HELYETTESÍTÉSES INTEGRÁLÁS (K-VÁLTOZÓS)

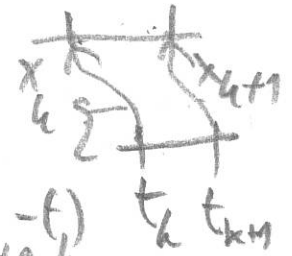
1-dim

$$\int_a^b f(x)\,dx \underset{\uparrow}{=} \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt$$

$x=\varphi(t) \quad a=\varphi(\alpha) \quad b=\varphi(\beta)$

$dx = \varphi'(t)\,dt$

Leibniz: $\varphi' = \frac{d\varphi}{dt} \qquad dx = d\varphi = \frac{d\varphi}{dt}\,dt$



$x_{k+1}-x_k \approx \varphi'(t_k)\,(t_{k+1}-t_k)$

UAZ, MINT TÖBB DIM-BAN $(K=N)$

$$\int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_k}^{\beta_k} f(Q(t_1,\dots,t_k))\,\det(Q'(t_1,t_k))\,dt_k\cdots dt_1 =$$

$$= \int_Q f(x_1,\dots,x_k)\,dx_1\dots dx_k =$$

$$= \int_{Q([\alpha_1,\beta_1]\times\cdots\times[\alpha_k,\beta_k])} f(x_1,\dots,x_k)\,dx_1\dots dx_k$$

Nehézség 1-dim-ben $\quad \varphi : \mathrm{INTV} \to \mathrm{INTV}$

K-dim $\quad Q : \mathrm{INTV}$ (KOCKA) $\to$ ÁLT ALAKZAT

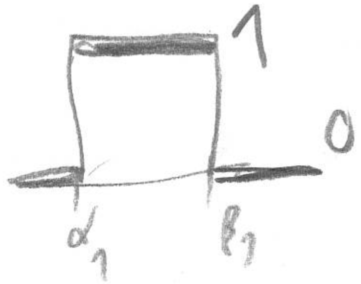
Mértékelmélet / (Geom. integrál)

$k \ge 1$ egész szám

$\exists\, \mathcal{M} \subset \{\mathbb{R}^k \to \mathbb{R}$ fgv-ek$\}$ ÚN ALTÉR $(\lambda\cdot, \pm)$

$\exists\, \mathcal{M} \ni f \mapsto \int_{\mathbb{R}^k} f$ LIN MŰVELET

(1) $1_{[\alpha_1,\beta_1]\times\dots\times[\alpha_k,\beta_k]} \in \mathcal{M}$



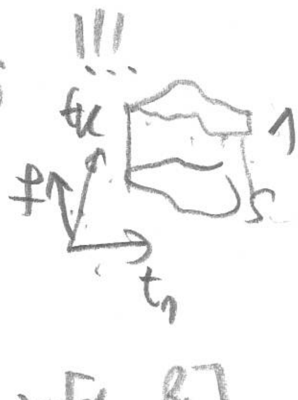
$\int_{\mathbb{R}^k} 1_{[\alpha_1,\beta_1]\times\dots\times[\alpha_k,\beta_k]} = (\beta_1-\alpha_1)\cdots(\beta_k-\alpha_k)$

(2) $f_1, f_2, \dots \in \mathcal{M}$ $\quad f_1 \le f_2 \le \dots \nearrow f$ $\quad \int_{\mathbb{R}^k} f_n \le c$

ESETÉN $f \in \mathcal{M}$, $\int_{\mathbb{R}^k} f_n \nearrow f$ $(n\to\infty)$ $\int_{\mathbb{R}^k} f \le c$

Megjegyzés $\quad \mathcal{M} \ne \{$összes $\mathbb{R}^k \to \mathbb{R}$ fgv$\}$

k-dim térfogat $\quad \mathrm{Vol}_k(S) = \int_{\mathbb{R}^k} 1_S$!!!



$\int_S f := \int_{\mathbb{R}^k} 1_S \cdot f$

f FOLYT és $f(p)=0$ ha $p \notin [\alpha_1,\beta_1]\times\dots\times[\alpha_k,\beta_k]$

ESETÉN $\int_{\mathbb{R}^k} f = \int_{x_1=\alpha_1}^{\beta_1} \left[\int_{x_2=\alpha_2}^{\beta_2} \cdots \left[\int_{x_k=\alpha_k}^{\beta_k} f(x_1,\dots,x_k)\, dx_k \right] \cdots \right] dx_1$

Tétel $f: \mathbb{R}^K \to \mathbb{R}$ mérhető $(\in \mathcal{M})$

$G, S \subset \mathbb{R}^K$ mérhető $(1_G, 1_S \in \mathcal{M})$

$Q: S \leftrightarrow G \quad \det(Q') \neq 0$

esetén $\int_G f(x_1,\dots,x_k)\,dx_1\cdots dx_k =$

$$= \int_S f(Q(t_1,\dots,t_k))\,|\det(Q'(t_1,\dots,t_k))|\,dt_1\cdots dt_k =$$

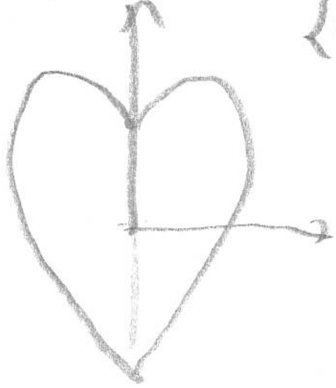
Külső formalizmus

Ha G összefüggő

$$\int_G f(x_1,\dots,x_k)\,dx_1\cdots dx_k = \pm\int_Q f\,dx_{t_1}\wedge\dots\wedge dx_{t_k}$$

NEM KELL DETERMINÁNST SZÁMOLNI

Példa



$\{p \in \mathbb{R}^2 : |x(p)| - \sqrt{1-x(p)^2} \leq$

$\leq y(p) \leq |x(p)| + \sqrt{1-x(p)^2}\} =$

$= (|x| - \sqrt{1-x^2} \leq y \leq |x| + \sqrt{1-x^2}) =$

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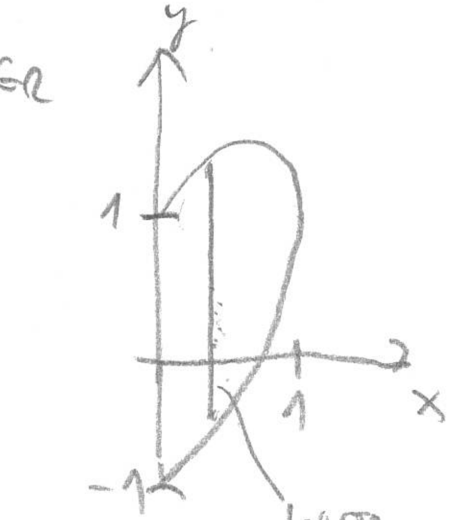
$= (||x| - y| \leq \sqrt{1-x^2}) =$

$= ((|x|-y)^2 \leq 1-x^2) =$

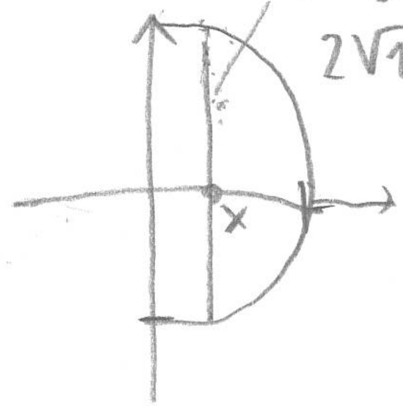
$= (2x^2 + y^2 - 2|x|y \leq 1)$

TERÜLET, SÚLYPONT, TEHETETLENSÉGI NYOM. (origóra)

$\frac{1}{2}$ TER



HOSSZ = $2\sqrt{1-x^2}$



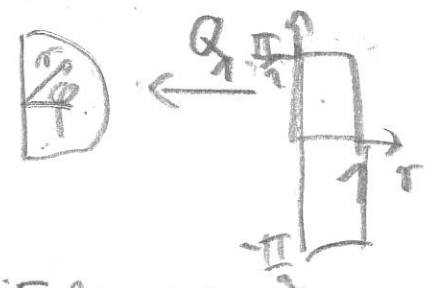
$$\frac{1}{2}\text{TER} = \int_{x=0}^{1} 2\sqrt{1-x^2}\,dx = \pi/2$$

MÉRTÉKELM. VERZIÓ

$$V = \int_{x=-\infty}^{\infty} \ell(x)\,dx$$

DIFF GEOM VERZIÓ

Félkör



$$Q(r,\varphi) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$$

$$\frac{1}{2}\text{TER} = \int_Q dx \wedge dy =$$

$$= \int_{r=0}^{1}\int_{\varphi=-\pi/2}^{\pi/2} d(r\cos\varphi) \wedge d(r(\sin\varphi + \cos\varphi))$$

Feltorzítás

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ y+|x| \end{bmatrix}$$

$$Q(r,\varphi) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi + r\cos\varphi \end{bmatrix}$$

$$= \int_{r=0}^{1}\int_{\varphi=-\pi/2}^{\pi/2} [\cos\varphi\, dr + r(-\sin\varphi)\,d\varphi] \wedge$$

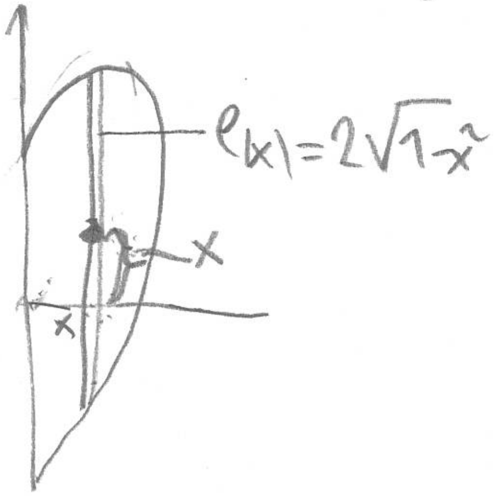
$$\wedge \left[(\sin\varphi + \cos\varphi)\,dr + r(\cos\varphi - \sin\varphi)\,d\varphi\right] =$$

$= \text{HF}$

LÁTSZÓLAG NEHÉZKES,

DE A FIZIKAI EGYENLETEK GYAKRAN DIFF GEOM ALAKÚAK

SÚLYPONT Csak y-koord (x koord $= 0 \in$ szimm.)



$$y(\text{súlyp}) = \frac{1}{\left(\frac{1}{2}\,\text{TER}\right)} \int_{x=0}^{1} x \cdot \ell(x)\, dx =$$

$$= \frac{2}{\pi} \cdot 2 \int_{x=0}^{1} x\sqrt{1-x^2}\, dx$$

$x = \cos\varphi$ ⌋ helyettesítés

Diff geom: $$y(\text{súlyp}) = \frac{1}{\left(\frac{1}{2}\,\text{TER}\right)} \int_{Q} y \, dx \wedge dy$$

$$Q = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi + r\cos\varphi \end{bmatrix}$$

$$dx = d\, r\cos\varphi = \cos\varphi\, dr + r\sin\varphi\, d\varphi$$
$$dy = (\sin\varphi + \cos\varphi)\, dr + r(\cos\varphi - \sin\varphi)\, d\varphi$$

$$dx \wedge dy = \cos\varphi \cdot r(\cos\varphi - \sin\varphi)\, dr \wedge d\varphi +$$
$$+ r \cdot \sin\varphi\, (\sin\varphi + \cos\varphi)\, d\varphi \wedge dr =$$
$$= r(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1})\, dr \wedge d\varphi = r\, dr \wedge d\varphi$$

$$y(\text{súlyp}) = \frac{4}{\pi} \int_{r=0}^{1} \int_{\varphi=-\pi/2}^{\pi/2} r(\sin\varphi + \cos\varphi)\, r\, d\varphi\, dr =$$

$$= \frac{4}{\pi} \int_{r=0}^{1} r^2\, dr \int_{\varphi=-\pi/2}^{\pi/2} (\sin\varphi + \cos\varphi)\, d\varphi = \frac{4}{\pi} \cdot \frac{1}{3} \cdot 2 = \underline{\underline{\frac{8}{3\pi}}}$$

TEHETETLENSÉGI NYOMATÉK

$$\Theta\left(\tfrac{1}{2}\Delta(u)\right) = \int_Q (x^2+y^2)\, dx \wedge dy =$$

$$= \int_{r=0}^{1} \int_{\varphi=-\pi/2}^{\pi/2} \left[r^2\cos^2\varphi + r^2(\sin\varphi+\cos\varphi)^2\right] r\, d\varphi\, dr =$$

$$= \int_{r=0}^{1} r^2\, dr \int_{\varphi=-\pi/2}^{\pi/2} [2\cos^2\varphi + \sin^2\varphi + 2\underbrace{\cos\varphi\sin\varphi}_{\sin 2\varphi \in \int = 0}]\, d\varphi =$$

$$= \frac{1}{3}\cdot\left[2\cdot\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi}{2}$$

hertelkedvélétlő vetszé (HF)

HAT TÖRT CONTOSSÁGÚ

$\int_5^6 e^{-x^2} dx$ NEM ELEMI FGV (nincs véges kifejezés sin, cos, e, ...)

$$\int_{\boxed{-\infty}}^{\boxed{\infty}} e^{-x^2} dx = ? = \sqrt{2\pi}$$

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx\, dy =$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)} dx \wedge dy = \lim_{R\to\infty} \int_{Q_R} e^{-(x^2+y^2)} dx \wedge dy \overset{x = r\cos\varphi}{\underset{y = r\sin\varphi}{=}}$$

$$= \lim_{R\to\infty} \int_{\varphi=0}^{2\pi} \int_{r=0}^{R} e^{-r^2} r\, dr\, d\varphi \underset{s=r^2}{=} 2\pi \lim_{R\to\infty} \int_0 e^{-s} ds = 2\pi$$