INTEGRÁLOK PARAMÉTEREZETT FELÜLETEN

$Q : [a_1, b_1] \times \cdots \times [a_K, b_K] \to \mathbb{R}^N$ folyt diff $0 < K \le N$

$T = \begin{pmatrix} t_1 \\ \vdots \\ t_K \end{pmatrix}$: koord $\mathbb{R}^K$-n $\quad t_i : \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_K \end{pmatrix} \mapsto \tau_i$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ koord $\mathbb{R}^N$-en $\quad x_j : \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} \mapsto \xi_j$.

$Q = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{bmatrix} q_1(t_1, \ldots, t_K) \\ \vdots \\ q_N(t_1, \ldots, t_K) \end{bmatrix} \quad q_j = x_j(Q)$

$$\left.\begin{array}{l} a_1 = \tau_{1,0} < \tau_{1,1} < \cdots < \tau_{1,n} = b_1 \\ a_2 = \tau_{2,0} < \tau_{2,1} < \cdots < \tau_{2,n} = b_2 \\ \quad\vdots \qquad\qquad\qquad\quad \vdots \qquad \vdots \\ a_K = \tau_{K,0} < \tau_{K,1} < \cdots < \tau_{K,n} = b_K \end{array}\right\} \text{egyenletes beosztások}$$

$g_1, \ldots, g_K : \mathbb{R}^N \to \mathbb{R}$ folyt diff

pl. $g_1 = x_{j_1}, \ldots, g_K = x_{j_K}$ (LEHET AZONOS $g_1 = \ldots g_K = x_j$)

$f : \mathbb{R}^N \to \mathbb{R}$ folyt diff $\quad f = f(x_1, \ldots, x_N)$

$$\sum_{i_1, \ldots, i_K = 0}^{N-1} f(Q(\underbrace{\tau_{i_1}, \ldots, \tau_{i_K}}_{\vec{\tau}_i})) \prod_{d=1}^{K} \Big[g_d(Q(\tau_{i_1}, \ldots, \tau_{i_d+1}, \ldots, \tau_{i_K})) - \underbrace{g_d(Q(\tau_{i_1}, \ldots, \tau_{i_K}))}_{\Delta_d g_d(\vec{\tau}_i)} \Big] =$$

$$= \sum_i f(Q(\vec{t}_i)) \prod_{d=1}^{K} \Delta_d g_d(\vec{t}_i) =$$

$$= \sum_i f(Q(\vec{t}_i)) \; \Delta_1 g_1(\vec{t}_i) \otimes \cdots \otimes \Delta_K g_K(\vec{t}_i) \rightarrow$$

$$\xrightarrow[n\to\infty]{} \int_Q f \, dg_1 \otimes dg_2 \otimes \cdots \otimes dg_K =$$

Pl $$= \int_{t_1=a_1}^{b_1} \cdots \int_{t_K=a_K}^{b_K} f(Q(t_1,\ldots,t_K)) \prod_{d=1}^{K} \frac{\partial g_d(Q(t_1,\ldots,t_K))}{\partial t_d} \, dt_1 \ldots dt_K =$$

Pl: 

8-ad GÖMB $\quad Q : [0, \pi/2]^2 \to \mathbb{R}^3$

$$\begin{pmatrix} \varphi \\ \vartheta \end{pmatrix} \mapsto \begin{pmatrix} \cos\varphi \cos\vartheta \\ \sin\varphi \cos\vartheta \\ \sin\vartheta \end{pmatrix}$$

$$\int_Q xy \, dx \otimes dz = \int_{\varphi=0}^{\pi/2} \int_{\vartheta=0}^{\pi/2} [\cos\varphi \sin\varphi \cos^2\vartheta] \frac{\partial x}{\partial \varphi} \frac{\partial z}{\partial \vartheta} \, d\varphi \, d\vartheta =$$

$$x = x(Q) = \cos\varphi \cos\vartheta$$
$$y = y(Q) = \sin\varphi \cos\vartheta$$
$$z = z(Q) = \sin\vartheta$$

$$= \int_{\varphi,\vartheta=0}^{\pi/2} [\cos\varphi \sin\varphi \cos^2\vartheta][-\sin\varphi \cos\vartheta][\cos\vartheta] \, d\varphi \, d\vartheta =$$

$$= \int_{\varphi,\vartheta=0}^{\pi/2} [-\sin^2\varphi \cos^2\varphi \cos^4\vartheta] \, d\varphi \, d\vartheta = -\frac{3}{256}\pi^2$$

UGYANAKKOR $\int_Q xy\, dz\otimes dx = \int [\cos\varphi \sin\varphi \cos\vartheta] \frac{\partial z}{\partial \varphi}\frac{\partial x}{\partial \vartheta}\, d\varphi\, d\vartheta$

$= 0$ úgyh. 0

Differenciál-formák

$f\, d\vartheta_1\otimes\cdots\otimes d\vartheta_k \; : \; Q \mapsto \int_Q f\, d\vartheta_1\otimes\cdots\otimes d\vartheta_k$

„FELÜLETRE ÉRTHETEN" a forma

Külső formák

$$f\, d\vartheta_1\wedge\cdots\wedge d\vartheta_k = \sum_{\pi\in[\{1,\dots,k\}\text{ PERMUTÁCIÓI}]} (-1)^{\mathrm{par}(\pi)} f\, d\vartheta_{\pi(1)}\otimes\cdots\otimes d\vartheta_{\pi(k)}$$

Külső derivált [CSAK ALAPKOORDINÁTÁKKAL]

$\Omega = f\, dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ külső k-forma $\mathbb{R}^N$-en

$$d\Omega := \sum_{d=1}^{N} (-1)^{d-1}\frac{\partial f}{\partial x_d}\, dx_d\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

STOKES TÉTEL $\int_{\partial Q}\Omega = \int_Q d\Omega$

∂Q — Q HATÁRA

Pl $(|x|+|y|+|z|<1)$ oktaéder határán $v=\begin{pmatrix}2z\\2z\\z\end{pmatrix}$ sebességgel víz megy át

$\Phi = [\text{KIFOLYÁSI SEBESSÉG}] = ?$

$$\Phi = \int_{\partial Q} [\underbrace{x(V)\, dy\wedge dz + y(V)\, dz\wedge dx + z(V)\, dx\wedge dy}_{\Omega}] =$$

$$= \sum_{8\,\text{LAP}} \int_{\text{LAP}_i} \Omega$$ NAGYON SOK MUNKA

Stokes ⇒ $\Phi = \int_Q d\Omega = \int_Q d[2z\, dy\wedge dz + 2z\, dz\wedge dx + z\, dx\wedge dy] =$

$$= \int_Q [\underbrace{2\, dz\wedge dy\wedge dz}_{0} + \underbrace{2\, dz\wedge dz\wedge dx}_{0} + dz\wedge dx\wedge dy] =$$

$$= \int_Q dz\wedge dx\wedge dy = \int_Q [-dx\wedge dz\wedge dy] = \int_Q dx\wedge dy\wedge dz =$$

$= [Q$ TÉRFOGATA$] =$  $= 2\cdot$ PIRAMIS $= 2\cdot\frac{1}{3}\sqrt{2}^2\cdot 1 =$

$= \underline{\underline{4/3}}$

Maxwell-egyenletek 3D-ben

1) $f: \mathbb{R}^3\to\mathbb{R}$ $\qquad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ GRADIENS

HAGYOMÁNYOSAN $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$F: \mathbb{R}^3\to\mathbb{R}^3$ ERŐTÉR! $\quad P: [a,b]\to\mathbb{R}^3$ út

$[F$ munkája P úton$] = \int_P [x(F)\,dx + y(F)\,dy + z(F)\,dz]$

VONALINTEGRÁL

Stokes $\Rightarrow$ $U: \mathbb{R}^3 \to \mathbb{R}$ potenciál-ra
∇U erőtér

$$U(b) - U(a) = \int_P dU = \int_P \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

2) $V: \mathbb{R}^3 \to \mathbb{R}^3$ folyási sebesség

$S: [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}^3$ felület

Fluxus

$$\Phi = [V \text{ átfolyása } S\text{-en}] = \int_S [x(V)\, dy \wedge dz + y(V)\, dz \wedge dx + z(V)\, dx \wedge dy]$$

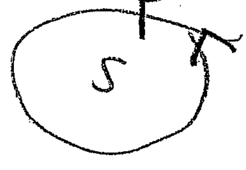
Stokes $\Rightarrow$ HA S EGY B TEST HATÁRA, AKKOR

$$[V \text{ átfolyása } S\text{-en}] = \int_B d[x(V)\, dy \wedge dz + \cdots] =$$

$$= \int_B \underbrace{\left[\frac{\partial x(V)}{\partial x} + \frac{\partial y(V)}{\partial y} + \frac{\partial z(V)}{\partial z}\right]}_{} dx \wedge dy \wedge dz$$

DIVERGENCIA $\langle \nabla, V \rangle = \mathrm{div}(V)$

3) ROTÁCIÓ

HA 

AZ S FELÜLETNEK HATÁRA P, AKKOR

$$[F \text{ munkája } P\text{-n}] = \int_P (x(F)\, dx + y(F)\, dy + z(F)\, dz) = \int_S d[x(F)\, dx + \cdots] =$$

$$= \int_S [\text{[illegible]} - \text{[illegible]}]\, dz \wedge dx +$$

$$= \int_S \left[\frac{\partial x(F)}{\partial y} dy \wedge dx + \frac{\partial x(F)}{\partial z} dz \wedge dx + \right.$$

$$+ \frac{\partial y(F)}{\partial x} dx \wedge dy + \frac{\partial y(F)}{\partial z} dz \wedge dy +$$

$$\left. + \frac{\partial z(F)}{\partial x} dx \wedge dz + \frac{\partial z(F)}{\partial y} dy \wedge dz \right] =$$

$$= \int_S \left[\left(\frac{\partial y(F)}{\partial x} - \frac{\partial x(F)}{\partial y} \right) dx \wedge dy + \left(\frac{\partial z(F)}{\partial y} - \frac{\partial y(F)}{\partial z} \right) dy \wedge dz + \right.$$

$$\left. + \left(\frac{\partial x(F)}{\partial z} - \frac{\partial z(F)}{\partial x} \right) dz \wedge dx \right]$$

$\uparrow$

$\mathrm{rot}(F) = \nabla \times F$ komponensei:

x komponens: $dy \wedge dz$

y komponens: $dz \wedge dx$

z komponens: $dx \wedge dy$ együtthatója

Többváltozós helyettesítés

1 vált
$$\int_{\Omega} f(x)\,dx \underset{x = g(t)}{\overset{\uparrow}{=}} \int_{g^{-1}(\Omega)} f(\underbrace{g(t)}_{x}) \underbrace{\frac{\partial g}{\partial t} dt}_{dx}$$

k vált
$$\int_{x \in \Omega} f(x) \underbrace{dx_1 \dots dx_k}_{dx_1 \wedge \dots \wedge dx_k} \underset{\substack{x_1 = g_1(t_1, \dots, t_k) \\ \vdots \\ x_k = g_k(t_1, \dots, t_k)}}{\overset{\uparrow}{=}} \int_{t \in g^{-1}(\Omega)} f(g(t))\, \underbrace{dg_1(t)}_{\frac{\partial g_1}{\partial t_1} dt_1 + \dots + \frac{\partial g_1}{\partial t_k} dt_k} \wedge \dots \wedge dg_k(t) =$$

$$= \int_{t \in g^{-1}(\Omega)} f(g(t)) \left[\det \frac{\partial g}{\partial t} \right] dt_1 \dots dt_k \qquad x = g(t)$$