\mathbf{a} \in \mathrm{dom}(\Phi')$, we can pass to the Möbius-equivalent semigrp

$$[\Psi^t : t \in \mathbb{R}_+], \quad \Psi^t = \Theta_{-a} \circ \Phi^t \circ \Theta_a \quad \text{with} \ \ 0 \in \mathrm{dom}(\Phi').$$

2) [Ves87,Ves 94] suggests to determine Ψ^t by means of the Riccati ODE

$$\dot{x}(t) = b - \{x(t)b^* x(t)\} + iAx(t), \quad x(0) = 0$$

to establish first the orbit $t \mapsto a(t) = \Psi^t(0)(= x(t))$. Then construct lin. ops. U^t for $\mathcal{G}^t = \mathcal{M}_{a(t)}(U^t \oplus \mathrm{Id}_{\mathbb{C}})$ with $\mathcal{G}' = \mathcal{A}$.

No explicit formulas. For final formulas with usual techniques:

complicated Dyson-Philips series for a sgr with gen $\begin{bmatrix} iA & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}$.

FIXED POINTS

Vesentini 1987-94:

1) **E** reflexive JB*-triple, $[\Phi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Iso}(d_{B(\mathbf{E})})$ with lin. Möbius part

$$\Phi^t \longrightarrow \overline{\Phi}^t \quad \text{weak*-cont extension to } \overline{\mathbf{B}},$$

$[\overline{\Phi}^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr wrt. $\|\cdot\|$, $\quad \bigcap_{t\in\mathbb{R}_+} \mathrm{Fix}(\overline{\Phi}^t) \neq \emptyset$

2) **F** refl. Cartan factor, $[\Psi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Iso}(d_{B(\mathbf{F})})$,

$a \in \mathrm{Fix}[\Psi^t : t \in \mathbb{R}_+]$ with $\|a\| < 1 \ \Rightarrow$
$\Phi^t = \Theta_a \circ U^t \circ \Theta_{-a}$ with $[U^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of lin **F**-isometries.

Question. Is U^t linear for generic JB*-triple **E** ? (NO: end of talk)

ADJUSTMENT WITH FIXED POINTS

Stachó [JMAA 2014]

$\mathbf{E} = \mathbf{H}$ Hilbert sp., $e \in \bigcap_t \mathrm{Fix}\overline{\Phi}^t$, $\|e\| = 1$

Thm. $\Phi^t = F(\mathcal{G}^t)$, $\mathcal{G}^t = \begin{bmatrix} A_t & b_t \\ c_t^* & d_t \end{bmatrix}$ $\Rightarrow$

(1) $\mathcal{G}^t \begin{bmatrix} e \\ 1 \end{bmatrix} = \lambda_t \begin{bmatrix} e \\ 1 \end{bmatrix}$ $\lambda_t = \left[\mathcal{G}^t \begin{bmatrix} e \\ 1 \end{bmatrix}\right]_{\mathbb{C}} = \langle e | c_t \rangle + d_t$

(2) $\mathcal{K}^t := \lambda_t^{-1} \mathcal{G}^t$ $(t \in \mathbb{R}_+)$ $\mathcal{C}_0$-sgr, $F(\mathcal{K}^t) = F(\mathcal{G}^t) = \Phi^t$

(2a) $\Phi^t = \Theta_{a_t} \circ U^t$, $t \mapsto a_t = \Phi^t(0)$ cont, $t \mapsto U^t \in \mathrm{Isom}(\mathbf{H})$ str cont;

(2b) $t \mapsto S_t T_t$ str cont if $t \mapsto S_t \in \mathcal{L}(\mathbf{X}_1, \mathbf{X}_2)$, $t \mapsto T_t \in \mathcal{L}(\mathbf{X}_2, \mathbf{X}_3)$

are unif bded str cont functions, $\mathbf{X}_k$ normed sp.

STRUCTURAL CONSEQUENCE

$0 \in \mathrm{dom}(\Phi')$ up to Möbius-equiv

$\Phi^t = F(\mathcal{G}^t) = F(\mathcal{K}^t)$, $[\mathcal{K}^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H} \oplus \mathbb{C})$

$\Phi'(x) = b - \langle x|b\rangle x + iAx$, $\mathrm{dom}(\Phi') = \mathrm{dom}(A)$

$iA = U'$, $[U^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of lin $\mathbf{H}$-isom

$\overline{\Phi}^t(e) = e \in \partial\mathbf{B}$ $(t \in \mathbb{R}_+)$, $e \in \mathrm{dom}(A)$, $\mathcal{K}^t \begin{bmatrix} e \\ 1 \end{bmatrix} = e^{\nu t} \begin{bmatrix} e \\ 1 \end{bmatrix}$

$\mathcal{A} = \mathcal{K}' = \begin{bmatrix} iA & b \\ b^* & 0 \end{bmatrix}$, $\mathrm{dom}(\mathcal{A}) = \mathrm{dom}(A) \oplus \mathbb{C}$ no-loss-gen

$$\mathcal{T} = \begin{bmatrix} \mathrm{Id}_{\mathbf{H}} & e \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \mathrm{Id}_{\mathbf{H}_0} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{H} = \mathbf{H}_0 \oplus \mathbb{C}e$$

$$\mathcal{M}^t = \mathcal{T}^{-1}\mathcal{K}^t\mathcal{T}, \quad \mathcal{M}' = \mathcal{T}^{-1}\mathcal{A}\mathcal{T} = \begin{bmatrix} -\overline{\nu} & 0 & 0 \\ -b_0 & iA_0 & 0 \\ \nu & b_0^* & \nu \end{bmatrix}, \quad \begin{matrix} b_0 = P_{\mathbf{H}_0} b, \\ \nu = \langle e | b \rangle \end{matrix}$$

$A_0 = P_{\mathbf{H}_0} A|_{\mathbf{H}_0}$, $iA_0 = U_0'$ with $[U_0^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of $\mathbf{H}_0$ isom

Thm. There exist constants $\lambda, \mu \in \mathbb{R}$ such that, by setting

$$S = A - i\mu I, \quad S_0 = P_{\mathbf{H}_0} S \big| \mathbf{H}_0, \quad V_0^t = e^{i\mu t} U_0^t, \quad \widetilde{b}_0 = P_{\mathbf{H}_0}(iSb),$$

$$P_e \Phi^t(x) = \Big[\varphi_{\lambda,\mu}(t, P_{\mathbf{H}_0} x, \xi)^{-1}(\xi - 1)e^{-2\lambda t} + 1\Big] e,$$

$$P_{\mathbf{H}_0} \Phi^t(x) = \frac{1}{\varphi_{\lambda,\mu}(t, P_{\mathbf{H}_0} x, \xi)} \Big[e^{-\lambda t} V_0^t P_{\mathbf{H}_0} x - (\xi - 1) e^{-2\lambda t} \big(\textstyle\int_0^t e^{\lambda\tau} V_0^\tau \, d\tau \big) \widetilde{b}_0 \Big],$$

$$\varphi_{\lambda,\mu}(t, z, \xi) := \Big\langle \big(\textstyle\int_0^t e^{-\lambda\tau} V_0^\tau d\tau \big) z \Big| \widetilde{b}_0 \Big\rangle + (\xi - 1)\big(\lambda + i\mu\big) e^{-2\lambda t} + 1 -$$

$$-(\xi - 1) \Big\langle \big(\textstyle\int_0^t e^{-2\lambda\tau} \int_0^\tau e^{\lambda\sigma} V_0^\sigma d\sigma d\tau \big) \widetilde{b}_0 \Big| \widetilde{b}_0 \Big\rangle \; .$$

Corollary. Each str.cont. 1-prsg $[\Psi^t : t \in \mathbb{R}_+]$ of hol. Carathéodory **B**-isometries with exactly two joint boundary fixed points is Möbius equivalent to some semigroup $[\Phi^t : t \in \mathbb{R}_+]$ of the form

$$P_e \Phi^t(x) = \frac{(1-\lambda)(1-\xi)e^{-2\lambda t} + 1}{1-(1-\xi)\lambda e^{-2\lambda t}} e,$$

$$P_{\mathbf{H}_0} \Phi^t(x) = \frac{e^{-\lambda t}}{1-(1-\xi)\lambda e^{-2\lambda t}} V_0^t P_{\mathbf{H}_0} x \ .$$

Dilation Thm. $\exists\, [\widehat{\Phi}^t : t \in \mathbb{R}]$ $\mathcal{C}_0$**-group in** $\mathrm{Aut}(B(\widehat{\mathbf{H}}))$ **with some Hilbert space** $\widehat{\mathbf{H}} \supset \mathbf{H}$ **(as subspace) such that** $\Phi^t = \widehat{\Phi}^t \big| \mathbf{B} \quad (t \in \mathbb{R}_+)$.

Proof: $\exists$ **unitary dilation** $[\widehat{V}_0^t : r \in \mathbb{R}]$ **of the isom sgr** $[V_0^t : t \in \mathbb{R}_+]$**:**

TRO CASE: $\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2),\ \dim(\mathbf{H}_2) < \infty$

Projective representation:

$$F \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \mapsto (AX + B)(CX + D)^{-1} \qquad (\mathbf{H}_1 \oplus \mathbf{H}_2\text{-matrices})$$

Vesentini 1994 $+$ Khatskevich-Reich-Shoikhet 2001

If $[\kappa_t \mathcal{G}^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H}_1 \leftarrow \mathbf{H}_2)$ with $\Phi^t = F(\kappa_t \mathcal{G}_t) \in \mathrm{Iso}(d_{\mathbf{B}})$

then $[\kappa \mathcal{G}]' = \begin{bmatrix} U_1' + \nu I_1 & B \\ B^* & U_2' + \nu I_2 \end{bmatrix}$ with $[U_k^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathcal{L}(\mathbf{H}_k)$

Adjustment κ_t: JORDAN theoretic approach

Möbius decomposition:

$$\mathcal{G}^t = \mathcal{M}_{a(t)}\mathcal{U}_t = \begin{bmatrix} A_t & B_t \\ C_t & B_t \end{bmatrix}, \quad a(t) = \Phi^t(0), \quad \mathcal{U}_t = \begin{bmatrix} U_t & 0 \\ 0 & V_t \end{bmatrix} \quad \begin{matrix} U_t, V_t \\ \text{isometr} \end{matrix}$$

$$\mathcal{M}_a = \begin{bmatrix} (1-aa^*)^{-1/2} & 0 \\ 0 & (1-a^*a)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \qquad (a \in \mathbf{B})$$

1) $\exists\ t \mapsto \mu_t$ with $t \mapsto \mu_t U_t, t \mapsto \mu_t V_t$ str.cont. [Stachó 2012]

2) Assume $t \mapsto U_t, V_t$ str.cont. (no-loss-gen), $E \in \bigcap_t \mathrm{Fix}(\overline{\Phi}^t)$,
Define $S_t := A_t E + B_t$.

Then $\mathcal{G}^t \begin{bmatrix} E \\ I_2 \end{bmatrix} = \begin{bmatrix} E \\ I_2 \end{bmatrix} S_t, \quad S_t S_h = \lambda(t,h) S_{t+h};$

trace argument $\Rightarrow$ $[S_t]_{t \geq 0}$ Abelian, invertible

$\kappa_t = 1/M(S_t)$ with mult.lin. functional $0 \neq M : [S_t]_{t \geq 0} \to \mathbb{C}$

STRUCTURE

No-loss-gen: $[\mathcal{G}^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr, $\Phi^t = F(\mathcal{G}^t)$, $0 \in \mathrm{dom}(\Phi')$, $b = \Phi'(0)$

$$\mathcal{T} = \begin{bmatrix} I_1 & E \\ 0 & I_2 \end{bmatrix}, \quad \mathcal{T}^{-1} = \begin{bmatrix} I_1 & -E \\ 0 & I_2 \end{bmatrix}, \quad F(\mathcal{T}) : X \mapsto X + E;$$

$$\mathcal{T}^{-1}\mathcal{G}^t\mathcal{T} = \begin{bmatrix} A_t EC_t & 0 \\ C_t & S_t \end{bmatrix}, \qquad \mathcal{T}^{-1}\mathcal{G}'\mathcal{T} = \begin{bmatrix} U' - Eb^* & 0 \\ b^* & b^*E - V' \end{bmatrix}.$$

Triangularity $\Rightarrow$ $[W^t : t \in \mathbb{R}_+], [S^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr with

$$W^t = A_t - EC_t,\ W' = U' - Eb^*,$$
$$S^t = S_t = A_t E + B_t,\ S' = b^* E - V'$$

$$\mathcal{T}^{-1}\mathcal{G}^t\mathcal{T} = \begin{bmatrix} W^t & 0 \\ \int_0^t S^{t-h} b^* W^h \, dh & S^t \end{bmatrix}$$

Thm. Since $F(\mathcal{T}) : X \mapsto X + E$, up to Möbius-equivalence

$$\Phi^t(X) = E + W^t(X-E)\Big[\int_0^t S^{t-h} b^* W^h(X-E) + S^t\Big]^{-1}$$

Thm. [Stachó 2017] Up to Möbius-equiv, E tripotent, $0 \in \mathrm{dom}(\Phi')$

Corollary. Triang. wrt. $\mathbf{H}_{10} \oplus \mathbf{H}_{11} \oplus \mathbf{H}_2$, $\mathbf{H}_{10} = \mathrm{range}(E) \longrightarrow$

$$W^t = \begin{bmatrix} W_0^t & 0 \\ \int_0^t W_1^{t-s}\big[P_{11}(U' - Eb^*)P_{10}\big] W_0^s\, ds & W_1^t \end{bmatrix}, \quad \begin{matrix} P_{10} = P_{\mathrm{ran}(E)} = EE^* \\ P_{11} = \mathrm{Id}_{\mathbf{H}_1} - P_{10}, \end{matrix}$$

$W_0^t = \exp\big(tP_{10}[U' - Eb^*]P_{10}\big)$ finite dim,

$[W_1^t : t \in \mathbb{R}_+] \subset \mathcal{L}\big(\mathbf{H}_{11}\big)$, $\mathcal{C}_0$-sgr of isometries, $W_1' = P_{11}U'|\mathbf{H}_{11}$.

SPIN FACTORS

$(\mathbf{H}, \langle\cdot|\cdot\rangle)$ Hilbert space, $x \mapsto \overline{x}$ conjugation, $\langle x|y\rangle^{-} = \langle\overline{x}|\overline{y}\rangle$

$\mathcal{S} := \mathcal{S}(\mathbf{H}, \bar{\cdot})$ JB*-triple

$$\{xa^*y\} = \langle x|a\rangle y + \langle y|a\rangle x - \underbrace{\langle x|\overline{y}\rangle}_{\langle y|\overline{x}\rangle} \overline{a}$$

$e = \{eee\}$ TRIPOTENT:

(1) $e = \lambda v, \quad \lambda \in \mathbb{T}, \quad v \in \mathrm{Re}(\mathbf{H}), \quad \|v\| = 1;$

(2) $e = \lambda u + iv, \quad \lambda \in \mathbb{T}, \quad u, v \in \mathrm{Re}(\mathbf{H}), u \perp v, \quad \|u\| = \|v\| = \frac{1}{2}.$

$\mathcal{S}$-unitary op.s: $U_t = \kappa_t V_t$: $\quad V_t : \mathrm{Re}(\mathcal{S}) \to \mathrm{Re}(\mathcal{S})$ $\langle\cdot|\cdot\rangle$-unitary, $\kappa_t \in \mathbb{T}$.

History. Pauli matrices $\sigma_x = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $\sigma_y = \left[\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right]$, $\sigma_z = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$

$\longrightarrow$ $\mathbb{S}$ cl. selfadj. subs.$\subset \mathcal{L}(\mathbf{H})$, $s^2 \in \mathbb{C}\mathrm{Id}$ $(s \in \mathbb{S})$.

Fractional lin. form for some $\mathcal{C}_0$-groups of inner automorphisms.

Vesentini 1989. No fr.lin. form for gen. $\boldsymbol{\Phi}$ in $\mathbb{S}$-setting

Hierzbruch 1965. finite dim. $\rightarrow$ **Vesentini 1992.**

$$\Phi^t = R\big(G^t\big), \qquad [G^t : t \in \mathbb{R}_+] \quad \mathcal{C}_0 \text{ sgr. in } \mathcal{L}(\mathbf{H} \oplus \mathbb{C}^2).$$

Remark. Continuity adjustment HARMLESS.

$$G^t = \begin{bmatrix} M_t & B_t \\ C_t^{\mathrm{T}} & E^t \end{bmatrix} \qquad B_t = [b_1^t, b_2^t] \in \mathbf{H}^2,\ C_t^{\mathrm{T}} = \begin{bmatrix} \overline{c_1^t}^* \\ \overline{c_2^t}^* \end{bmatrix},\ E = \big[E_{k\ell}\big]_{k,\ell=1}^2$$

MATRIX REPRESENTATION:

$$\Phi^t(x) = R(G^t)(x) = F^t(x)/\varphi^t(x)$$

$$F^t(x) = (b_1^t - ib_2^t) + 2M_t x + (x^{\mathrm{T}}x)(b_1^t + ib_2^t)$$

$$\varphi^t(x) = (E_{11}^t + E_{22}^t - iE_{12}^t + iE_{21}^t) + 2(c_1^t + ic_2^t)^{\mathrm{T}} x + (E_{11}^t - E_{22}^t + iE_{12}^t + iE_{21}^t) x^{\mathrm{T}} x$$

Alg. constrains:

$$[G^t]^* \mathrm{diag}(I, -I_2) G^t = \mathrm{diag}(I, -I_2), \ \ \det(E^t) > 0 \quad (t \in \mathbb{R}_+),$$

$$C_t E^t = M_t^{\mathrm{T}} B_t, \ \ M_t^{\mathrm{T}} = I + C_t C_t^{\mathrm{T}}, \ \ [E^t]^{\mathrm{T}} E^t = I_2 + B_t^{\mathrm{T}} B_t.$$

Proposition. If (up-to Möbius-equiv) $0 \in \mathrm{dom}(\Phi')$ then

$\Phi'(x) = a + iAx - \{xa^*x\}$ is of Kaup's type and

$$G' = \begin{bmatrix} iA - i\varepsilon I & 2\,\mathrm{Re}(a) & -2\,\mathrm{Im}(a) \\ 2\,\mathrm{Re}(a)^{\mathrm{T}} & 0 & -\varepsilon \\ -2\,\mathrm{Im}(a)^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}$$

$\varepsilon \in \mathbb{R}$, $\quad iA = U'$ with $[U^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr of real **H**-isom

TRIANGULARIZATION WITH FIXED POINTS

$0 \neq e \in \bigcup_t \mathrm{Fix}(\overline{\Phi}^t)$ common fixed point

Assumption up to Möbius equiv: e TRIPOTENT

$\Phi'(x) = a + iAx - \{xa^*x\} =$
$\left(\frac{1}{2}b_1 - \frac{i}{2}b_2\right) + M'x + i\varepsilon x - \langle x|b_1 - ib_2\rangle x + \langle x|\overline{x}\rangle\left(\frac{1}{2}b_1 + \frac{i}{2}b_2\right),$

$G' = \begin{bmatrix} M' & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix}$ where $b_1 := 2\mathrm{Re}(a), b_2 := -2\mathrm{Im}(a),$
$M' = \overline{M'} = -[M']^{\mathrm{T}}, \ \varepsilon \in \mathbb{R}.$

Cases up to lin. equiv.

1) $e = \overline{e}$, $\langle e|e\rangle = 1$ (real extreme point),

2) $e \perp \overline{e}$, $\langle e|e\rangle = \frac{1}{2}$ (face middle point).

Case (1): $\mathbf{H}\oplus\mathbb{C}^2=[\mathbb{C}e]\oplus\mathbf{H}_0\oplus\mathbb{C}\oplus\mathbb{C}$ matrix decomposition

$$G'=\begin{bmatrix}M' & b_1 & b_2\\ b_1^{\mathrm{T}} & 0 & -\varepsilon\\ b_2^{\mathrm{T}} & \varepsilon & 0\end{bmatrix}=\begin{bmatrix}0 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon\\ -x_1 & M_1' & x_1 & x_2\\ \rho_1 & x_1^{\mathrm{T}} & 0 & -\varepsilon\\ -\varepsilon & x_2^{\mathrm{T}} & \varepsilon & 0\end{bmatrix}$$

Quasi-triangular form

$$T:=\begin{bmatrix}1/2 & 0 & 0 & 1\\ 0 & I_1 & 0 & 0\\ -1/2 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix},\quad T^{-1}G'T=\begin{bmatrix}-\rho_1 & 0 & 0 & 0\\ -x_1 & M_1' & x_2 & 0\\ -\varepsilon & x_2^{\mathrm{T}} & 0 & 0\\ 0 & x_1^{\mathrm{T}} & -\varepsilon & \rho_1\end{bmatrix}.$$

Remark. (1) $M_1'=U_0'$ with $[U_1^t : t\in\mathbb{R}_+]$ $\mathcal{C}_0$-sgr.

(2) G' triang if $y=\langle z|y\rangle z-M_1'z$ has solution $z\in e^{\perp}$, e.g. if $y\in\mathrm{range}(M_1')$

Case (2): $e = \frac{1}{2}u + \frac{i}{2}v, \quad u \perp v, \ u, v \in \mathrm{Re}(\mathbf{H}), \ \langle u\rangle^2 = \langle v\rangle^2 = 1.$

$$0 = \Phi'(e) = \left(\tfrac{1}{2}b_1 - \tfrac{i}{2}b_2\right) + M'e + i\varepsilon e - \langle e|b_1 - ib_2\rangle e.$$

$$G' = \begin{bmatrix} M' & b_1 & b_2 \\ b_1^{\mathrm{T}} & 0 & -\varepsilon \\ b_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2\rho_2-\varepsilon & x_1^{\mathrm{T}} & \rho_1 & \rho_2 \\ \varepsilon-2\rho_2 & 0 & -x_2^{\mathrm{T}} & \rho_2 & -\rho_1 \\ -x_1 & x_2 & M_2' & x_1 & x_2 \\ \rho_1 & \rho_2 & x_1^{\mathrm{T}} & 0 & -\varepsilon \\ \rho_2 & -\rho_1 & x_2^{\mathrm{T}} & \varepsilon & 0 \end{bmatrix} \begin{matrix} \}\mathbb{C}u \oplus \mathbb{C}v \\ \leftarrow \{u,v\}^{\perp} \\ \}\mathbb{C} \oplus \mathbb{C} \end{matrix}$$

Quasi-triangular $\mathbb{C}^2 \oplus \mathbf{H}_0 \oplus \mathbb{C}^2$**-form** $T^{-1}G'T$

$$\begin{bmatrix} -\rho_1 & \varepsilon-\rho_2 & 0 & 2\varepsilon & 0 \\ \rho_2-\varepsilon & -\rho_1 & 0 & 0 & 2\varepsilon \\ x_2 & -x_1 & M_2' & 0 & 0 \\ \rho_2 & \rho_1 & x_1^{\mathrm{T}} & \rho_1 & -\varepsilon-\rho_2 \\ -\rho_1 & \rho_2 & x_2^{\mathrm{T}} & \rho_2+\varepsilon & \rho_1 \end{bmatrix}, \ T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & I_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Conclusion.
$[G^t : t \in \mathbb{R}]$ can be expressed with finite formulas
which are fractional lin. terms of

some $\mathcal{C}_0$-sgr. of Hilbert space isometries,

Hilbert space operators of rank 1,

some classical special functions,

the solution of a Volterra type scalar convolution equation

admitting closed form with Laplace- and inverse Laplace transforms.

Dilation.
Using a Deddens type $\mathcal{C}_0$-group dilation (with enlarged Hilbert space),
we can construct a $\mathcal{C}_0$-group dilation for $[\Phi^t : t \in \mathbb{R}_+]$
on the unit ball of a suitable covering spin factor.

Open problem. Simplify the procedure with Laplace transform.

TOWARD A HOLOMORPHIC HILLE-YOSIDA THEORY

[Stachó, Rev. Roum. Acad. Sci. 2018]

Basic $\mathcal{C}_0$-principles $\sim$ [Engel-Nagel, Ch. 2]

$D \subset E$ domain, $[\Phi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Hol}(\mathbf{D})$, generic type

$x \in \mathrm{dom}(\Phi') \iff t \mapsto \Phi^t(x)$ differentiable.

$\mathrm{dom}(\Phi')$ is Φ^t-invariant, $\Phi'\big(\Phi^t(x)\big) = \Phi^t\big(\Phi'(x)\big)$;

$\mathrm{graph}(\Phi')$ is rel.closed in $\mathbf{D} \times \mathbf{E}$;

Φ^t is unambigously defd on $\overline{\mathrm{dom}(\Phi')}$.

Proof. With Cauchy estimates $\not\approx$ linear argument

Open problem. $\exists$? $[\Phi^t : t \in \mathbb{R}_+]$ nowhere diff. in t?

Remark. $\exists$ real non-linear dyn. system without inf.gen.

0-FIXING $d_{B(\mathbf{E})}$-ISOMETRIES

$\mathbf{D} = \mathbf{B} = B(\mathbf{E})$ unit ball

Question. Cartan's Linearity Thm. with holomoric $d_{\mathbf{B}}$-isometries?

Counter-ex. [Vesentini 1992]:

$\mathbf{E} := c_0, \ \ \Phi(\zeta_0, \zeta_1, \zeta_2 \ldots) := (\zeta_0^2, \zeta_0, \zeta_1, \zeta_2, \ldots)$

Not suited directly for constructing non-lin $\mathcal{C}_0$-sgr counter-ex.

Proposition. If $\Phi(0) = 0$ then Φ differs from its linear part
only with vectors of tangential directions.

New counter-ex with $\mathcal{C}_0$-sgr: with $\mathbf{E} = C_0(\mathbb{R}_+, \mathbb{C})$,

$$\Phi^t(x) : \mathbb{R}_+ \ni \tau \mapsto \left[\frac{2x(0)}{\left(1 - e^{2(t-\tau)}\right)x(0) + 2e^{2(t-\tau)}} \text{ if } \tau \leq t, \ \ x(\tau - t) \text{ if } \tau \geq t \right]$$

COMMON FIXED POINTS IN JB*-TRIPLES

Setting: **E** JB*-triple,

$M_a = $ [Kaups' Möbius trf $0 \mapsto a$],

$[\Phi^t : t \in \mathbb{R}_+]$ $\mathcal{C}_0$-sgr in $\mathrm{Iso}(d_{B(\mathbf{D})})$.

Thm. Assume (i) $\Phi^t = M_{a(t)} \circ U_t \ (t \in \mathbb{R}_+)$, (ii) $\bigcap_t \mathrm{Fix}(\overline{\Phi}^t) \neq \emptyset$.

Then either $\mathrm{dom}(\Phi')$ dense in $B(\mathbf{D})$,

or $\mathrm{dom}(\Phi') = \emptyset$.

Proof: $\widetilde{M}_a : \|a\|^{-1}\mathbf{B} \to \mathbf{E}$ well-def. hol. extension for M_a [Kaup 1983],

Jordan calculations with the Fréchet derivatives

$\Lambda_t = \widetilde{M}^{[1]}_{a(t)}(e), \quad a(t) = \Phi^t(0)$.

Thanks for your attention

THANKS, VESENTINI

Thanks S.N.S.