Iterációk, iterált pontsorozatok

▪ 1D

n−edik iterált
Legyen f, x rögzített.

x nulladik iterált: $f^{(0)}[x]$

1. iterált $f^{(1)}[x] = f[x]$
2. iterált $f^{(2)}[x] = f[f[x]]$

```
Nest[f, x, 3]

f[f[f[x]]]

NestList[f, x, 3]

{x, f[x], f[f[x]], f[f[f[x]]]}
```

vektorértékû függvényekre is $F : \mathbb{R}^2 \to \mathbb{R}^2$!

```
NestList[F, {x0, y0}, 2]

{{x0, y0}, F[{x0, y0}], F[F[{x0, y0}]]}
```

Kaczmarz−Steinhaus iteráció lin. egyr. megoldására

▪ 2D a

Steinhaus Iteráció, (gépi korlátok, buta gép)

Ax=b alak helyett <a1,x>=b1, <a2,x>=b2; megoldás hipersíkok metszete

Közelítés merôleges vetítések egymás után történô elvégzésével

V1: elsô egyenesre vetítés
V2: második egyenesre vetítés

Iterált sorozat valamely kezdeti $x^{(0)}$ vektorból:
$\{x^{(0)},\ V1[x^{(0)}],\ V2[V1[x^{(0)}]],\ V1[V2[V1[x^{(0)}]]],\ \ldots\}$

Példa:
−x1+97x2=97
x1+x2=2

```
NSolve[{-x1 + 97 x2 == 97, x1 + x2 == 2}, {x1, x2}]
```

$\{\{x1 \to 0.989796,\ x2 \to 1.0102\}\}$

```
Solve[{{y1 - x1, y2 - x2}.{97, 1} == 0, -y1 + 97 y2 == 97}, {y1, y2}]
```

$$\left\{\left\{y1 \to \frac{97\,(-1 + 97\,x1 + x2)}{9410},\ y2 \to \frac{9409 + 97\,x1 + x2}{9410}\right\}\right\}$$

P(x1,x2) V2−re vetített képének koordinátái: V2[P]={y1,y2}

```
Solve[{{y1 - x1, y2 - x2}.{1, -1} == 0, y1 + y2 == 2}, {y1, y2}]
```

$$\left\{\left\{y1 \to \frac{1}{2}\,(2 + x1 - x2),\ y2 \to \frac{1}{2}\,(2 - x1 + x2)\right\}\right\}$$

V1−re vetítés helyett csak y=1−re vetítünk:

```
In[13]:= V11[{x_, y_}] := {x, 1};
```

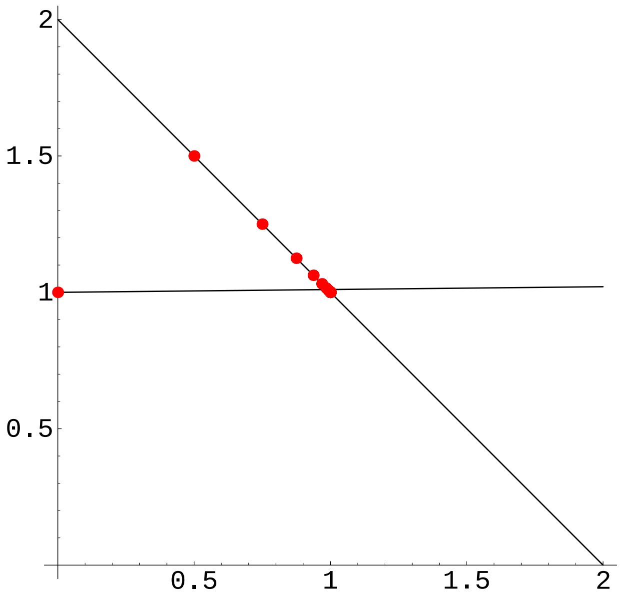
$$\texttt{In[14]:= V2[\{x\_, y\_\}] :=}\ \left\{\frac{1}{2}\,(2 + x - y),\ \frac{1}{2}\,(2 - x + y)\right\}$$

```
In[15]:= NestList[Composition[V2, V11], {0, 1}, 20] // N // TableForm
```

Out[15]//TableForm=

0.	1.
0.5	1.5
0.75	1.25
0.875	1.125
0.9375	1.0625
0.96875	1.03125
0.984375	1.01563
0.992188	1.00781
0.996094	1.00391
0.998047	1.00195
0.999023	1.00098
0.999512	1.00049
0.999756	1.00024
0.999878	1.00012
0.999939	1.00006
0.999969	1.00003
0.999985	1.00002
0.999992	1.00001
0.999996	1.
0.999998	1.
0.999999	1.

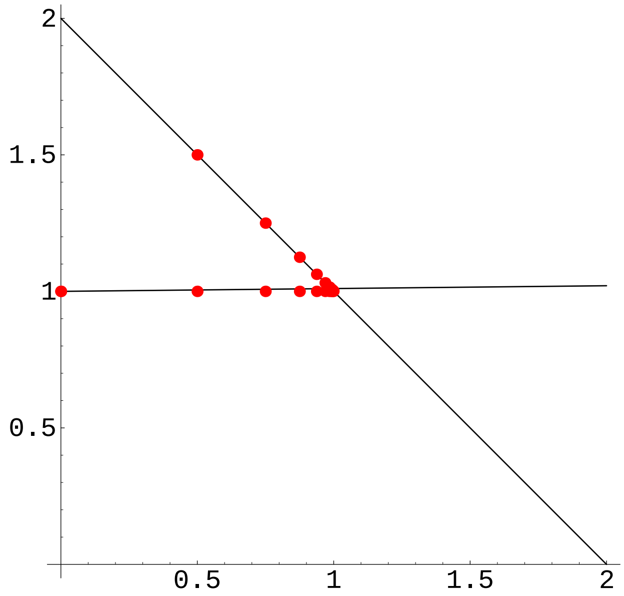
```
In[16]:= Plot[{2 - x, (97 + x) / 97}, {x, 0, 2}, Epilog →
           {RGBColor[1, 0, 0], PointSize[.02], (NestList[Composition[V2, V11], {0, 1}, 20] /.
              {x_?NumberQ, y_?NumberQ} → Point[{x, y}])}, AspectRatio → 1];
```




```
L = True;
```

```
In[17]:= V[v_] := If[L, L = False; V11[v], L = True; V2[v]]
```

```
In[18]:= Plot[{2 - x, (97 + x) / 97}, {x, 0, 2},
           Epilog → {RGBColor[1, 0, 0], PointSize[.02], (NestList[V, {0, 1}, 20] /.
               {x_?NumberQ, y_?NumberQ} → Point[{x, y}])}, AspectRatio → 1];
```



Miért mûködik? Kontrakció, távolságok csökkenek.

▪ 2D b

Elôzô variánsa: Most ténylegesen az elsô egyenesre vetítünk

Ax=b alak helyett <a1,x>=b1, <a2,x>=b2; megoldás hipersíkok metszete
V1: elsô egyenesre vetítés
V2: második egyenesre vetítés

Példa:
−x1+97x2=97
x1+x2=2

```
NSolve[{-x1 + 97 x2 == 97, x1 + x2 == 2}, {x1, x2}]
{{x1 → 0.989796, x2 → 1.0102}}
```

In[7]:= $\mathtt{V1[\{x\_, y\_\}] := \left\{\frac{97\,(-1 + 97\,x + y)}{9410}, \frac{9409 + 97\,x + y}{9410}\right\};}$

In[3]:= V2[{x_, y_}] := $\left\{\frac{1}{2}(2+x-y), \frac{1}{2}(2-x+y)\right\}$

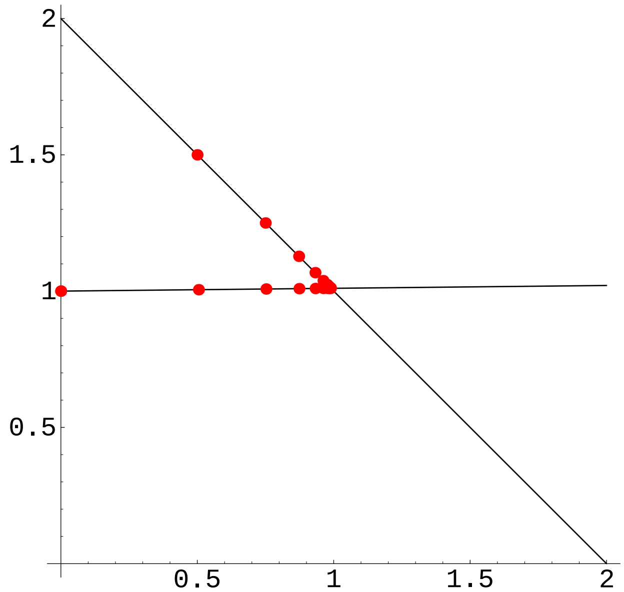
```
(IS2b = NestList[Composition[V2, V1], {0, 1}, 20] // N) // TableForm
```

0.	1.
0.5	1.5
0.749947	1.25005
0.872344	1.12766
0.932281	1.06772
0.961631	1.03837
0.976004	1.024
0.983042	1.01696
0.986489	1.01351
0.988176	1.01182
0.989003	1.011
0.989408	1.01059
0.989606	1.01039
0.989703	1.0103
0.98975	1.01025
0.989774	1.01023
0.989785	1.01022
0.989791	1.01021
0.989793	1.01021
0.989795	1.01021
0.989795	1.0102

In[11]:=
```
L = True;
```

In[8]:=
```
V[v_] := If[L, L = False; V1[v], L = True; V2[v]]
```

In[12]:=
```
Plot[{2 - x, (97 + x) / 97}, {x, 0, 2},
  Epilog → {RGBColor[1, 0, 0], PointSize[.02], (NestList[V, {0, 1}, 20] /.
      {x_?NumberQ, y_?NumberQ} → Point[{x, y}])}, AspectRatio → 1];
```



Összevetés:

```
NSolve[{-x1 + 97 x2 == 97, x1 + x2 == 2}, {x1, x2}]
{{x1 → 0.989796, x2 → 1.0102}}
Last[IS2b]
{0.989795, 1.0102}
```

Miért mûködik? Kontrakció, távolságok csökkenek.

```
In[21]:= Composition[V2, V11][{x, y}]
Out[21]= {(1 + x)/2, (3 - x)/2}
In[26]:= Norm[{1, 2} - {0, 0}]
Out[26]= √5
In[27]:= N[%]
Out[27]= 2.23607
In[28]:= Norm[Composition[V2, V11][{1, 2}] - Composition[V2, V11][{0, 0}]]
Out[28]= 1/√2
```

Képek távolsága már 1-nél kisebb!

```
In[29]:= N[%]
Out[29]= 0.707107
```

1. egyenlet.
<a1,x>=b1
a11 x1+ a12 x2 −b1=0

Ekkor a geometriai megfontolások alapján a vetület koordinátái (ort. proj.+eltolás): n−egyenes normálvektora (pl. itt {a11,a12})

P ’=V(P) =x−<x n>/<n,n> n+b1/<n,n>n $\Longrightarrow x_{n+1} = x_n + \frac{\text{b1}-\langle \text{a1},x\rangle}{\langle \text{a1},,\text{a1}\rangle}$ a1

Példa a második egyenesre: x1+x2=1 a2={a21,a22}={1,1}

```
In[81]:= a2 = {1, 1}; b2 = 2;
In[82]:= v2[{x1_, x2_}] := {x1, x2} + (b2 - a2.{x1, x2}) / Norm[a2]^2
```

In[83]:= `v2[{x1, x2}] // Expand`

Out[83]= $\left\{1+\frac{x1}{2}-\frac{x2}{2},\ 1-\frac{x1}{2}+\frac{x2}{2}\right\}$

In[84]:= `u = a2 / Norm[a2]`

Out[84]= $\left\{\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right\}$

In[85]:= `P = IdentityMatrix[2] - (Transpose[{u}].{u})`

Out[85]= $\left\{\left\{\frac{1}{2},\ -\frac{1}{2}\right\},\ \left\{-\frac{1}{2},\ \frac{1}{2}\right\}\right\}$

In[86]:= `q2 = b2 / Norm[a2] u;`

In[87]:= `P.{x1, x2} + q2`

Out[87]= $\left\{1+\frac{x1}{2}-\frac{x2}{2},\ 1-\frac{x1}{2}+\frac{x2}{2}\right\}$

Ez ugyanaz, mint a V2 képlete. lehetôség mátrixokkal való kifejezére!

Q2=E−u uT+q2

▪ 3D

In[56]:=

$$A = \begin{pmatrix} 7 & 5 & 2 \\ 8 & 4 & 9 \\ 11 & 3 & 3 \end{pmatrix};\ b = \{1, 2, 3\};$$

In[57]:= `LinearSolve[A, b]`

Out[57]= $\left\{\frac{39}{115},\ -\frac{34}{115},\ \frac{6}{115}\right\}$

In[58]:= `N[%]`

Out[58]= {0.33913, −0.295652, 0.0521739}

A sorainak normái

In[59]:= $\texttt{Sqrt[Plus @@ A[[1]]}^2\texttt{] // N}$

Out[59]= 8.83176

In[60]:= $\texttt{MyNorm[v\_] :=} \sqrt{\sum_{i=1}^{\texttt{Length[v]}} \texttt{v[[i]]}^2}$

In[61]:= `MyNorm[A[[1]]]`

Out[61]= $\sqrt{78}$

```
In[62]:= N[%]

Out[62]= 8.83176

In[63]:= norma1 = N[MyNorm[A⟦1⟧]];
         norma2 = N[MyNorm[A⟦2⟧]];
         norma3 = N[MyNorm[A⟦3⟧]];

In[66]:= norma3

Out[66]= 11.7898

In[67]:= 1 / norma1 A⟦1⟧

Out[67]= {0.792594, 0.566139, 0.226455}

In[89]:= 1 / norma2 A⟦2⟧

Out[89]= {0.630488, 0.315244, 0.709299}

In[90]:= 1 / norma3 A⟦3⟧

Out[90]= {0.933008, 0.254457, 0.254457}

In[91]:= e1 = A⟦1⟧ / norma1; e2 = A⟦2⟧ / norma2; e3 = A⟦3⟧ / norma3;

In[92]:= β1 = b⟦1⟧ / norma1; β2 = b⟦2⟧ / norma2; β3 = b⟦3⟧ / norma3;

In[93]:= {e1}

Out[93]= {{0.792594, 0.566139, 0.226455}}

In[94]:= Q1 = IdentityMatrix[3] - (Transpose[{e1}].{e1}); q1 = β1 e1;

In[95]:= Q2 = IdentityMatrix[3] - (Transpose[{e2}].{e2}); q2 = β2 e2;

In[96]:= Q3 = IdentityMatrix[3] - (Transpose[{e3}].{e3}); q3 = β3 e3;

In[97]:= q = Q3.Q2.q1 + Q3.q2 + q3;

In[98]:= QCore[x_] := Q3.Q2.Q1.x + q;

In[99]:= x = {0, 0, 0};

In[100]:=
        NestList[QCore, x, 100] // TableForm

Out[100]//TableForm=
        0              0              0
        0.217603       0.10795        0.0941737
        0.223509       0.0654835      0.114984
        0.230971       0.026077       0.127029
        0.23925        -0.0100844     0.132833
        0.247826       -0.042975      0.134278
        0.256341       -0.0726768     0.132759
        0.264557       -0.099341      0.129298
        0.272324       -0.123161      0.124642
        0.279553       -0.144354      0.119325
        0.286205       -0.163143      0.113726
        0.292266       -0.179752      0.108108
```

0.297749	−0.194396	0.102648
0.302678	−0.207279	0.0974586
0.307086	−0.218592	0.0926082
0.311012	−0.22851	0.088132
0.314495	−0.237192	0.0840422
0.317576	−0.244782	0.0803353
0.320295	−0.251411	0.0769974
0.322687	−0.257195	0.0740079
0.324789	−0.262237	0.0713426
0.326633	−0.266628	0.0689754
0.328247	−0.270452	0.0668796
0.329659	−0.273778	0.0650293
0.330892	−0.27667	0.0633995
0.331968	−0.279183	0.0619671
0.332906	−0.281367	0.0607102
0.333724	−0.283264	0.0596091
0.334436	−0.28491	0.0586458
0.335055	−0.28634	0.057804
0.335594	−0.28758	0.0570692
0.336062	−0.288655	0.0564284
0.336469	−0.289588	0.05587
0.336822	−0.290397	0.0553837
0.337129	−0.291099	0.0549605
0.337395	−0.291707	0.0545924
0.337626	−0.292234	0.0542724
0.337826	−0.292691	0.0539943
0.338	−0.293087	0.0537528
0.338151	−0.29343	0.053543
0.338282	−0.293728	0.053361
0.338395	−0.293985	0.053203
0.338493	−0.294208	0.0530659
0.338579	−0.294402	0.052947
0.338652	−0.294569	0.0528439
0.338716	−0.294714	0.0527545
0.338772	−0.29484	0.0526769
0.33882	−0.294949	0.0526097
0.338861	−0.295043	0.0525515
0.338897	−0.295125	0.052501
0.338929	−0.295195	0.0524573
0.338956	−0.295257	0.0524193
0.338979	−0.29531	0.0523865
0.338999	−0.295355	0.052358
0.339017	−0.295395	0.0523334
0.339032	−0.29543	0.052312
0.339045	−0.29546	0.0522935
0.339057	−0.295485	0.0522775
0.339067	−0.295508	0.0522636
0.339075	−0.295527	0.0522516
0.339083	−0.295544	0.0522412
0.339089	−0.295558	0.0522322
0.339095	−0.295571	0.0522244
0.339099	−0.295582	0.0522176
0.339104	−0.295591	0.0522118
0.339107	−0.295599	0.0522067
0.33911	−0.295607	0.0522023
0.339113	−0.295613	0.0521985
0.339115	−0.295618	0.0521952
0.339117	−0.295623	0.0521923
0.339119	−0.295627	0.0521899

0.339121	−0.29563	0.0521877
0.339122	−0.295633	0.0521859
0.339123	−0.295636	0.0521843
0.339124	−0.295638	0.0521829
0.339125	−0.29564	0.0521817
0.339126	−0.295641	0.0521806
0.339126	−0.295643	0.0521797
0.339127	−0.295644	0.052179
0.339127	−0.295645	0.0521783
0.339128	−0.295646	0.0521777
0.339128	−0.295647	0.0521772
0.339128	−0.295648	0.0521767
0.339129	−0.295648	0.0521764
0.339129	−0.295649	0.052176
0.339129	−0.295649	0.0521758
0.339129	−0.29565	0.0521755
0.339129	−0.29565	0.0521753
0.33913	−0.29565	0.0521751
0.33913	−0.295651	0.0521749
0.33913	−0.295651	0.0521748
0.33913	−0.295651	0.0521747
0.33913	−0.295651	0.0521746
0.33913	−0.295651	0.0521745
0.33913	−0.295651	0.0521744
0.33913	−0.295651	0.0521743
0.33913	−0.295652	0.0521743
0.33913	−0.295652	0.0521742
0.33913	−0.295652	0.0521742
0.33913	−0.295652	0.0521742
0.33913	−0.295652	0.0521741

Newton−Raphson

Adott f[x]=0 egy. továbbá egy x^* valós gyök "durva" közelítése

Pl x^2−2=0 x0=1

F[x]=x−f[x] / ’f ’ [x]

Áll. Bizonyos feltételek mellett $F^{(n)}$[x0] $\longrightarrow x^*$

Példa.

In[101]:=

```
NewtonRaphsonStep[x_] := x - f[x] / f'[x];
```

In[108]:=

```
f[x_] := x^2 - 2;
x0 = 2;
```

In[104]:=
```
NestList[NewtonRaphsonStep, x0, 6]
```

Out[104]=

$$\left\{2, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \frac{886731088897}{627013566048}, \frac{1572584048032918633353217}{1111984844349868137938112}\right\}$$

In[105]:=
```
N[%]
```

Out[105]=
```
{2., 1.5, 1.41667, 1.41422, 1.41421, 1.41421, 1.41421}
```

In[106]:=

$$N\left[\sqrt{2}\right]$$

Out[106]=
```
1.41421
```

In[112]:=
```
Clear[x]
```

In[113]:=
```
NewtonRaphsonStep[x] // Simplify
```

Out[113]=

$$\frac{1}{x} + \frac{x}{2}$$

In[114]:=
```
NestList[NewtonRaphsonStep, 17 / 12, 10] // N
```

Out[114]=
```
{1.41667, 1.41422, 1.41421, 1.41421, 1.41421,
 1.41421, 1.41421, 1.41421, 1.41421, 1.41421, 1.41421}
```

In[115]:=
```
Resolve[ForAll[x, x > Sqrt[2], 1 / x + x / 2 > Sqrt[2]], Reals]
```

Out[115]=
```
True
```

In[116]:=
```
Resolve[ForAll[x, 0 < x < Sqrt[2], 1 / x + x / 2 > Sqrt[2]], Reals]
```

Out[116]=
```
True
```

In[117]:=
```
Resolve[ForAll[x, x > Sqrt[2], 1 / x + x / 2 - Sqrt[2] < x - Sqrt[2]], Reals]
```

Out[117]=
```
True
```

■ Vessük össze!

In[118]:=
```
Solve[6 - x^2 == 3 x^2 - 3]
```

Out[118]=

$$\left\{\left\{x \to -\frac{3}{2}\right\}, \left\{x \to \frac{3}{2}\right\}\right\}$$

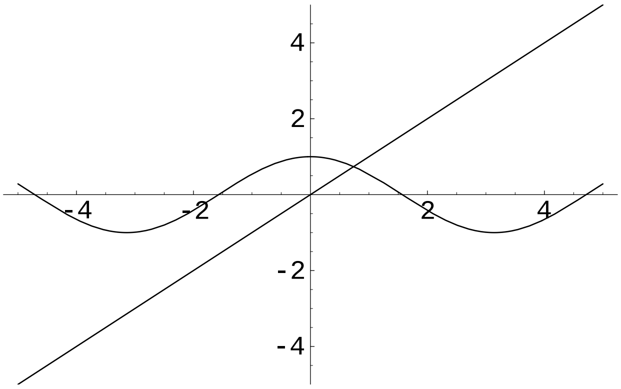
In[119]:=
```
Solve[Cos[x] == x]
```

```
Solve::tdep :
 The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More…
```

Out[119]=
```
Solve[Cos[x] == x]
```

Oldjuk meg numerikusan, közelítsük a zérushely(ek)et iterációval, készítsünk ábrát a kezdõérték megállapítására!
Használhatjuk az ábrát egy kezdõérték abszcisszájának leolvasására is!

In[120]:=
```
Plot[{Cos[x], x}, {x, -5, 5}, PlotRange → {-5, 5}];
```



Legyen f kétszer diff. ∃x∈[a,b]: f[x]=0? pl folytonossági megfontolások
Alapötlet: legyen x_0 egy jó közelítése a gyöknek és használjuk a deriváltat egy jobb közelítés megadására

$$f'[x_n] = (0 - f[x_n])/(x_{n+1} - x_n) \Longrightarrow x_{n+1} = x_n + \frac{-f[x_n]}{f'[x_n]}$$

Egy lépés az iterációban (f globális)

In[121]:=
```
x0 = 1;
f[x_] := Cos[x] - x;
```

In[123]:=
```
NewtonRaphsonStep[x_] := N[x - f[x] / f'[x]];
```

In[124]:=
```
Nest[NewtonRaphsonStep, x0, 2]
```

Out[124]=
```
0.739113
```

In[125]:=
```
FindRoot[f[x] == 0, {x, x0}]
```

Out[125]=
```
{x → 0.739085}
```

In[126]:=
```
NestList[NewtonRaphsonStep, x0, 3]
```

Out[126]=
```
{1, 0.750364, 0.739113, 0.739085}
```

Ezt egy kicsit nehezebb megérteni, mindenki próbálkozzon azért!

In[127]:=
```
NestWhile[NewtonRaphsonStep, x0, Unequal, 2]
```

Out[127]=
```
0.739085
```

Mathematica függvény

In[128]:=
```
FindRoot[f[x] == 0, {x, x0}]
```

Out[128]=
```
{x → 0.739085}
```

■ Feladat

■ Projektmunka: NR Vizualizáció

Egy lépés viz.

In[129]:=
```
Clear[f, g]
```

In[130]:=
```
f[x_] := x^2 - 5;
```

In[131]:=
```
g[x_] := x^3 - 2;
```

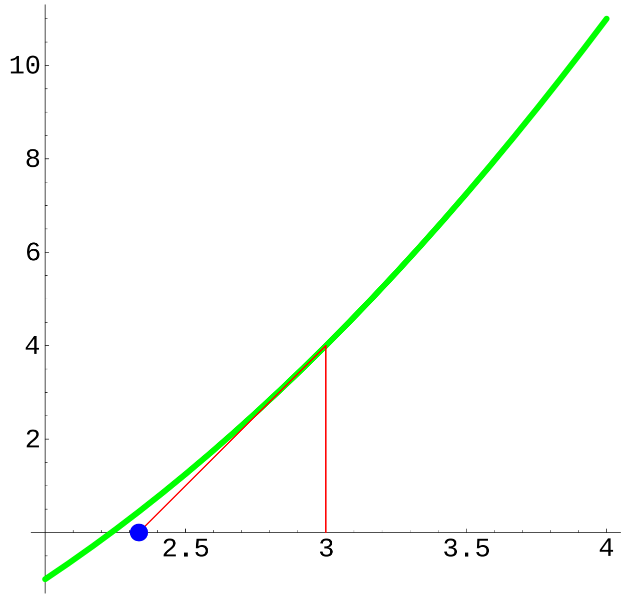
In[132]:=
```
D[f[y], y]
```

Out[132]=
```
2 y
```

In[133]:=
```
NewtonRaphsonVis[f_, x0_] := Module[{x},
  Plot[f[x], {x, x0 - 1, x0 + 1}, PlotStyle → {Thickness[.01], RGBColor[0, 1, 0]},
   AspectRatio → 1, Epilog → {RGBColor[1, 0, 0], Line[
      {{x0, f[x0]}, {-f[x0] / (D[f[x], x] /. x → x0) + x0, 0}}], Line[{{x0, 0}, {x0, f[x0]}}],
     RGBColor[0, 0, 1], PointSize[.03], Point[{-f[x0] / (D[f[x], x] /. x → x0) + x0, 0}]}]]
```

In[134]:=

```
NewtonRaphsonVis[f, 3];
```



In[135]:=

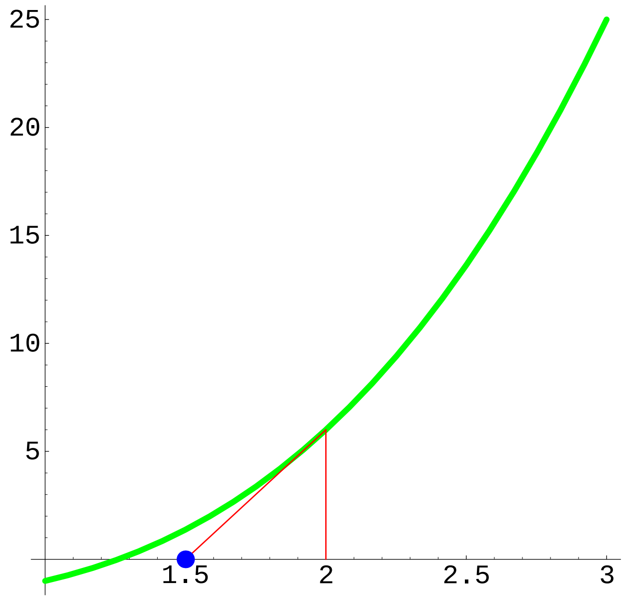
```
Nest[NewtonRaphsonStep, 3, 1]
```

Out[135]=

```
2.33333
```

In[136]:=

```
NewtonRaphsonVis[g, 2];
```



f globális

In[137]:=
```
Clear[f]; f = g;
```

In[138]:=
```
Nest[NewtonRaphsonStep, 2, 1]
```

Out[138]=
```
1.5
```

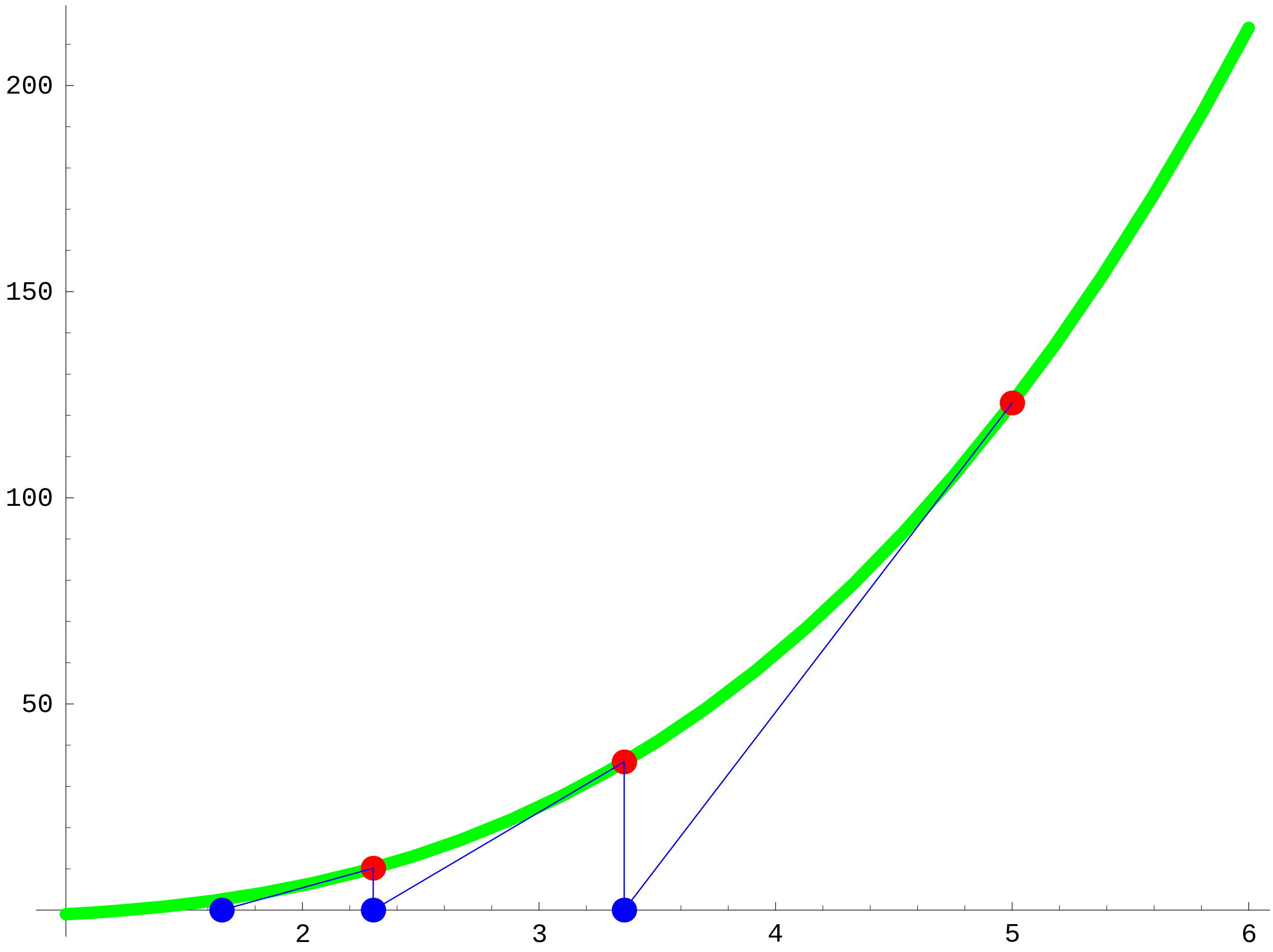
Több lépés

In[139]:=
```
f[x]
```

Out[139]=
$-2 + x^3$

In[141]:=
```
NewtonRaphsonVis[f_, x0_, step_] :=
 Module[{loctable = Transpose[NestList[NewtonRaphsonStep, x0, step] /. x_ -> {x, f[x]}],
   loctable2}, loctable2 = Drop[loctable, 1] /. {x_?NumericQ, y_} → {x, 0};
  Plot[f[x], {x, x0 - 4, x0 + 1}, PlotStyle → {Thickness[.01], RGBColor[0, 1, 0]},
   AspectRatio → .75, Epilog → {RGBColor[1, 0, 0], PointSize[.02],
     Drop[loctable, -1] /. {x_?NumericQ, y_?NumericQ} → Point[{x, y}],
     RGBColor[0, 0, 1], loctable2 /. {x_?NumericQ, 0} → Point[{x, 0}],
     Map[Line[#] &, Transpose[{Drop[loctable, -1], loctable2}]],
     Map[Line[#] &, Transpose[{Drop[loctable, 1], loctable2}]]}, ImageSize → {600, 600}]]
```

In[142]:=
```
NewtonRaphsonVis[f, 5, 3];
```



In[143]:=
```
NestList[NewtonRaphsonStep, 5, 3]
```

Out[143]=
```
{5, 3.36, 2.29905, 1.65883}
```

2D köv óra