VALÓJÁBAN A NEWTON-LEIBNIZ-tétel ÁLLT.

j-EDIK LAPPÁR ∂Q-ban

$$L_1^j = [(t_1,\dots,\cancel{t_j},\dots,t_k) \mapsto Q(t_1,\dots,a_j,\dots,t_k)]$$

$$L_2^j = [(t_1,\dots,\cancel{t_j},\dots,t_k) \mapsto Q(t_1,\dots,b_j,\dots,t_k)]$$

$$\int_{L_2^j} f\, dx_{i_1}\wedge\dots\wedge dx_{i_{k-1}} - \int_{L_1^j} f\, dx_{i_1}\wedge\dots\wedge dx_{i_{k-1}} =$$

$$= \int_{t_1=a_1}^{b_1}\dots\cancel{\int_{t_j=a_j}^{b_j}}\dots\int_{t_k=a_k}^{b_k} f(Q(t_1,\dots,b_j,\dots,t_k))\, \det\left[\partial Q_{i_p}/\partial t_r\right]_{p=1,\ r=1,\ r\neq j}^{k-1,\ k}\, dt_1\dots\cancel{dt_j}\dots dt_k$$

↑ EGY k MÁTRIX − j. OSZLOPA

$|a_k|$

− UAZ

$$= \int_{t_1=a_1}^{b_1}\dots\cancel{\int_{t_j=a_j}^{b_j}}\dots\int_{t_k=a_k}^{b_k}\int_{t_j=a_j}^{b_j}\left[\frac{d}{dt_j} f(Q(t_1,t_k))\right] dt_j \cdot \det\left[\partial Q_{i_p}/\partial t_r\right]_{r\neq j} \cdot dt_1\dots\cancel{dt_j}\dots dt_k =$$

↑ NEWTON LEIBNIZ

$$= \int_{t_1=a_1}^{b_1}\dots\int_{t_k=a_k}^{b_k}\left[\frac{\partial f}{\partial x_1}\frac{\partial Q_1}{\partial t_j}+\frac{\partial f}{\partial x_2}\frac{\partial Q_2}{\partial t_j}+\dots+\frac{\partial f}{\partial x_N}\frac{\partial Q_N}{\partial t_j}\right]\det\left[\frac{\partial Q_{i_p}}{\partial t_r}\right]_{r\neq j} dt_1\dots dt_k$$

MÁTRÉSZTŐL

$$\int_Q df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} =$$

$$= \int_Q \left[\frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_N} dx_N\right] \wedge dx_{i_1} \wedge \dots \wedge dx_{k-1} =$$

$$= \sum_{i=1}^{N} \int_Q \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} =$$

$$= \sum_{i=1}^{N} \int_{t_1=a_1}^{b_1} \dots \int_{t_k=a_k}^{b_k} \frac{\partial f}{\partial x_i} \det \begin{bmatrix} \partial Q_i / \partial t_1 & \cdots & \partial Q_i / \partial t_k \\ \partial Q_{i_1} / \partial t_1 & \cdots & \partial Q_{i_1} / \partial t_k \\ \vdots & & \\ \partial Q_{i_{k-1}} / \partial t_1 & \cdots & \partial Q_{i_{k-1}} / \partial t_k \end{bmatrix} dt_k \dots dt_1$$

EZT KIFEJTVE → AZ 1. SOR SZERINT

KAPJUK, HOGY

$$\int_Q df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} = \sum_{j=1}^{k} \left(\int_{L_2^j} - \int_{L_1^j} \right) f \, dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}$$

<u>Példa</u> $\Omega = 2xy^3 dx + 3x^2 y^2 dy$

Észrevétel $d\Omega = 0$!!!

EZÉRT $\int_K \Omega = \int_{K \text{ BELSEJE}} d\Omega = 0.$

K  ZÁRT GÖRBE 

BONYOLULT INTEGRÁLÁSOK HELYETT

Egyszerűbben: $\Omega = 2x^2y^3\,dx + 3x^2y^2\,dy = \underline{d\Psi}$

$$\Psi = x^2y^3$$

$\underline{\underline{\text{I}}}$ Állítás: $d^2\Psi = d(d\Psi) = 0$ (HA Ψ 2x DIFF)

$\underline{\underline{\text{biz}}}$ Elég csak $\Psi = f\,dx_{i_1}\wedge\dots\wedge dx_{i_k}$-ra

$$d\Psi = \left(\frac{\partial f}{\partial x_1}dx_1 + \dots + \frac{\partial f}{\partial x_N}dx_N\right)\wedge dx_{i_1}\wedge\dots\wedge dx_{i_k} =$$

$$= \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} dx_i \wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}$$

$$d^2\Psi = \sum_{i=1}^{N} \left(d\frac{\partial f}{\partial x_i}\right)\wedge dx_i \wedge dx_{i_1}\wedge\dots\wedge dx_{i_k} =$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i \wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}$$

f 2x FOLYT DIFF $\Rightarrow \dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$

MÁSRÉSZT $dx_i\wedge dx_j = -dx_j\wedge dx_i$

$$d^2\Psi = \sum_{1\le i<j\le N}\left[\frac{\partial^2 f}{\partial x_i \partial x_j} dx_j\wedge dx_i \wedge dx_{i_1}\wedge\dots\wedge dx_{i_k} + \frac{\partial^2 f}{\partial x_j \partial x_i} dx_i\wedge dx_j \wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}\right] = 0$$

+ KIEJTIK EGYMÁST

POINCARÉ LEMMA (Jegyzetből tétel)

$$d\Omega = 0 \Rightarrow \exists \Psi \quad \Omega = d\Psi$$

<u>Példa</u> (volt) $\Omega = 2xy^3\,dx + 3x^2y^2\,dy =$

$$= d(x^2y^3)$$

KLASSZIKUS GRAD, ROT, DIV

1) $f : \mathbb{R}^3 \to \mathbb{R}$ fgv

$$\operatorname{grad} f = \nabla f \simeq \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{bmatrix}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

1a) VONALINTEGRÁL

$Q : [a,b] \to \mathbb{R}^3$ görbe

$V : \mathbb{R}^3 \to \mathbb{R}^3$ vektormező

$$\int_Q V = \int_{t=a}^{b} \langle V(Q(t)) \,|\, Q'(t) \rangle\, dt =$$

$$= \int_Q V_1\,dx + V_2\,dy + V_3\,dz \qquad V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

1b) Tétel $\int_Q \nabla f = Q(b)-Q(a)$

biz: $\int_Q \nabla f = \int_Q \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right) = \int_Q df = \int_{\partial\Omega} f$

Tétel: $\frac{\partial V_i}{\partial x_j} = \frac{\partial V_j}{\partial x_i}$ $(i,j=1,2,3) \Rightarrow \exists f \quad V=\nabla f$

biz: JELENTÉSE $d(V_1 dx + V_2 dy + V_3 dz) = 0$

Poincaré $\Rightarrow \exists f \quad V_1 dx + V_2 dy + V_3 dz = df =$

$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

2) $V: \mathbb{R}^3 \to \mathbb{R}^3$ vektormező

V rotációja $\mathrm{rot}\, V = \mathrm{curl}\, V = \underbrace{\nabla}_{\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{bmatrix}} \times V =$

[ÖRVÉNYERŐSSÉG] $= \begin{bmatrix} \frac{\partial V_2}{\partial z} - \frac{\partial V_3}{\partial y} \\ \frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z} \\ \frac{\partial V_1}{\partial y} - \frac{\partial V_2}{\partial x} \end{bmatrix}$ — BONYOLULT

$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ rot V kompaktabban

$d(V_1 dx + V_2 dy + V_3 dz)$ -ben

$= (\mathrm{rot}\, V)_1\, dy \wedge dz + (\mathrm{rot}\, V)_2\, dz \wedge dx +$

$+ (\mathrm{rot}\, V)_3\, dx \wedge dy$

2a) K  ZÁRT GÖRBE $K = \partial Q$

Q = KÖRLAP $\to \mathbb{R}^3$

Tétel $\int_K V = \int_Q \langle \operatorname{rot} V \mid \text{normál vektor} \rangle$

V: Vektormező

Az EREDETI STOKES TÉTEL VOLT

Biz: $\int_K V = \int_{\partial Q} (V_1 dx + V_2 dy + V_3 dz) =$

$= \int_Q d[V_1 dx + V_2 dy + V_3 dz]$

2b) $\operatorname{rot} \operatorname{grad} f = 0$ MIVEL $d(df) = 0$

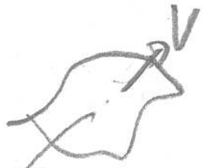
2c) $\operatorname{rot} V = 0 \Rightarrow \exists f \;\; V = \operatorname{grad} f$ [VS 16]

3) DIVERGENCIA (FORRÁSERŐSSÉG)

$$\operatorname{div} V = \langle \nabla \mid V \rangle = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

$$\|$$

$$d(V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy)$$

3a) ÁTFOLYÁS  Q felület V sebességi fgv.

$$\Phi = \int_Q \langle V \mid \text{Normál} \rangle$$

$$\phi = \int_Q \left(V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy \right)$$

3b) Ha F ∂Q Q 3D test határa, akkor

$$\int_F \langle V, \text{norm.} \rangle = \int_{\partial Q} \left(V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy \right) =$$

$$= \int_Q d\left[V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy \right] =$$

$$= \int_Q \operatorname{div} V$$

3d) $\operatorname{div} V = 0 \Rightarrow \exists\, U \quad V = dU$ VEKTORPOTENCIÁL

biz: $d(V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy) = 0$

Poincaré $\Rightarrow \exists\, U_1, U_2, U_3 \quad d(U_1\, dx + U_2\, dy + U_3\, dz) =$

$= (V_1\, dy \wedge dz + V_2\, dz \wedge dx + V_3\, dx \wedge dy)$

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EGY VEKTORPOTENCIÁLLAL ÍRHATÓ LE