1. Introduction

Recently some novel interest seems to be raised toward the symbolic LU decomposition of Vandermonde matrices. Several explicit formulas at various levels along with matrix subfactorizations are well-known for them and their inverses. Our aim in this paper to extend these results to the matrices associated with interpolation problems with Hermite type.

2. Preliminaries

2.1. First we recall briefly the results spread in the literature concerning the LU decomposition of the *Vandermonde* matrix

$$V = V(\mathbf{x}) := \begin{bmatrix} 1 & x_0 & \dots & x_0^N \\ 1 & x_1 & \dots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \dots & x_N^N \end{bmatrix}, \qquad \begin{matrix} \mathbf{x} := \big[x_0, \dots, x_N\big] \\ x_i \neq x_j \ \text{ for } i \neq j \ . \end{matrix}$$

Henceforth N is a fixed positive integer, and we shall consider $(N+1)^2$-matrices with indices ranging from 0 to N. In terms of the formal vectors

$$\delta(\mathbf{x}) := \begin{bmatrix} \delta_{x_0} \\ \delta_{x_1} \\ \vdots \\ \delta_{x_N} \end{bmatrix}, \qquad \mathbf{e}(x) := \big[1, x, x^2, \dots, x^N\big]$$

where x is a variable symbol and δ_a denotes the evaluation functional $p \mapsto p(a)$ defined for polynomials in x of degree$\leq N$, we can write

$$V(\mathbf{x}) = \delta(\mathbf{x})\mathbf{e}(x).$$

Consider the Lagrange interpolation polynomial $p := p_{\mathbf{x},\mathbf{y}}$ defined by the requirements $p(x_n) = y_n$ $(n = 0, \dots, N)$ (with $\mathbf{y} := [y_0, \dots, y_n]^{\mathrm{T}}$). Since we can write p in the form $p = \sum_n p_n x^N = \mathbf{e}(x)\mathbf{p}$, it follows $\mathbf{y} = \delta(\mathbf{x})p_{\mathbf{x},\mathbf{y}}(x) = \delta(\mathbf{x})\mathbf{e}(x)\mathbf{p}_{\mathbf{x},\mathbf{y}} = V(\mathbf{x})\mathbf{p}_{\mathbf{x},\mathbf{y}}$ that is

$$p_{\mathbf{x},\mathbf{y}}(x) = \mathbf{e}(x)V(\mathbf{x})^{-1}\mathbf{y}.$$

The Newtonian form $p(x) = \sum_{n=0}^{N} p(x_0, \dots, x_n)\omega_n(x)$ of this polynomial with

$$\omega_n(x) := \prod_{k:k<n} (x - x_k), \qquad p(x_0, \dots, x_n) = \sum_{j=0}^{n} p(x_j) \prod_{i:j\neq i\leq n} (x_j - x_i)^{-1}$$

(convention: $\omega_0 \equiv 1$) yields the relation $p_{\mathbf{x},\mathbf{y}}(x) = \mathbf{e}(x)\Omega(\mathbf{x})\Delta(\mathbf{x})\mathbf{y}$ with the upper resp. lower triangular matrices

$$\Omega(\mathbf{x}) := \big[\text{coeffs of } \omega_n \text{ in column } n\big]_{n=0}^{N}, \qquad \Delta(\mathbf{x}) := \Big[\prod_{i:j\neq i\leq n} (x_j - x_i)^{-1}\Big]_{\substack{n,j=0 \\ n\geq j}}^{N}.$$

Hence the following triangular decompositions are immediate

$$V(\mathbf{x})^{-1}=\Omega(\mathbf{x})\Delta(\mathbf{x}),\quad V(\mathbf{x})=\Delta(\mathbf{x})^{-1}\Omega(\mathbf{x})^{-1}$$

along with the explicit closed formula

$$\Delta(\mathbf{x})^{-1}=V(\mathbf{x})\Omega(\mathbf{x})=\delta(\mathbf{x})\mathbf{e}(x)\Omega(\mathbf{x})=\big[\omega_n(x_k)\big]_{k,n=0}^{N}.$$

Not stated explicitly in [Monthly] but the arguments can be continued to achieve a shortcut to a closed (i.e. recursion free) formula for the entries of the upper triangular term $\Omega(\mathbf{x})^{-1}$. Indeed, we have

$$\begin{aligned}\Omega(\mathbf{x})^{-1}&=\Delta(\mathbf{x})V(\mathbf{x})=\Delta(\mathbf{x})\delta(\mathbf{x})\mathbf{e}(x)=\Big[\big[\sum_{j=0}^{n}\prod_{i:j\neq i\leq n}(x_j-x_i)^{-1}\big]\delta_{x_j}\Big]_{n=0}^{N}\mathbf{e}(x)=\\ &=\Big[\big[\sum_{j=0}^{n}\prod_{i:j\neq i\leq n}(x_j-x_i)^{-1}\big]\delta_{x_j}(x^\nu)\Big]_{\substack{n,\nu=0\\ n\leq\nu}}^{N}=\begin{bmatrix}\text{Newton difference}\\ x^\nu(x_0,\dots,x_n)\end{bmatrix}_{\substack{n,\nu=0\\ n\leq\nu}}^{N}=\\ &=\Big[\sum_{\substack{i_0+\dots+i_n=\nu-n\\ i_0,i_1,\dots,i_n\geq 0}}x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}\Big]_{\substack{n,\nu=0\\ n\leq\nu}}^{N}.\end{aligned}$$

For any fixed degree ν, the last formula can be obtained by induction on n from the identities $p(x_k,\dots,x_{k+s+1})=\big[p(x_k+1,\dots,x_{k+s+1})-p(x_k,\dots,x_{k+s})\big]/(x_{k+s+1}-x_k)$.

2.2. Next we recall the concept of Hermite (or Hermite-Vandermonde) matrices along with their relationship to Hermite approximation. Henceforth we fix numbers $m_1,\dots,m_r$ such that $(m_1+1)+\ldots+(m_r+1)=N+1$. Given an r-tuple $\mathbf{a}:=[a_1,\dots,a_r]$, along with row matrices $\mathbf{b}_k:=\big[b_k^{(0)},\dots,b_k^{(m_k)}\big]$ $(k=1,\dots,r)$, the Hermite interpolation polynomial $q(x):=q_{\mathbf{a},\mathbf{b}_1,\dots,\mathbf{b}_r}(x)$ is defined as the unique polynomial of degree$\leq N$ satisfying

$$\frac{d^sq}{dx^s}\Big|_{x=a_k}=b_k^{(s)}\qquad(k=1,\dots,r;\quad 0\leq s\leq m_k).$$

For later matrix operations, we divide the integer interval $I:=\{0,1,\dots,N\}$ into consecutive segments

$$I_k:=\{\nu_k^{(0)},\dots,\nu_k^{(m_k)}\},\quad \nu_k^{(s)}:=s+\sum_{\ell:\ell<k}(m_\ell+1)$$

with inverse indices $\kappa(n):=\big[k:n\in I_k\big]$, $\sigma(n):=\big[\text{position of } n \text{ in } I_{\kappa(n)}\big]=\big[s:n=\nu_{\kappa(n)}^{(s)}\big]$. The *Hermite-Vandermonde matrix* over the base point system $\mathbf{a}=[a_1,\dots,a_r]$ of multiorder $\mathbf{m}:=[m_1,\dots,m_r]$ is the $(N+1)^2$-matrix $H=H^{\mathbf{m}}(\mathbf{a})$ of the system of linear equations of the form $\mathbf{q}H^{\mathbf{m}}(\mathbf{a})=[\mathbf{b}_1,\dots,\mathbf{b}_r]$ for the coefficient vector $\mathbf{q}:=[q_0,\dots,q_N]$ of the polynomial $q_{\mathbf{a},\mathbf{b}_1,\dots,\mathbf{b}_r}$. In terms of the linear functionals

$$\delta_a^{(s)}:p\mapsto\frac{d^sp}{dx^s}\Big|_{x=a},\qquad \delta^{\mathbf{m}}(\mathbf{a}):=\big[\delta_{a_1}^{(0)},\dots,\delta_{a_1}^{(m_1)},\dots,\delta_{a_r}^{(0)},\dots,\delta_{a_r}^{(m_r)}\big]^{\mathrm{T}},$$

analogously as in the Vandermonde case ($r = N+1, m_1 = \cdots = m_{N+1} = 0$), we can write

$$H^{\mathbf{m}}(\mathbf{a}) = \delta^{\mathbf{m}}(\mathbf{a})\mathbf{e}(x), \qquad q_{\mathbf{a},\mathbf{b}_1,\ldots,\mathbf{b}_r}(x) = \mathbf{e}(x)H^{\mathbf{m}}(\mathbf{a})^{-1}\mathbf{b} \;\text{ with }\; \mathbf{b} := [\mathbf{b}_1,\ldots,\mathbf{b}_r]^{\mathrm{T}}.$$

It is well known that Hermite interpolation polynomials admit also a Newtonian form

$$q(x) = \sum_{n=0}^{N} q(a_{\nu_0^{(0)}},\ldots,a_{\nu_k^{(s)}})\omega_n(x), \qquad \omega_n(x) := \prod_{i:i<n} (x - a_{\kappa(i)})$$

in terms of generalized Newton differences. As outlined in [Monthly] hence we can get again a triangular decomposition of the form

$$H^{\mathbf{m}}(\mathbf{a})^{-1} = \Omega^{\mathbf{m}}(\mathbf{a})\Delta^{\mathbf{m}}(\mathbf{a})$$

where $\Delta^{\mathbf{m}}(\mathbf{a})$ is a lower triangular matrix whose row with index n contains the coefficients of $q_{\mathbf{a},\mathbf{b}_1,\ldots,\mathbf{b}_r}(a_{\nu_0^{(0)}},\ldots,a_{\nu_k^{(s)}})$ with respect to the variables $b_k^{s)}$ with $\nu(k,s) \le n$, while column n of $\Omega^{\mathbf{m}}(\mathbf{a})$ consists of the coefficients of the polynomial $\omega_n(x)$. Similarly as in the Vandermonde case, we can conclude that

$$\Delta^{\mathbf{m}}(\mathbf{a})^{-1} = \Omega^{\mathbf{m}}(\mathbf{a})H^{\mathbf{m}}(\mathbf{a})) = \delta^{\mathbf{m}}(\mathbf{a})\mathbf{e}(x)\Omega^{\mathbf{m}}(\mathbf{a}) = \Big[\frac{d^{\sigma(i)}}{dx^{\sigma(i)}}\Big|_{x=a_{\kappa(i)}} \omega_n(x)\Big]_{i,n=0}^{N}$$

in the standard LU decomposition $H^{\mathbf{m}}(\mathbf{a}) = \Delta^{\mathbf{m}}(\mathbf{a})^{-1}\Omega^{\mathbf{m}}(\mathbf{a})^{-1}$.

3. Combinatorial formulas of the upper triangular factors

Closed formulas for the entries of the matrices $\Omega^{\mathbf{m}}(\mathbf{a}), \Omega^{\mathbf{m}}(\mathbf{a})^{-1}$ can be obtained simply by a formal substitution of the tuple $\mathbf{x} = (x_0,\ldots,x_N)$ (supposed to have pairwise different entries in Subsection 2.1 with $\mathbf{x} := \mathbf{a}^{\mathbf{m}} = \big(\underbrace{a_0,\ldots,a_0}_{m_0+1},\ldots,\underbrace{a_r,\ldots,a_r}_{m_r+1}\big)$. Namely the diagonal entries are 1 in both cases and, for for the entries of indices $i < j$ we have

$$\begin{aligned}
\big[\Omega^{\mathbf{m}}(\mathbf{a})\big]_{ij} &= \Big[\text{coeff. of } x^i \text{ in} \prod_{k=0}^{j-1}(x - x_k)\Big]\Big|_{\mathbf{x}=\mathbf{a}^{\mathbf{m}}} = \\
&= (-1)^{j-i} \sum_{0\le \ell_1<\ell_2<\cdots<\ell_{j-i}<j} x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{j-i}}\Big|_{\mathbf{x}=\mathbf{a}^{\mathbf{m}}} = \\
&= (-1)^{j-i} \sum_{(k_0,\ldots,k_r)\in K_{ij}} \varrho^{ij}_{(k_0,\ldots,k_r)}\; a_0^{k_0}\cdots a_r^{k_r}
\end{aligned}$$

with the index sets

$$K_{ij} := \Big\{(k_0,\ldots,k_r) \in \Big[\mathop{\times}_{\alpha<\kappa(j-1)}[0,m_\alpha]\Big] \times \big[0,\sigma(\alpha)\big] \times \big\{0\big\}^{r-\kappa(j)} :\; k_0 + \cdots + k_r = j - i\Big\}$$

and respective weight coefficients $\varrho^{ij}_{(k_0,\dots,k_r)} := \Big[\prod_{\alpha<\kappa(j-1)} \binom{m_\alpha}{k_\alpha}\Big]\binom{\sigma(j)}{k_{\kappa(j-1)}}$.

Similarly, for $n<\nu$ we have

$$\big[\Omega^{\mathbf{m}}(\mathbf{a})^{-1}\big]_{n\nu} = \sum_{\substack{i_0+\cdots+i_n=\nu-1\\ i_0,i_1,\dots,i_n\ge 0}} x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}\Big|_{\mathbf{x}=\mathbf{a}^{\mathbf{m}}} = \sum_{(k_0,\dots,k_r)\in\widetilde{K}_{n\nu}} \widetilde{\varrho}^{n\nu}_{(k_0,\dots,k_r)}\, a_0^{k_0}\cdots a_r^{k_r}$$

with $\widetilde{K}_{n\nu} := \Big\{(k_0,\dots,k_r)\in\Big[\mathop{\times}_{\alpha<\kappa(j-1)}[0,m_\alpha]\Big]\times\big[0,\sigma(\alpha)\big]\times\{0\}^{r-\kappa(j)} : k_0+\cdots+k_r=\nu-1\Big\}$,

$$\widetilde{\varrho}^{n\nu}_{(k_0,\dots,k_r)} = \#\Big\{\text{functions } \phi:\{0,\dots,n\}\to\mathbb{Z}_0 \text{ with } \sum_{i\in I_s}\phi(i)=k_s\ \ (s=0,\dots,\kappa(n))\Big\} =$$
$$= \Big[\prod_{s=0}^{\kappa(n-1)}\mu(m_s+1,k_s)\Big]\mu\big(\sigma(n)+1,k_{\kappa(n)}\big)$$

where $\mu(\ell,k)$ denotes the number of all functions $\psi:\{1,\dots,\ell\}\to\mathbb{Z}_+$ with $\sum_i\psi(i)=k$.

4. Combinatorial formulas of the upper triangular factors

In accordance with the partition $I=\bigcup_{k=1}^r I_k$, we partition the vectors $\mathbf{z}:=[z_0,\dots,z_N]$ into subvectors

$$\mathbf{z}_k := \big[z_{\nu(k,0)},z_{\nu(k,1)},\dots,z_{\nu(k,m_k)}\big] \qquad (k=,\dots,r)$$

and consider the corresponding Newton difference matrices

$$\Delta(\mathbf{x}_k) := \Big[\prod_{i:j\ne i\le n}(x_{\nu(k,j)}-x_{\nu(k,i)})^{-1}\Big]^{m_k}_{\substack{n,j=0\\ n\ge j}}.$$

It is well-known from classical analysis [??] that differentiations can be obtained as limits of Newton differences: given any smooth function $\phi:\mathbb{R}\to\mathbb{R}$ along with a net $\mathbf{x}^{(\alpha)}\to\mathbf{a}^{\mathbf{m}}$ with pairwise different terms $x_n^{(\alpha)}\ne x_\nu^{(\alpha)}$ $(0\le n<\nu\le N)$, we have

$$\Delta_k\big(\mathbf{x}_k^{(\alpha)}\big)\Big[\phi\big(x^{(\alpha)}_{\nu(k,0)}\big),\dots,\phi\big(x^{(\alpha)}_{\nu(k,m_k)}\big)\Big]^{\mathrm{T}} \longrightarrow \Big[\phi(a_k),\phi'(a_k),\dots,\frac{d^{m_k}}{dx^{m_k}}\Big|_{a=a_k}\phi(x)\Big]^{\mathrm{T}}.$$

5. Hermite interpolation as limit of Lagrange interpolations

As far as we know no explicit formulas were published for the entries of both $\Delta^{\mathbf{m}}(\mathbf{a})$ and its inverse. We achieve them below by a limiting process from Vandermonde cases.

Proposition. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is a function being $\mathcal{C}^{m_k}$-smooth in suitable neighborhoods of the points a_k with $d^j/dx^j|_{x=a_k}f(x)=b_k^{(j)}$ $(k=1,\dots,r;\ 0\le j\le m_k)$. Assume a net of $(N+1)$-tuples $\mathbf{x}^{[\alpha]}$ consists of points with pairwise different coordinates ($x_i^{[\alpha]}\ne x_j^{[\alpha]}$ for $i\ne j$) and converges to $\mathbf{a}^m$. Then $\mathcal{L}f|\{x_0^{[\alpha]},\dots,x_N^{[\alpha]}\}(x)\to\mathcal{H}\mathbf{F}$.*

Proof. Let $g := f - \mathcal{H}\mathbf{F}$. Notice that $d^j/dx^j|_{x=a_k} f(x) = 0$ $(k = 1,\dots,r;\ 0 \le j \le m_k)$. Since $\mathcal{H}\mathbf{F}$ is a polynomial of degree N, we have $\mathcal{L}\mathcal{H}\mathbf{F} = \mathcal{H}\mathbf{F}$ and hence it suffices to see that

$$\mathcal{L}g|\{x_0^{[\alpha]},\dots,x_N^{[\alpha]}\}(x) \to 0.$$

This can be done by showing that for all its Newton differences,

$$(*) \qquad g\big(x_n^{(\alpha)},\dots,x_{n+s}^{(\alpha)}\big) \to 0 \qquad (s = 0\dots,N;\ 0 \le n \le N-s).$$

We verify this statement by induction on the order index s. For $s = 0$ and fixed $n = (m_1+1)+\dots+(m_k+1)+j-1$ we have $g(x_n^{(\alpha)}) \to g(a_{k+1}) = 0$ because of the continuity of g at the points $a_1,\dots,a_r$ and since $x^{(\alpha)}_{(m_1+1)+\dots+(m_k+1)+j-1} \to a_{k+1}$ by assumption. Assuming $(*)$ for some s, we consider the behavior of $g\big(x_n^{(\alpha)},\dots,x_{n+s+1}^{(\alpha)}\big)$ in two cases: (1) if $x_n^{(\alpha)} \to a_i, x_{n+s+1}^{(\alpha)} \to a_j$ with $i \ne j$ (i.e. $a_i \ne a_j$); (2) if $x_n^{(\alpha)}, x_{n+s+1}^{(\alpha)} \to a_i$. In this case also $x_{n+1}^{(\alpha)},\dots,x_{n+s}^{(\alpha)} \to a_i$. In case (1) we have

$$g\big(x_{n+1}^{(\alpha)},\dots,x_{n+s+1}^{(\alpha)}\big) = \frac{g\big(x_{n+1}^{(\alpha)},\dots,x_{n+s+1}^{(\alpha)}\big) - g\big(x_n^{(\alpha)},\dots,x_{n+s}^{(\alpha)}\big)}{x_{n+s+1}^{(\alpha)} - x_n^{(\alpha)}} \to$$

$$\to \frac{0-0}{a_j - ia_i} = 0.$$

In case (2) we apply the fact that a Newton difference of order $(s+1)$ can be expressed by a derivation of or order $(s+1)$ taken at some location between the most left and right base points:

$$g\big(x_{n+1}^{(\alpha)},\dots,x_{n+s+1}^{(\alpha)}\big) = \frac{d^{s+1}}{dx^{s+1}}\Big|_{x=\theta_\alpha} g(x) \to \frac{d^{s+1}}{dx^{s+1}}\Big|_{x=a_i} g(x) = 0$$

with a suitable net $\theta_\alpha \to a_i$.

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Corollary. *For any $r \in \mathbb{R}$, let $\mathbf{x}^{[t]} := \big[a_k + jt :\ k = 1,\dots,r;\ 0 \le j \le m_k\big]$ $\big($that is $\mathbf{x}^{[t]} = (a_1, a_1+t,\dots,a_1+m_1t, a_2,\dots,a_r+m_rt)\big)$ and let $\mathbf{y}^{[t]} := \big[\sum_{i=0}^{m_k} b_k^{(i)} \frac{[(j-1)t]^i}{i!} :\ 0 \le j \le m_k\big]$. Then, for some $\varepsilon > 0$, the components of the tuples $\mathbf{x}^{[t]}$ are different if $|t| < \varepsilon$. Fixing such a value of ε, with the function germs $\mathbf{f}^{[t]} := \big[x_0^{[t]} \mapsto y_0^{[t]},\dots,x_N^{[t]} \mapsto y_N^{[t]}\big]$ $(-\varepsilon < t < \varepsilon)$ we have $\mathcal{H}\mathbf{F} = \lim_{t\to 0} \mathcal{L}\mathbf{f}^{[t]}$ and $H(\mathbf{a}^m)^{-1}\mathbf{b} = \lim_{t\to 0} V(\mathbf{x}^{[t]})^{-1}\mathbf{y}^{[t]}$.*

Proof.

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Lemma. *With the upper triangular Newton matrices* $N_k := \left[\binom{j}{i}\right]_{0\le i\le j\le m_k}$ *we have*

$$H(\mathbf{a}^{\mathbf{m}}) = \lim_{t\to 0} V(\mathbf{x}^{[t]})\Big[\bigoplus_{k=1}^{r}\big[N_k\ \mathrm{diag}(1^{-1},t,\ldots,t^{-m_k})\big]\Big].$$

Proof.
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5. Closed formulas for the entries of the upper triangular factors.

Lemma. *For the subdiagonal matrix* $S := \big[1_{\{k\}}(\ell+1)k\big]_{k,\ell=0}^{N} = \sum_{k=1}^{N} k\ \mathbf{e}_k\mathbf{e}_{k-1}^{\mathrm{T}}$ *we have*

$$\exp(tS) = \left[t^{k-\ell}\binom{k}{\ell}\right]_{k,\ell=0}^{N} = \sum_{k=0}^{N}\sum_{\ell=0}^{k} t^{k-\ell}\binom{k}{\ell}\mathbf{e}_k\mathbf{e}_\ell^{\mathrm{T}}.$$

Proof. The n-th power of a subdiagonal matrix has non-zero entries only in the n-th skew row below the diagonal. Thus we can write

$$S^n := \Big[1_{\{k\}}(\ell+n)\sigma_\ell^{(n)}\Big]_{k,\ell=0}^{N} = \sum_{\ell=0}^{N-n}\sigma_\ell^{(n)}\mathbf{e}_{\ell+n}\mathbf{e}_\ell^{\mathrm{T}}.$$

We have the recursion

$$\begin{aligned}
&\sum_{\ell=0}^{N-(n+1)}\sigma_\ell^{(n+1)}\mathbf{e}_{\ell+n+1}\mathbf{e}_\ell^{\mathrm{T}} = S^{n+1} = \\
&= S^nS = \Big[\sum_{\ell=0}^{N-n}\sigma_\ell^{(n)}\mathbf{e}_{\ell+n}\mathbf{e}_\ell^{\mathrm{T}}\Big]\Big[\sum_{k=1}^{N}k\ \mathbf{e}_k\mathbf{e}_{k-1}^{\mathrm{T}}\Big] = \sum_{\ell=1}^{N-n}\sigma_\ell^{(n)}\ell\ \mathbf{e}_{\ell+n}\mathbf{e}_{\ell-1}^{\mathrm{T}};\\
&\sigma_\ell^{(n+1)} = \sigma_{\ell+1}^{(n)} \qquad (\ell=0,\ldots,N-(n+1)).
\end{aligned}$$

Taking into account the definition of S implying $\sigma_\ell^{(1)} = \ell+1$, we conclude by induction on n that $\sigma_\ell^{(n)} = (\ell+1)(\ell+2)\cdots(\ell+n) = (\ell+n)!/\ell!$ in all cases. It follows $\exp(tS) = \sum_{n=0}^{N}\frac{t^n}{n!}S^n = \sum_{n=0}^{N}t^n\sum_{\ell=0}^{N-n}\frac{(\ell+n)!}{n!\ell!}\mathbf{e}_{\ell+n}\mathbf{e}_\ell^{\mathrm{T}} = \sum_{k=0}^{N}\sum_{\ell=0}^{k}t^{k-\ell}\binom{k}{\ell}\mathbf{e}_k\mathbf{e}_\ell^{\mathrm{T}}$.
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References

$\mathbf{v}(x) := \big[1, x, x^2, \ldots, x^N\big]$
$\mathbf{x} := \big[x_0, x_1, x_2, \ldots, x_N\big]$
$\mathbf{y} := \big[y_0, y_1, y_2, \ldots, y_N\big]$
$\mathbf{f} := \big[(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)\big]$ **function germ**
$V(\mathbf{x}) := \begin{bmatrix} \mathbf{v}(x_0) \\ \vdots \\ \mathbf{v}(x_N) \end{bmatrix}$ **Vandermonde matrix**
$\mathcal{L}\mathbf{f}(x) := \big[$**Lagrange polynomial of f with symbolic variable** $x\big]$
$\mathcal{L}\mathbf{f}(x) = c_0 + c_1 x + \cdots c_N x^N = \mathbf{v}(x)\mathbf{c}$ **where** $\mathbf{c} := \big[c_0, c_1, c_2, \ldots, c_N\big]^{\mathrm{T}}$
$V(\mathbf{x})\mathbf{c} = \mathbf{y}$ **interpolation equations**
$\mathcal{L}\mathbf{f}(x) = \mathbf{v}(x)V(\mathbf{x})^{-1}\mathbf{y}$
$\mathcal{L}\mathbf{f}(x) = \mathbf{f}(x_0) + \mathbf{f}(x_0, x_1)(x - x_0) + \cdots + \mathbf{f}(x_0, \ldots, x_N)(x - x_0)\cdots(x - x_{N-1})$ **Newton form**
$\mathbf{f}(x_0, \ldots, x_k) = \sum_{j=0}^{k} y_j \prod_{i: j \neq i \leq k} (x_k - x_i)^{-1} = \mathbf{d}_k(\mathbf{x})\mathbf{y}$ **Newton difference quotients**
$\omega_k(x) := (x - x_0)\cdots(x - x_{k-1})$, $\omega_0(x) := 1$
$\omega_k(x) = \sum_{j=0}^{k} w_{k,j} x^j = \mathbf{v}(x)\mathbf{w}_k$
$\Omega(\mathbf{x}) := \big[\mathbf{w}_0(\mathbf{x}), \ldots, \mathbf{w}_N(\mathbf{x})\big]$ **upper triangular matrix**
$D\mathbf{f}(\mathbf{x}) := \begin{bmatrix} \mathbf{d}_0(\mathbf{x}) \\ \vdots \\ \mathbf{d}_N(\mathbf{x}) \end{bmatrix}$ **lower triangular matrix**
$\mathcal{L}\mathbf{f}(x) = \mathbf{v}(x)\Omega(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{y}$
$V(\mathbf{x})^{-1} = \Omega(\mathbf{x})D\mathbf{f}(\mathbf{x})$ **triangular decomposition**

$\mathbf{a} := \big[a_1, \ldots, a_r\big]$, $r \leq N + 1$
$\mathbf{b} := \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_r \end{bmatrix}$, $\mathbf{b}_k := \begin{bmatrix} b_k^{(0)} \\ \vdots \\ b_k^{(m_k)} \end{bmatrix}$
$(m_1 + 1) + \cdots + (m_r + 1) = N + 1$
$\mathbf{F} := \big[\big((a_1, \mathbf{b}_1), \ldots, (a_r, \mathbf{b}_r)\big]$ **function germ with derivatives**
$\mathcal{H}\mathbf{F}(x) := \big[$**Hermite polynomial of F with symbolic variable** $x\big]$
$\frac{d^j}{dx^j}\Big|_{x=a_k} \mathcal{H}\mathbf{F}(x) = b_k^{(j)}$ $(k = 1, \ldots, r;\ 0 \leq j \leq m_k)$ **interpolation equations**
Lagrangian case: $r = N + 1,\ m_1 = \cdots = m_{N+1} = 0$
$\mathcal{H}\mathbf{F}(x) = \mathbf{v}(x)\mathbf{c}$ **classical polynomial form,** $\mathbf{c} := \big[c_0, \ldots, c_N\big]$
$\mathbf{m} := \big[m_1, \ldots, m_r\big]$
$T_m(a) := \begin{bmatrix} \mathbf{v}(a) \\ d/dx|_{x=a}\mathbf{v}(x) \\ \vdots \\ d^m/dx^m|_{x=a}\mathbf{v}(x) \end{bmatrix}$ **Taylor matrices**

$$H(\mathbf{a^m}) := \begin{bmatrix} T_{m_1}(a_1) \\ \vdots \\ T_{m_r}(a_r) \end{bmatrix} \quad \textbf{Hermite matrix}$$

$H(\mathbf{a^m})\mathbf{c} = \mathbf{b}$ **interpolation equations**

$\mathcal{H}\mathbf{F}(x) = \mathbf{v}(x)H(\mathbf{a^m})^{-1}\mathbf{b}$

We get the Newtonian form of $\mathcal{H}\mathbf{F}(x)$ as a limit of Vandermode cases

$\mathbf{a^m}(t) := \big[a_1, a_1+t, \ldots, a_1+m_1t, \ldots, a_r, a_r+t, \ldots, a_r+m_rt\big]$

$\mathbf{b}(t) := \Big[b_1^{(0)}, b_1^{(0)}+b_1^{(1)}t, \ldots, \sum_{j=0}^{m_1} b_1^{(j)}t^j/j!, \ldots, b_r^{(0)}, b_r^{(0)}+b_r^{(1)}t, \ldots, \sum_{j=0}^{m_r} b_r^{(j)}t^j/j!\Big]^{\mathrm{T}}$

$\mathbf{f}_t := \Big[\big(a_1, b_1^{(0)}\big), \big(a_1, b_1^{(0)}+b_1^{(1)}(t)\big), \ldots, \big(a_r+m_rt, \sum\limits_{j=0}^{m_1} b_1^{(j)}t^j/j!\big)\Big]$ **pairing of $\mathbf{a^m}(t), \mathbf{b}(t)$**

$\mathcal{H}\mathbf{F}(x) = \lim\limits_{t\to 0} \mathcal{L}\mathbf{f}_t(x)$