KÜLSŐ SZORZAT

$1 \leq k \leq N$

$u_1, \dots, u_k \in \mathbb{R}^N$ $\qquad u_j = \begin{bmatrix} u_{1j} \\ u_{2j} \\ \vdots \\ u_{Nj} \end{bmatrix}$

$$u_1 \wedge u_2 \wedge \dots \wedge u_k = \left[\det \begin{bmatrix} u_{i_1 1} & u_{i_1 2} & \cdots & u_{i_1 k} \\ u_{i_2 1} & u_{i_2 2} & \cdots & u_{i_2 k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i_k 1} & u_{i_k 2} & \cdots & u_{i_k k} \end{bmatrix} : 1 \leq i_1 < i_2 < \dots < i_k \leq N \right]$$

LEXIKOGRAFIKUSAN

Példa

$k = 2,\ N = 4 \to \binom{4}{2} = 6$ dim vektor

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \wedge \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} \det \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \\ \det \begin{bmatrix} 1 & 1 \\ 3 & 9 \end{bmatrix} \\ \det \begin{bmatrix} 1 & 1 \\ 4 & 16 \end{bmatrix} \\ \det \begin{bmatrix} 2 & 4 \\ 3 & 9 \end{bmatrix} \\ \det \begin{bmatrix} 2 & 4 \\ 4 & 16 \end{bmatrix} \\ \det \begin{bmatrix} 3 & 9 \\ 4 & 16 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 12 \\ 6 \\ 16 \\ 12 \end{bmatrix}$$

Algebrai kalkulus

$$u_j = u_{1j} e_1 + u_{2j} e_2 + \dots + u_{Nj} e_N$$

$$N \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right. \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\wedge$ ASSZOCIATÍV, DISZTRIBUTÍV, ANTIKOMMUTATÍV

$u \wedge v = -v \wedge u$ $\qquad (\alpha u) \wedge v =$

$u \wedge u = 0$ $\qquad = \alpha\, u \wedge v$

Példák

$$(e_1+2e_2+3e_3+4e_4)\wedge(e_1+4e_2+9e_3+16e_4)=$$

$$=e_1\wedge[e_1+4e_2+9e_3+16e_4]+2e_2\wedge[e_1+4e_2+9e_3+16e_4]+$$
$$+3e_3\wedge[e_1+4e_2+9e_3+16e_4]+4e_4\wedge(e_1+4e_2+9e_3+16e_4)=$$

$$=e_1\wedge e_1+4e_1\wedge e_2+9e_1\wedge e_3+16e_1\wedge e_4+$$
$$+2e_2\wedge e_1+8e_2\wedge e_2+18e_2\wedge e_3+32e_2\wedge e_4+(\ldots)+$$
$$+4e_4\wedge e_1+16e_4\wedge e_2+36e_4\wedge e_3+64e_4\wedge e_4=$$

$$=0+4e_1\wedge e_2+9e_1\wedge e_3+16e_1\wedge e_4+$$
$$+2\cdot(-e_1\wedge e_2)+8\cdot 0+18e_2\wedge e_3+32e_2\wedge e_4+$$
$$+3(-e_1\wedge e_3)+12(-e_2\wedge e_3)+27\cdot 0+48e_3\wedge e_4+$$
$$+4(-e_1\wedge e_4)+16(-e_2\wedge e_4)+36(-e_3\wedge e_4)+64\cdot 0=$$

$$=(\underbrace{4-2}_{2})e_1\wedge e_2+(\underbrace{9-3}_{6})e_1\wedge e_3+(\underbrace{16-4}_{12})e_1\wedge e_4+$$
$$+(\underbrace{18-12}_{6})e_2\wedge e_3+(\underbrace{32-16}_{16})e_2\wedge e_4+(\underbrace{48-36}_{12})e_3\wedge e_4$$

$$\begin{bmatrix}2\\6\\12\\6\\16\\12\end{bmatrix}\begin{array}{l}e_1\wedge e_2\\e_1\wedge e_3\\e_1\wedge e_4\\e_2\wedge e_3\\e_2\wedge e_4\\e_3\wedge e_4\end{array}$$

KÜLSŐ DIFFERENCIÁL MÉRTÉK

$K \leq N \quad Q:[a_1,b_1]\times\dots\times[a_k,b_k]\to\mathbb{R}^N$ k-dim felületdb

$$\hat{F}:\mathbb{R}^N\to\underbrace{\mathbb{R}^N\wedge\mathbb{R}^N\wedge\dots\wedge\mathbb{R}^N}_{K}=\Lambda^K\mathbb{R}^N$$

<u>Def</u> $\int F : Q\mapsto\int_Q\left\langle\hat{F}\,\middle|\,\frac{\partial Q}{\partial t_1}\wedge\dots\wedge\frac{\partial Q}{\partial t_k}\right\rangle :=$

$$=\int_{a_1}^{b_1}\cdots\int_{a_k}^{b_k}\left\langle F(Q(t_1,..,t_k))\,\middle|\,\frac{\partial Q}{\partial t_1}\wedge\dots\wedge\frac{\partial Q}{\partial t_k}\right\rangle dt_k\,dt_{k-1}\dots dt_1$$

<u>Példa</u> Átfolyás felületdb-on (FLUXUS)

$$\Phi=\int_{\substack{s=a_1\\ |\\ t_1}}^{b_1}\int_{\substack{t=a_2\\ |\\ t_2}}^{b_2}\left\{F_1(Q(s,t))\det\begin{bmatrix}\partial Q_2/\partial s & \partial Q_2/\partial t\\ \partial Q_3/\partial s & \partial Q_3/\partial t\end{bmatrix}+\dots\right\}dt\,ds$$

$x_1 = x(s,t) = Q_1(s,t)$ [$= Q$ x KOORD]
$x_2 = y(s,t) = Q_2(s,t)$ [$= Q$ y KOORD]
$x_3 = z(s,t) = Q_3(s,t)$ [$= Q$ z KOORD]

$$\hat{F}=\begin{bmatrix}\hat{F}_{12}\\ \hat{F}_{13}\\ \hat{F}_{23}\end{bmatrix}=\begin{bmatrix}F_3\\ -F_2\\ F_1\end{bmatrix}$$

$$\Phi=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left\{F_1(Q)\det\begin{bmatrix}\partial x_2/\partial t_1 & \partial x_2/\partial t_2\\ \partial x_3/\partial t_1 & \partial x_3/\partial t_2\end{bmatrix}+\right.$$

$$\left.+F_2(Q)\det\begin{bmatrix}\partial x_3/\partial t_1 & \partial x_3/\partial t_2\\ \partial x_1/\partial t_1 & \partial x_1/\partial t_2\end{bmatrix}+\dots\right\}dt_2\,dt_1$$

→ jó SORRENDHEZ CSERE

$$\det\begin{bmatrix}\partial x_{i_1}/\partial t_1 & \partial x_{i_1}/\partial t_2 \\ \partial x_{i_2}/\partial t_1 & \partial x_{i_2}/\partial t_2\end{bmatrix} =$$

$$= \det\begin{bmatrix}\partial x_{i_1}/\partial t_1 & \partial x_{i_2}/\partial t_1 \\ \partial x_{i_1}/\partial t_2 & \partial x_{i_2}/\partial t_2\end{bmatrix} =$$

$$\nabla x_{i_1} \qquad \nabla x_{i_2}$$

$$= \left(\frac{\partial x_{i_1}}{\partial t_1} e_1 + \frac{\partial x_{i_1}}{\partial t_2} e_2\right) \wedge \left(\frac{\partial x_{i_2}}{\partial t_1} e_1 + \frac{\partial x_{i_2}}{\partial t_2} e_2\right)$$

$$\Phi = \int_{t_1=a_1}^{b_1}\int_{t_2=a_2}^{b_2}\left[\hat{F}_{12}\cdot(\nabla x_1)\wedge(\nabla x_2) + F_{13}\cdot(\nabla x_1)\wedge(\nabla x_3) + F_{23}(\nabla x_2)\wedge(\nabla x_3)\right] dt_2\, dt_1$$

RÖVIDÍTETT JELÖLÉS

$$(\nabla x_{i_1}) \wedge (\nabla x_{i_2})\, dt_2\, dt_1 = dx_{i_1} \wedge \partial x_{i_2}$$

$$\boxed{\Phi = \int_Q \sum_{1 \le i_1 < i_2 \le 3} \hat{F}_{i_1 i_2}\, dx_{i_1}\, dx_{i_2}}$$ ALAKÚ

ÁLTALÁBAN IS EGY KÜLSŐ DIFFERENCIÁLT

$$\int_Q f \cdot dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$$ ALAKÚ TAGOK ÖSSZEGE

Kalkulus

MEGLEPŐEN EGYSZERŰ

$Q : [a_1, b_1] \times \dots \times [a_k, b_k] \to \mathbb{R}^N$

$f : \mathbb{R}^N \to \mathbb{R}$ fgv

$\int_Q f \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$ KISZÁMÍTÁSA

$x_1 = Q_1(t_1, \dots, t_k), \dots, x_N = Q_N(t_1, \dots, t_k)$

$$\int_Q f \, dx_{i_1} \wedge \dots \wedge dx_{i_k} = \int_{t_1 = a_1}^{b_1} \dots \int_{t_k = a_k}^{b_k} f(x_1(t_1, \dots, t_k), \dots, x_N(t_1, \dots, t_k)) \cdot$$

$$\cdot \left(\frac{\partial x_{i_1}}{\partial t_1} e_1 + \dots + \frac{\partial x_{i_1}}{\partial t_k} e_k \right) \wedge \dots \wedge \left(\frac{\partial x_{i_k}}{\partial t_1} e_1 + \dots \right) dt_k \dots dt_1$$

$\uparrow$

NEM DET

Példa

ÁTFOLYÁS 8-AD GÖMBÖN

$$Q(\vartheta, \varphi) = \begin{bmatrix} \sin\vartheta \cos\varphi \\ \sin\vartheta \sin\varphi \\ \cos\vartheta \end{bmatrix} \qquad 0 \le \vartheta, \varphi \le \frac{\pi}{2}$$

$$F = \begin{bmatrix} x^2 y \\ y^2 z \\ z^4 x \end{bmatrix}$$

$$\Phi = \int_Q \left[x^2 y \, dy \wedge dz + y^2 z \, dz \wedge dx + z^4 x \, dx \wedge dy \right]$$

$$\int_Q x^2 y \, dy \wedge dz = \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} [\text{BEHELYETTESÍTÉS}]$$

$x = \sin\vartheta \cos\varphi \quad y = \sin\vartheta \sin\varphi \quad z = \cos\vartheta$

$$dx = \underbrace{\frac{\partial x}{\partial \vartheta}}_{(\cdot e_1)} d\vartheta + \underbrace{\frac{\partial x}{\partial \varphi}}_{(\cdot e_2)} d\varphi = \cos\vartheta\cos\varphi \, \underline{d\vartheta} - \sin\vartheta \sin\varphi \, \underline{d\varphi}$$

$(\cdot e_2)$ — NEM KELL KIÍRNI

$$dy = \cos\vartheta \sin\varphi \, d\vartheta + \sin\vartheta \cos\varphi \, d\varphi$$

$$dz = -\sin\vartheta \, d\vartheta$$

$$\int_Q x^2 y \, dy \wedge dz = \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} (\sin\vartheta\cos\varphi)^2 (\sin\vartheta\sin\varphi) \cdot$$

$$\cdot (\cos\vartheta\sin\varphi \, d\vartheta + \sin\vartheta\cos\varphi \, d\varphi) \wedge$$

$$\wedge (-\sin\vartheta \, d\vartheta) =$$

$$= \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} \sin^3\vartheta \cos^2\varphi \sin\varphi \cdot [\cos\vartheta \sin\vartheta \sin\varphi \underbrace{d\vartheta \wedge d\vartheta}_{0} +$$

$$+ \sin\vartheta\cos\varphi(-\sin\vartheta) \, d\varphi \wedge d\vartheta] =$$

$$= \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} [-\sin^2\vartheta \cos\varphi \, d\varphi \wedge d\vartheta] =$$

→ SORREND BEÁLLÍTÁSA

$$= \int_{\vartheta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} [+\sin^2\vartheta \cos\varphi \underbrace{d\vartheta \wedge d\varphi}_{d\vartheta \, d\varphi}]$$