2. Ea

lineáris
parciális diff.egyenlet
valószínűség blokk

LU-felbontás
QR-felbontás
Főtengely tétele
SVD felbontás
Moore-Penrose inverz

Felbontások: azért használjuk, mert

mátrixok használata: mátrixos jó volt, mikor zakat kellett számolni.

$\downarrow$
XIX. század vége

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

mátrix mit ír le: lineáris leképezés: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+4y \\ 5x+6y \end{bmatrix}$

$3\times 2 \quad 2\times 2$

pl.: $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \to \begin{bmatrix} (1-6)=-5 \\ -9 \\ -17 \end{bmatrix}$

$1 = x$
$y = -3$

$Ax = b$

$\downarrow$

$\begin{bmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

Többváltozós lgr-eket mátrixxal lehet leírni.

Klasszikus gát:

$\pi \in R$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} : \mathbb{R}^2$ ⟨végtelen tizedes törtek⟩ $\mapsto \mathbb{R}^3$

$\mathbb{Z}$ egész $\qquad$ $\mathbb{Q} \to \mathbb{Z}$ egész nem hányadosa racionális számok

$Ax=b$ megoldhatósága

R helyett lehetne $\mathbb{Q}$

↓

{végtelen periódikus tizedes törtek}

↓

lehet más számrendszer

pl: 2-20.

$x = 12,72\underbrace{6868}68... \to \frac{P}{Q}$

$100x = 1272,\underbrace{6868}68...$

$1000000x = 127268,6868$

$100\,000x - 100x = 127268 - 1272 = 125996$

$x = \frac{125996}{99900}$

$Ax=b$ megoldható hogy $+, -, \times, \div$ benne maradjon → elvégezhető legyen.

$\mathbb{Z}_p$ prím $= p$

→ $\{0, 1, 2 \dots p-1\}$

$\text{modulo } p(x) = \frac{x}{p}$ maradéka $\to 0, 1 \dots p-1$

$\text{mod}_7(133) = 0$

$+, -, \cdot$ mod 7 alapján megy

$132 \overset{7}{\cdot} 134 = 6 \overset{7}{\cdot} 1 = 6$

TEST: $+, -, \cdot, \div$ nem vezet ki.

Mátrix elemek testbeli elemekből lehetnek.

$6 \overset{7}{/} 8 = 6$

$A \overset{7}{/} B = X$

$X \overset{7}{\cdot} B = A$

TALÁLGATÁS

$1 \overset{7}{\cdot} 8 = 1$

$2 \overset{7}{\cdot} 8 = 2$

$3 \overset{7}{\cdot} 8 = 3$

⋮

$6 \overset{7}{\cdot} 8 = 6$

$$\det_7 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 4 \end{bmatrix} = \cancel{32 + 12 = 44 - 44} = 0$$

$\cancel{-20 - 24}$

$$20 + 24 - 32 - 12 = 0$$

$$\det [a_{ij}]_{i,j=1}^{n} = \sum_{\pi = \text{permutáció } (1 \ldots n)} (-1)^{\text{Par}(\pi)} \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \cdot \ldots$$

$n = 3$

$\text{Par}(\pi) \Rightarrow$ hány cserével oldható meg π — Ha páros akkor 0, ha páratlan akkor 1

$\pi = [1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3]$ — 0

$\pi = [1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2]$ — 1

$\pi = [2 \mapsto 1, 1 \mapsto 3, 3 \mapsto 2]$ — 0

$\pi = [2 \mapsto 2, 1 \mapsto 3, 3 \mapsto 1]$

$\pi = [1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3]$

$\pi = [2 \mapsto 3, 1 \mapsto 2, 3 \mapsto 1]$

mod 7: 0, 0; 1, 1; 6, 6; −1, 6; −2, 5; −3, 4

$n! \to$ lehetőség $\Rightarrow$ nagyon nagy

GE

$$\det_7 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 4 \end{bmatrix} = \det_7 \begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \end{bmatrix} \begin{matrix} \to 2.\text{ sor} - 4 \cdot (1.\text{ sor}) \\ \to 3.\text{ sor} - 0 \cdot (1.\text{ sor}) \end{matrix} = \det_7 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 - 4 \cdot 2 & 6 - 4 \cdot 3 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\det_7 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 2 & 4 \end{bmatrix} = \det_7 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \prod_7 \text{ átló} =$$

$1 \cdot 4 \cdot 4 = 16$ mod 7 = 2

3. sor − $\frac{2}{4}$ 2. sor

$1 \cdot 4 = 4$

$2 \cdot 4 = 1$

$3 \cdot 4 = 5$

$4 \cdot 4 = 2$

3-as nénérdes ZH feladat ZH-ban

det CRAMER $\to$ $Ax = b$ megoldható

$A = LU$ $\quad Ax = b$ $\quad LUx = b$

$Ux = L^{-1}b \to$ GE az $Ax = b$

$Ux = L^{-1}b = \tilde{b}$

$x = U^{-1}\tilde{b}$ visszahelyettesítés

$A = QR$ = (ortogon) $\begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$

$\quad\ U$

$Q^{-1} = Q^T$

3ZH

$Ax = b$

$QRx = b$

$Rx = Q^Tb$

$x = R^{-1}[Q^Tb]$

$\boxed{\text{2. gyak}}$

$\mathbb{Z}_2 = \{0, 1\}$ $\quad a \overset{2}{\square} b = \bmod_2\ a \square b$

$1 \overset{2}{+} 1 = 0$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = LU$$

Algoritmus: GE A-ra $\to U$ $\quad L = ?$

$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ & & \end{bmatrix}$ ez nem vezet sehova

GE mikor működik: ha a sarokdeterminánsok $\neq 0$

MOST $\det \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$

$$A_0 = A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & x & x \\ 0 & x & x \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \nabla = U$$

$$\rightarrow \begin{bmatrix} A_2 \ 2. \text{ sor} \\ A_2 \ 3. \text{ sor} - A_2 \ 2. \text{ sor} \end{bmatrix}$$

Mátrix nyelv:

Mátrix hatás

$$L = [l_{ij}]_{i,j=1}^{n} \quad A = [a_{ij}]_{i,j=1}^{n} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \begin{matrix} [A \ 1. \text{ sora}] \\ \\ [A \ n. \text{ sora}] \end{matrix}$$

L balról a sorokra hat

$$LA = \begin{bmatrix} l_{11} a_1 + l_{12} \cdot a_2 + \dots + l_{1n} \cdot a_n \\ l_{21} a_1 + l_{22} \cdot a_2 \dots + l_{2n} \cdot a_n \\ l_{n1} a_1 + \dots + l_{nn} \cdot a_n \end{bmatrix}$$

$$\underset{L}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{?}$$

l_i. sora l_i. sorhoz

oszlopokkal

$$A \cdot R = [l_{ij}]_{i,j=1}^{n} \ [r_{ij}]_{i,j=1}^{n}$$

$\{\bar{a}_1 \dots \bar{a}_n\}$ // oszlophatás jobbról szorzás.

R i. oszlopa az i. ~~sorhoz~~ oszlophoz

$$= \begin{bmatrix} r_{11} \bar{a}_1 & r_{12} \bar{a}_1 + \dots & r_{1n} \cdot \bar{a}_n \\ r_{21} \bar{a}_2 & r_{22} \bar{a}_2 + \dots & r_{2n} \cdot \bar{a}_2 \\ \vdots & & \\ r_{n1} \bar{a}_n & r_{n2} \bar{a}_n + \dots & r_{nn} \cdot \bar{a}_n \end{bmatrix}$$

$A_0 = A \qquad A_1 = L_1 \cdot A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A_1$

Soramanipuláció A_0-on

1. sor marad $\quad 1 \cdot [1. sor] + 0 \cdot [2. sor] + 0 \cdot [3. sor]$

2. sor $- [1. sor]$ $\quad -1 [1. sor] + 1 \cdot [2. sor] + 0 \cdot [3. sor]$

3. sor marad $\quad 0 \cdot [1. sor] + 0 \cdot [2. sor] + 1 \cdot [3. sor]$

$A_2 = L_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L_2} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1}$$

Algebra

$A_0 = L_1 A$

$A_1 = L_1 A_0 = L_1 A_1$

$A_2 = L_2 A_1 = L_2 \cdot L_1 A$

$A_2 = L_2 A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A = (L_2 L_1)^{-1} U = \nabla U = LU$

$L = (L_2 L_1)^{-1} = L_1^{-1} L_2^{-1}$

$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\cancel{L = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}$

$\tilde{L} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ x & 0 & \end{bmatrix}$

$\tilde{L}^{-1} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0\,x & 0 & \end{bmatrix}$

$L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = L_1$

$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\mathbb{Z}_2$ ugyanaz

$$L = L_1^{-1} L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = L$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

QR felbontás

$Ax = b$ $\qquad x = A^{-1} b$

$A = QR$ $\qquad A^{-1} = R^{-1} Q^{-1} = R^{-1} Q^T$

Q: ortogonális, R: felső trianguláris (+) átló.

NUMERIKUSAN STABIL. De placcigényes $\sqrt{\ }$ miatt.

csak valós (vagy komplex testen van)

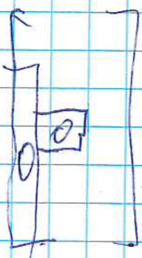
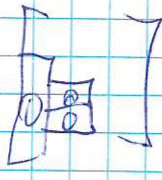
GIVENS-forgatások

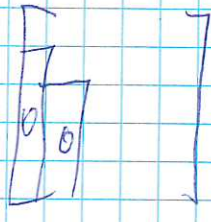
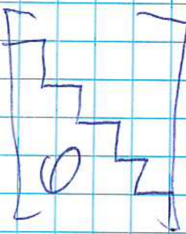
Elimináció

$A^{11} \rightarrow$ kinullázom az 11 elemét — 1. sor 1. oszlop

$A_0 = A \qquad A^{21} = Q^{21} A_0 = \begin{bmatrix} \boxed{0} & \\ & \end{bmatrix} \qquad A^{31} = Q^{31} \cdot A^{21} = \begin{bmatrix} \boxed{0} & \\ \boxed{0} & \end{bmatrix}$ ...

$A^{n1} = Q^{n1} A^{n-1,1} = \begin{bmatrix} \boxed{0} & \\ \boxed{0} & \\ \vdots & \\ \boxed{0} & \end{bmatrix}$

$A^{32} = Q^{32} A^{n1} =$  $A^{42} = Q^{42} A^{32} =$  $\ldots$ $A^{n2} = Q^{n2} A^{n1}$

 $\ldots\ldots$ $A^{n,n-1} = Q^{n,n-1} A^{n-1,n-1} =$  $= R$

átló alatt 0 van

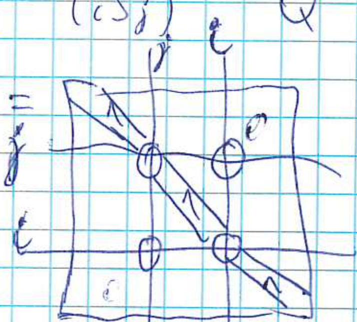
R felső triangulláris.

$Q^{n,n-1} \ldots\ldots Q^{21} A = R$

vissza sorrend

$$A = \underbrace{[Q^{21}]^T \ldots [Q^{n,n-1}]^T}_{Q} R$$

Q ortogonális.

$Q^{ij} =$ 

⊕ → forgatások.

4 db 0 kitüntetett elem.

sin α

1 → összesen választandó

1 normájú forgató mátrix

−sin α

cos α

ALG:

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \quad x^2 + y^2 = 1$$

olyan x, y meddig adott.

Givens tétel: vmelyik képezve van megfelelő 1 normájú, ami jól nulláz.

LU felbontás

$$A = \begin{bmatrix} a_{11} & \cdots & \vdots \\ a_{12} & \cdots & \vdots \\ a_{13} & \cdots & a_{33} \end{bmatrix}$$ szabály M1

$$\begin{bmatrix} A \end{bmatrix}$$

① $$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{12}}{a_{11}} & 1 & 0 \\ -\frac{a_{13}}{a_{11}} & 0 & 1 \end{bmatrix}$$

② $$A_1 = M_1 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{12}}{a_{11}} & 1 & 0 \\ -\frac{a_{13}}{a_{11}} & 0 & 1 \end{bmatrix} = A_1 = \begin{pmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}$$

③ szabály $$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{y_2}{y_1} & 1 \end{bmatrix}$$

④ $$M_2 \cdot A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{y_2}{y_1} & 1 \end{bmatrix} \cdot \begin{bmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} U \end{bmatrix}$$

$$U = M_1 \cdot M_2 \cdot A = A_2$$

⑤

$$L = L_1 \cdot L_2 = M_1^{-1} \cdot M_2^{-1}$$

$$L_1 = M^{-1}$$

$$L = L_1 \cdot L_2$$

$$L_2 = M^{-2}$$

⑥ $$L \cdot U = A$$

inverz: $\cancel{\begin{matrix} 1 & 2 & 3 \\ 8 & 9 & 10 \\ 5 & 7 & 6 \end{matrix}}$ $\Rightarrow$ $$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{5}{6} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{5}{6} & 0 & 1 \end{bmatrix}$$

hozzáadta 2. sorhoz az 1. sor felét

Még 1. sor $\frac{5}{6}$ -odik adjuk hozzá a 3. sorhoz.

Qr felbontás

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 8 & 0 \\ 3 & -3 & 2 \end{bmatrix}$$

$Q = [q_1 \; q_2 \; q_3]$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

$r_{jk} = \langle a_k, q_j \rangle$

$r_{kk} = \| a_k - \sum_{j=1}^{k-1} r_{jk} q_j \|_2$

$q_k = \frac{1}{r_{kk}} \left(a_k - \sum_{j=1}^{k-1} r_{jk} q_j \right)$

elírt hasonló

$r_{11} = \|a_1\|_2 = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7$

$q_1 = \frac{1}{r_{11}} \cdot a_1 = \frac{1}{7} \cdot \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$

$r_{1,2} = \langle a_2, q_1 \rangle = \frac{1}{7} \cdot (5 \cdot 2 + 8 \cdot 6 + -3 \cdot 3) = 7$

$a_2 - r_{12} q_1 = \begin{bmatrix} 5 \\ 8 \\ -3 \end{bmatrix} - 7 \cdot \frac{1}{7} \cdot \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$

$r_{22} = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

$q_2 = \frac{1}{7} \cdot \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix}$

$r_{13} = \langle a_3, q_1 \rangle = \frac{1}{7} \cdot (4 \cdot 2 + 0 \cdot 6 + 2 \cdot 3) = 2$

$r_{23} = \langle a_3, q_2 \rangle = \frac{1}{7}\cdot(4\cdot 5 + 0\cdot 2 + 2\cdot -6)$

QR felbontás

$r_{jk} = \langle a_k, q_j \rangle$

r_{11}, r_{22}, r_{33} $\sqrt{(\,1\,)^2}$

$r_{kk} = \| a_k - \sum_{j=1}^{k-1} r_{jk} q_j \|_2$

r_{1x}, r_{2x} $\rangle (x\, q_1, x_1 q_1 \ldots$

$q_k = \frac{1}{r_{kk}} \left(a_k - \sum_{j=1}^{k-1} r_{jk} q_j \right)$

$i > j$

$a_i - r_{ji} q_j$ = ez fontos

$\boxed{3.\ \text{óra}}$

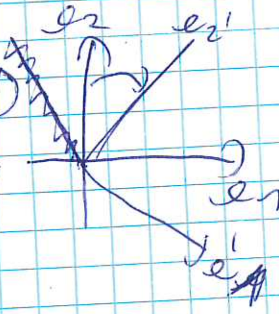
$$A = \begin{bmatrix} 36 & 192 & -517 \\ -48 & 394 & 581 \\ 25 & 480 & 170 \end{bmatrix}$$

QR felbontás Givens forgatással.

$$Q_1 = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 \times 1.\ \text{sor} & 0 \times 1.\ \text{sor} & 1 \times 3.\ \text{sor} \end{bmatrix}$$

$a^2 + b^2 = 1$

forgatási mátrix



$$A_1 = \begin{bmatrix} & & \\ 0 & & \\ 25 & 480 & 17 \end{bmatrix}$$

marad a 3. sor

$Q_1 \cdot A = A_1$

$$\begin{bmatrix} a & b & \\ -b & a & \\ & & 1 \end{bmatrix} \begin{bmatrix} 36 & 192 & \\ -48 & 394 & \\ & & \end{bmatrix} = \begin{bmatrix} -36b - 48a = 0 \\ \\ \end{bmatrix}$$

$$\left.\begin{array}{l} -36b - 48a = 0 \\ a^2 + b^2 = 1 \end{array}\right]$$

$$-b = \frac{48}{36}a = \frac{4}{3}a$$

$$a^2 + \left(\frac{4}{3}a\right)^2 = 1$$

$$a^2 + \frac{16}{9}a^2 = 1$$

$$\left(\frac{9+16}{9}\right)a^2 = 1$$

$$a^2 = \frac{9}{25} \qquad a = \pm\frac{3}{5} \quad \text{lehetőség. } (+)$$

$$\boxed{a = \frac{3}{5} \quad b = \frac{4}{5}}$$

$$Q_1 = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -\frac{4}{5} & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q_1 A = \begin{bmatrix} 60 & -200 & -775 \\ 0 & 390 & -65 \\ 25 & 480 & 170 \end{bmatrix}$$

kinullázni a 25-öt

$$A = Q_2 A_1 = \begin{bmatrix} & & \\ & & \\ \boxed{0} & & \end{bmatrix}$$

ortogonalitás miatt

$$Q_2 \Rightarrow \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{bmatrix}$$

$$\begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ -b & 0 & a \end{bmatrix}\begin{bmatrix} 60 & -200 & -775 \\ 0 & 390 & -65 \\ 25 & 480 & 170 \end{bmatrix} \quad 60a \quad \boxed{-60b \quad 25a}$$

csak 3. sora lesz érdekes.

$$-60b + 25a = 0$$
$$a^2 + b^2 = 1$$

$$a = \frac{12}{13}$$

$$b = \frac{5}{13}$$

$$A_2 = \begin{bmatrix} 65 & 0 & -650 \\ 0 & 390 & -65 \\ 0 & 520 & 455 \end{bmatrix}$$

nullázza a 32-est

$$A_3 = Q_3 A_2 = \begin{bmatrix} * & & \\ 0 & * & \\ 0 & \boxed{0} & * \end{bmatrix} \qquad Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}$$

$$\left.\begin{array}{l} -520b + 455a = 0 \\ a^2 + b^2 = 1 \end{array}\right\}$$

$$\boxed{a = \frac{3}{5} \quad b = \frac{4}{5}}$$

$$A_3 = \begin{bmatrix} 65 & 0 & -650 \\ 0 & 650 & 325 \\ 0 & 0 & 325 \end{bmatrix} = U = Q_3 Q_2 Q_1 \cdot A$$

$$A = Q_1^{-1} Q_2^{-1} Q_3^{-1} U = \underbrace{Q_1^T Q_2^T Q_3^T}_{Q} U =$$

$$Q = \begin{bmatrix} \frac{36}{65} & \frac{96}{325} & -253/325 \\ -48/65 & \frac{192}{325} & -\frac{96}{325} \\ 5/13 & \frac{48}{65} & \frac{36}{65} \end{bmatrix}$$

$A \cdot x = b \Rightarrow$ ~~ezt~~ ha ismerjük a QR felbontást, most megoldjuk.

$A = QR$

$Q \cdot Rx = b$

$Rx = Q^T b \rightarrow$ ált. megoldás.

$$\left.\begin{array}{l} r_{11}x_1 \dots r_{1n}x_n = b_1 \\ \ddots \\ r_{n-1}x_{n-1} + r_{nn}x_n = b_n \end{array}\right\} \rightarrow x_n = \frac{b_n}{r_{nn}}$$

$$r_{n-1,n-1}x_{n-1} \cdot \frac{b_n}{r_{nn}} = b_{n-1}$$

~~$x_{n-1} = b_{n-1} \, r_{n-1,n-1}$~~

$$x_{n-1} = \frac{b_{n-1} - r_{n-1,n}\frac{b_n}{r_{nn}}}{r_{n-1,n-1}}$$

A numerikus hibák nagyok lennének (csúnya helyettesítés)

$Ax = b$

$A = L \cdot U$

$LUx = b$

$Ux = L^{-1}b \rightarrow$ csúnya helyettesítési hiba (nagy numerikus hiba)

~~Főtengely transzformáció~~

SVD felbontás: Singular Value Decomposition.

Rayleigh-Ritz:

$A = Q \Lambda R^T$

A: $m \times n$, Q: $m \times m$ ortogonális, R: $n \times n$ ORT

$A \neq Q$

$r \leq \min(m,n)$, rang(A)

$A_R = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & 0 \end{bmatrix}$ $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots \geq \lambda_n \geq 0$

$Q = [q_1 | q_2 | \dots | q_m]$ $R = [r_1 | \dots | \dots | r_n]$

↳ egységvektorok $\mathbb{R}^m$-ben ↳ egységvektorok $\mathbb{R}^n$-ben

DESCARTES KOORDINÁTÁK

$\mathbb{R}^m$-ben $\mathbb{R}^n$-ben

$[p \text{ koord}] = \begin{bmatrix} \langle p | r_1 \rangle \\ \vdots \\ \langle p | r_n \rangle \end{bmatrix}$

$[p' \text{ koord}] = \begin{bmatrix} \langle p' | q_1 \rangle \\ \langle p' | q_m \rangle \end{bmatrix}$

$A : \mathbb{R}^n \to \mathbb{R}^m$

pl: $n=2$ $m=3$

Paralelogr.

Adaptált ⇒

Téglalapot.

Hogyan létezik SVD felbontás
═

Algebrai érvelés

$A = Q \Lambda R^T$

$A^T A = (Q \Lambda R^T)^T Q \Lambda R^T = \underbrace{R^{TT} \Lambda^T Q^T}Q \Lambda R^T = R \Lambda^T \Lambda R^T =$

$R \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} R^T$ $r_i \mapsto \lambda_i^2 r_i$

$Q^T Q = I$

$R^{TT} = R$

$A^T A =$

$B = A^T A$ r_i sajátvektora B-nek

λ_i sajátértékkel.

B főtengely TKF adja meg $\lambda_1^2 \dots \lambda_n^2$-et meg okat az $r_1 \dots r_n$

B sajátértékei B sajátvektorai

Emlékeztető: Főtengely tétel

$$B = \begin{bmatrix} 73 & -36 \\ -36 & 52 \end{bmatrix}$$

$S_p(B) = \{B \text{ sajátértékei}\} = \{\text{Ch pol}(B, \lambda) \text{ gyökei}\}$

$\text{Char pol}(B, \lambda) = \det(\lambda I - B) =$

$$\det\begin{pmatrix} \lambda-73 & 36 \\ 36 & \lambda-52 \end{pmatrix} =$$

$(\lambda-73)\cdot(\lambda-52) - 36^2$

$\lambda^2 - 73\lambda - 52\lambda + 73\cdot 52 -$

$36^2 = \lambda^2 - 125\lambda + 2500$

$$\lambda_{1,2} = \frac{125 \pm \sqrt{125^2 - 4\cdot 2500}}{2}$$

$$= \frac{125 \pm 75}{2} \nearrow 100 \searrow 25$$

"tekintsük"

Sajátvektor: $\begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix} = v_1$

$v_2 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

$\sqrt{4^2 + 3^2} = \sqrt{25} = 5$

SVD

$$A = \frac{1}{325}\begin{bmatrix} 1128 & -696 \\ -1329 & 2228 \\ 1120 & 210 \end{bmatrix} = Q \Lambda R^T$$

Q: ort 3×3; $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 0 & 0 \end{bmatrix}$ 3×2; R^T: ort 2×2

$A^T A = \cancel{R\Lambda R^T}$

$R \Lambda^T \Lambda R^T = B$

$\Lambda^T\Lambda = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}}_{R} \underbrace{\begin{pmatrix} 100 & 0 \\ 0 & 25 \end{pmatrix}}_{\Lambda^T\Lambda} \underbrace{\begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}}_{R^T}$$

Tráhl.

$A \cdot r_1 = Q \Lambda R^T r_1$

$R = [r_1 \quad r_2]$

$\downarrow$ 1. onlap $\quad \downarrow$ 2. onlap

$I = R^T R = [R^T r_1 \quad R^T r_2]$

$\downarrow$ 1. par $\quad \downarrow$ 2. par

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [R^T r_1 \; R^T r_2]$$

$$R^T r_i = e_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow i$$

$A r_1 = Q \Lambda^2 e_1$

$A r_2 = Q \Lambda^2 e_2$

$$\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\downarrow$ 100

$= \lambda_2^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\downarrow$ 25

~~$A r_i = Q \Lambda^2 r_i$~~

$A r_1 = Q \Lambda^2 e_1 = \lambda_1^2 Q e_1 = \lambda_1^2 q_1$

$A r_2 = \lambda_2^2 q_2 \qquad Q = [q_1, q_2 \cdots q_n]$

$\downarrow$ 1. onlap $\quad \downarrow$ 2. onlap $\quad \downarrow$ n. onlap

$$100 \quad \Sigma_1 = \frac{1}{325}\begin{bmatrix} 1 & 728 & \cdots \\ \vdots & \ddots & \vdots \\ \cdots & & 210 \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} \frac{3600}{65} \\ -48/65 \\ 25/65 \end{bmatrix} \Rightarrow q_1 = \begin{bmatrix} \frac{36}{65} \\ 48/100/65 \\ 25/100/65 \end{bmatrix}$$

Xenek/100.fme

$$25 \, q_1 = \begin{matrix} 190/325 \\ -12/325 \\ 125/325 \end{matrix}$$

$$q_2 = \frac{1}{325}\begin{bmatrix} 1728 & \cdots & \vdots \\ \cdots & & 210 \end{bmatrix}\begin{bmatrix} 3/5 \\ 4 \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 96/325 \\ 157/325 \\ 240/355 \end{bmatrix}$$

megérteni $\langle \{q_1, q_2\} \rangle$-öt teljes ortonormális rendszerré $\mathbb{R}^3$-ban.

$$q_3 = \pm q_1 \times q_2$$

$$A = Q \Lambda R^T = [q_1 q_2 q_3] \begin{bmatrix} 10 & 0 \\ 0 & \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

A^TA-nak

R^T

$$\begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix} \quad R = [r_1 \; r_2]$$

oszlop

Mátrix ALT (Moore–Penrose (féle) inverze)

$Ax = b$ hiteles

kísérlet

mérés

ismeretlen, hasznos adat

mérlésből adódó

hiteles mérések eredménye

$Ax = b$

$$\left.\begin{aligned} 2x + 4y - 5z &= 3 \\ 2x + 4y - 5z &= 2{,}5 \\ 2x + 4y - 5z &= 5 \end{aligned}\right\} \text{ellentmondás}$$

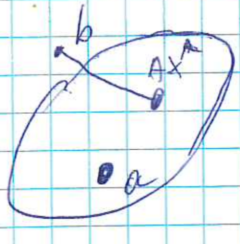
$A: x \mapsto Ax$ pontosan

mi lett a legjobb megoldás? (legkisebb négyzetes)

x

Képzeljük az összes lehető mérést.

$$\langle Ax : x \in \mathbb{R}^n \rangle \to \text{range}(A)$$



→ ALTÉR $\mathbb{R}^n$-ben

Legjobb becslés akkor Ax legközelebb van b-hez.

← bizonyítás $\qquad Ax_* = b$ megoldása vetületére Range(A)-ra

x_* → nem teljesen egyértelmű.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (c_1)\vec{i} + (c_2)\vec{j} + (c_3)\vec{k}$$

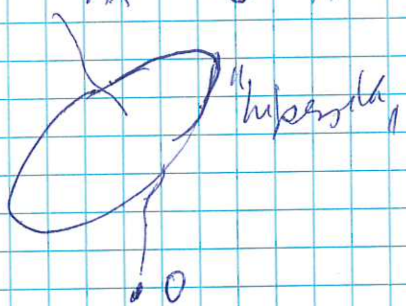
vektoriális szorzat

$Ax_* + u = Ax_*$ ha $Au = 0$

$$A = \begin{bmatrix} 2 & 4 & -5 \\ 2 & 4 & -5 \\ 2 & 4 & -5 \end{bmatrix} \quad u = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix} \text{ stb.}$$

$x_* + u$

$Ax_* - u = Ax$ ⇒ Affin altér



$$\begin{pmatrix} 1 & 2 & -6 \\ & 1 & 12 \end{pmatrix} =$$

Ezek mindegyike legjobb közelítést adó becslése.

Melyik a legjobb??? amelyik hossza a legrövidebb.

EGYÉRTELMŰ.

$$b \mapsto A^+(b) = \begin{bmatrix} x_1(b_1 \dots b_m) \\ x_n(b_1 \dots b_m) \end{bmatrix}$$

I $\quad b \mapsto A^+(b)$ lineáris és a mátrixa $= R\Lambda^+ Q^T$ ha

$A = Q \Lambda R^T_m$ SVD felbontással $\qquad (\lambda_1 \geq \dots \geq \lambda_r > 0)$

$\Lambda^+ = \left[\frac{1}{\lambda_1} \dots \frac{1}{\lambda_r}\right]_n$

I. olyan bázisvektorai akkal $\mathbb{R}^m$-ben $[q_1 \; q_m]$ Q oszlopai i

$\mathbb{R}^n$-ben $r_1 r_n$ R oszlopai.

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ & & \lambda_n \; 0 \end{bmatrix} \qquad \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{bmatrix}$$

$$\tilde{x}_1 = \frac{1}{\lambda_1}\tilde{b}_1 \quad i \quad \tilde{x}_r = \frac{1}{\lambda_r}\tilde{b}_m \quad \underbrace{\tilde{x}_{r+1} \ldots \tilde{x}_n}_{\text{hagyjuk}} = b$$

(1) Megjegyzés, ha A invertálható akkor $A^+ = A^{-1}$

(2) ~~A → SVD~~ SVD → A^+ nehézkes

3, jobb ~~eset~~ esetek $A = \underbrace{\boxed{\quad\quad\quad}}_{}$ (n oszlop) sor

$r = \mathrm{rank}(A) \leq n$

r független sor van benne.

oszt $A = \boxed{\quad}$ m sor, r oszlop

$r = \mathrm{rank}(A) \leq m$

r független oszlop van

$A^+ = ?$

$A A^T = \boxed{r \times r}$ invertálható $\quad (A A^T)^{-1}$ van

$A^+ = A^T (A A^T)^{-1}$

$A^T A = \boxed{r \times r}$ invertálható $\quad (A^T A)^{-1} A^T = A^+$

biz: $r = \underbrace{\mathrm{rank}(A)}_{\text{sorokra}} \leq n$ ossz. $A = Q\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix}$

$$AA^T = Q\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix} R^T R \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{bmatrix} Q^T =$$

$$Q \cdot \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{bmatrix} Q^T$$

$$\cancel{(AA^T)^{-1}} = Q\begin{bmatrix} 1/\lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\lambda_r^2 \end{bmatrix} Q^T \underbrace{\cancel{Q}}_{I} \cancel{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}} \cdot R^T \Rightarrow$$

$$\cancel{Q\begin{bmatrix} \lambda_1/\lambda_1^2 & & 0 \\ & \ddots & \\ & & \lambda_r/\lambda_r^2 \end{bmatrix}}$$

$$A^T(AA^T)^{-1} = R \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \\ & & \end{bmatrix} \underbrace{Q^T Q}_{I} \begin{bmatrix} 1/\lambda_1^2 & & \\ & \ddots & \\ & & 1/\lambda_r^2 \end{bmatrix} Q^T = R\begin{bmatrix} \lambda_1/\lambda_1^2 & & \\ & \ddots & \\ & & \lambda_r/\lambda_r^2 \\ & & \end{bmatrix}$$

$$\cdot Q^T = R\begin{bmatrix} 1/\lambda_1 & & \\ & \ddots & \\ & & \frac{1}{\lambda_r} \end{bmatrix} Q^T = A^+$$

4, Sajnos ált. nem igaz, hogy $(AB)^+ = \underbrace{\cancel{\quad}}_{B^+A^+}$ pedig ilyenkor $(n \times n)$-es mátrixra $(AB)^{-1} = B^{-1}A^{-1}$

5, $(F_1 F_2)^+ = F_2^+ F_1^+$ ha $F_1 = {}_m\overset{r}{\boxed{}}$ $\mathrm{rank}(F) = r$ és $F_2 = \overset{n}{\boxed{}}\, r$

$\mathrm{rank}(F) = r$

I feladatsor célja: A tetszőleges $m \times n$-es mátrix. CTETSZŐLEGES test felett, nem csak $\mathbb{R} / \mathbb{C} / \mathbb{Q}$ pl: $\mathbb{Z}_p$ felett

$r = \text{rank}(A) \Rightarrow$

$\exists\ F_1\ F_2$ mint r-o partban, hogy $A = F_1 \cdot F_2$

Pl Faktorizációs felbontás $\rightarrow$ $\boxed{\text{ált. inverz}}$

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 5 \end{bmatrix}$$

A számából $r = \text{rank}(A)$ lin. független sort akarok számolni.

(Gauss – Jordan elimináció)

$$A_1 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(1. sor) marad
(2. sor) − 2× 1. sor
(3. sor) − 1. sor
(4. sor − 1. sor)

~~(1. sor) − 3×4. sor)~~ → 1. sor nem változik
(2. sor) − 3.×4. sor)
(3. sor) − 2×4. sor
(4. sor)

egyes oszlop

$$A_2 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$F_1 = ?$

$$F_2 = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \qquad F_1 = \begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}$$

$F_1 \cdot F_2 = A$

lesznek 1-esek, meg 0-k

(Ezek lesznek 0)

$$\begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{matrix} a & b \\ c & d \\ e & f \\ g & h \end{matrix} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 3 & 12 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 1 & 5 \end{bmatrix} =$$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} = F_1$$

$A^+ = F_2^+ F_1^+$

① $A^T A$ sajátértékeit számolom ki.

Λ

$|B|$ — 2 találat — $\lambda_{1,2}$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\begin{bmatrix} b_{11}-\lambda_1 & b_{12} \\ b_{21} & b_{22}-\lambda_1 \end{bmatrix} \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$$

találni egy párt $\begin{bmatrix} z \\ u \end{bmatrix}$

$\sqrt{z^2+u^2} = k$

$r_1 = \begin{bmatrix} z/k \\ u/k \end{bmatrix}$ $\quad R = \begin{bmatrix} r_1 & r_2 \end{bmatrix}$ → sajátvektorok

$A \cdot r_i$

$A \cdot r_i = l_i \cdot g_i$ $\quad \boxed{g_i = A r_i / l_i}$

adott megvan erő.

$A^T A = R \; \Lambda^T \Lambda \; R^T = B$

$$\begin{bmatrix} \text{normált} \\ \text{vektorok} \end{bmatrix} \begin{bmatrix} \text{saját é.} & 0 \\ 0 & \text{saját é.} \end{bmatrix} \begin{bmatrix} \end{bmatrix}^T$$

$r_1 \quad r_2$

~~Arr~~ $g_i = A \cdot r_i / l_i$

mátrix sajátvektor

annyi a i. ~~sajátérték~~ sajátértékek.

szélsőségben A → gyök sajátértékek

$\boxed{\text{4. óra}}$

$\mathbb{R}$ felett legjobban közelítő megoldást is megtaláltuk.

$A = m \times n$

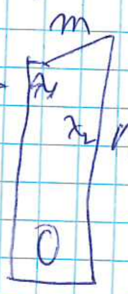
$A^+ = n \times m$

$A = Q \Lambda R^T$ (Q: $m \times m$, Λ: $m \times n$, $\lambda_1, \dots, \lambda_r, 0$)

$A \overset{SVD}{\rightsquigarrow} Q \Lambda R^T \rightsquigarrow R \Lambda^+ Q^T = A^+$ $\mathbb{R}^n$-en

(általános)

$R \Lambda^+ Q^T$



$\Lambda^+ = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \\ 0 & & \end{bmatrix}$ ($n \times m$)

ÁLG. inv. $\mathbb{R}^n$-en $\quad \mathbb{R}^n \sim \mathbb{K}^n$

A^+ ekvivalens tulajdonságai: $\quad [A \text{ invert} \iff A^{-1}A = AA^{-1} = I]$

$A^+ A A^+ = A^+$!

$A A^+ A = A$!

$\square_n \; \det \neq 0 \qquad A^{-1} A A^{-1} = A^{-1}$

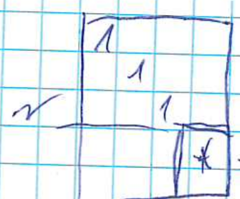
$A A^{-1} A = A$

Legjobb közelítés rendszeren.

$A^+ = Y \tilde{\Lambda} X^{-1}$

$X A Y^{-1} = \begin{bmatrix} \Lambda_r & \\ & 0 \end{bmatrix}$

$X = [x_1 \dots x_m] \quad Y = [y_1 \dots y_n] \quad \tilde{\Lambda}$



$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & * \end{bmatrix}$ (r) $\to$ itt más is lehet.

Frobenius felbontás:

$A = F_1 F_2 \qquad F_1 = m \times r$, $r = \text{rang}$ $\qquad A^+ = F_2^+ F_1^+$

$F_2 = r \times n$

$F_1 = \boxed{F_1}$ $\quad F_1^+ = \boxed{F_1^+} = \left[F_1^T F_1\right]^{-1} F_1^T$

$F_2 = \boxed{F_2}$ $\quad F_2^+ = \boxed{F_2^+} = F_2^T \left(F_2 F_2^T\right)^{-1}$

Q fölött nem A^+

VEKTOR analízis.

felületformula

Alapfogalmak $\mathbb{R}^n$ K dimenziós C^1 halmaz ($K \le n$)

integrálás C^1 halmazon.

Def: (C^1 halmaz) $Q : \underbrace{[a_1, b_1] \cdots [a_K, b_K]}_{\text{valós intervallumok}} \to \mathbb{R}^N$ $\quad$ /Q ~ halmaz: $\square$

$$Q(\tau_1, \dots, \tau_K) = \begin{bmatrix} Q_1(\tau_1 \dots \tau_K) \\ Q_2(\tau_2 \dots \tau_K) \\ \vdots \\ Q_N(\tau_N \dots \tau_K) \end{bmatrix} \qquad \begin{matrix} a_1 \le \tau_1 \le b_1 \\ a_2 \le \tau_2 \le b_2 \\ \vdots \\ a_N \le \tau_N \le b_N \end{matrix}$$

$Q_1 \dots Q_k$ folytonosan differenciálható (C^1 - cont)
continuity

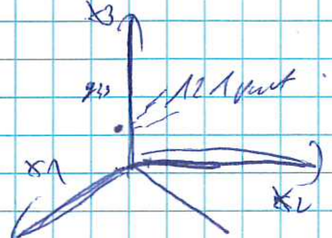
példa: Skalár $K=2$ $N=3$ $\quad$ 2 dimenziós felület 3D térben.

$Q(\tau_1, \tau_2) = \tau_1^2$ $\quad Q_2(\tau_1, \tau_2) = \tau_2^2$ $\quad Q_3(\tau_1, \tau_2) = \tau_1 + \tau_2$

$(a_1, b_1) = (a_2, b_2) = [0, 1]$

$\tau_1 = 0.0, 0.1, 0.2 \dots 0.9, 1.0,$

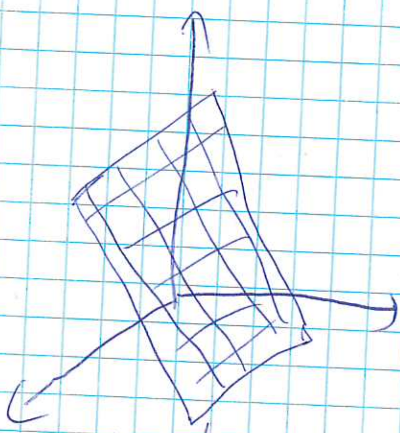
$\tau_2 = -\|-$ $\quad , 1.0,$



1,1,1 random pont

$q_{ij} = Q(\tau_i, \tau_j)$

pl: $q_{2,3} = (0{,}1\,,\, 0{,}2) = \begin{bmatrix}0{,}01\\0{,}04\\0{,}3\end{bmatrix}$



$i=2 \; j=3$

$x(q_{ij}) = 0{,}01 \quad \tau_i = \sqrt{0{,}01} = 0{,}1$

$\downarrow$

q_{ij} x koordinátája

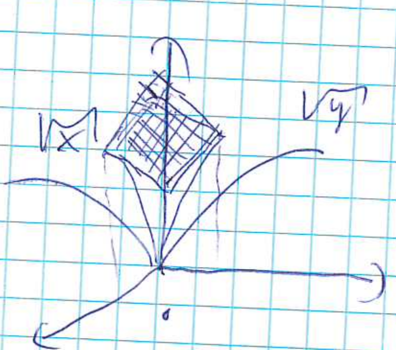
$y(q_{ij}) = 0{,}04 \quad \tau_j = \sqrt{0{,}04} = 0{,}2 = \sqrt{y(q_{ij})}$

$z(q_{ij}) = 0{,}3 = 0{,}1 + 0{,}2 = \tau_i + \tau_j = \sqrt{x(q_{ij})} + \sqrt{y(q_{ij})}$

~~[illegible]~~

~~[illegible]~~

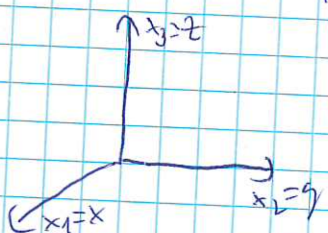
$z(q) = \sqrt{x(q)} + \sqrt{y(q)} \quad z = \sqrt{x} + \sqrt{y}$



Def: Q injektív $\iff (\tau_1, \tau_2) \neq (\bar{\tau}_1, \bar{\tau}_2)$

$Q \quad Q(\tau_1, \tau_2) \neq Q(\bar{\tau}_1, \bar{\tau}_2)$

Alapkoordináták: $\mathbb{R}^N$-en $x_1, x_2 \ldots x_N$:



$x_i : \begin{bmatrix}\xi_1\\ \vdots\\ \xi_n\end{bmatrix} \to \xi_i$

pl: $x_2 \begin{bmatrix}0\\2\\4\\6\end{bmatrix} = 2$

$x_4 \begin{bmatrix}0\\2\\4\\6\end{bmatrix} = 6$

Technikai okból $\mathbb{R}^k$ felett $\tau_1, \ldots, \tau_k$ $\quad \tau_j : \begin{bmatrix}\tau_1\\ \vdots\\ \tau_k\end{bmatrix} = \tau_j$

Pl: $Q(\tau_2, \tau_1) = \begin{bmatrix}\tau_1^2\\ \tau_2^2\\ \tau_1 + \tau_2\end{bmatrix} \quad Q = \begin{bmatrix}\tau_1^2\\ \tau_2^2\\ \tau_1 + \tau_2\end{bmatrix}$

$x_2(Q) = \tau_2^2$

Alapfogalom: PARALELLEPIPEDON

[p alapponttal, ~~$v_1 \dots v_k$~~ $v_1 \dots v_k$] oldalvektorokkal =

$\mathbb{R}^N$ k-ban ez.

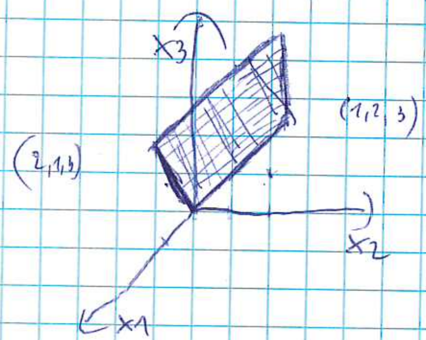
$\langle p + \tau_1 v_1 + \tau_2 v_2 \dots \tau_k v_k$

$0 \le \tau_1 \le \tau_i \le \tau_k \le 1$

pl: $p = 0$ → régi alappont.

$N=3$ $K=2$

oldal: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$



K-DEN Térfogat: ($\mathbb{R}^N$)

HA egy paralellepipedon oldalvektorai $x_1, \dots, x_K$ és egymásra merőlegesek (téglatest) $x_1 \dots x_k$ = térfogat.

Nagyobb test nagyobb térfogat → Nem MEGY

Járási kérdés: $\forall A \subset \mathbb{R}^3$ $\exists v = V(A)$ térfogat. → diszjunkt unió!

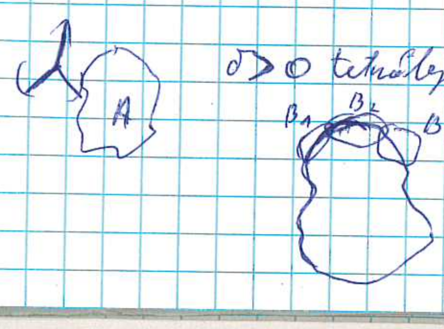
Neumann János Gömb = $G_1 \dot\cup G_2 \dot\cup G_3$ $\mathbb{R}^3$-ban

G_1, G_2, G_3 egybevágóak. $\exists G_1', G_3', G_2'$ egybevágóak

G_k-val $G_1' \dot\cup G_2' \dot\cup G_3'$ = [NAGYOBB GÖMB]

Hausdorff: térfogatmérés k ~~dim~~ dimenziós térben.

$A \subset \mathbb{R}^N$ $\delta > 0$ tetszőlegesen adott [vagy legalábbis $\delta = 0{,}1$, 0.01, 0.001, 0.0001]



$\delta \leq$ átmérő(d_i) halmazokkal lefedjük A-t. Minden lehetséges módon. $B = \{B_1 \dots B_N\}$ lefedéshez vegyük

$$Vol_{K\delta_B}(A) = \sum_{i=1}^{\infty} [\text{diam } B_i]^K$$

$$Vol_{K\delta}(A) = \text{infimum } Vol_{K,\delta,B}(A)$$

B lehetséges.

$$\overline{Vol}_K(A) = \lim_{\delta \to 0} Vol_{K,\delta}(A) \cdot \mu_K$$

μ_K választása: $\mu_K \, Vol_K([0,1]^K \times \langle 0 \rangle^{N-K}) = 1$

$\boxed{\text{4. óra}}$

$A \subset \mathbb{R}^n \quad K \in \mathbb{R}$ (eddig $K \leq N$ egész szám volt)

$$Vol_{K,\delta}(A) = \inf\left\{ \sum_{n=1}^{\infty} (\text{diam } B_n)^K : B_1 \cup B_2 \cup \dots \supset A \right\}$$

$$\overline{Vol}_K(A) = \lim_{\delta \to 0} Vol_{K,\delta}(A)$$

1, A, A' egybevágóak $\exists T: A \to A'$ $d(T(x), T(y)) = (x,y)$ $\forall x,y \in \mathbb{R}^n$)

ráforgatható.

$Vol_K(A') = Vol_K(A)$

2, $\lambda A = \langle \lambda a : a \in A \rangle$

$Vol_K(\lambda A) = \lambda^K \, \overline{Vol}_K(A)$

felnagyítom

3, lehúzós tétel: $T: A \to \mathbb{R}^n$ kontrakció ($d(T(a), T(b)) \leq d(a,b)$ $\forall a,b \in A$)

$\Downarrow$

1) $\overline{Vol}_K(TA) \le \overline{Vol}_K(A)$

gátlóban a Vol.

Kérdés: MELYEK a $\overline{Vol}_K$ szerint mérhető részhalmazok ???

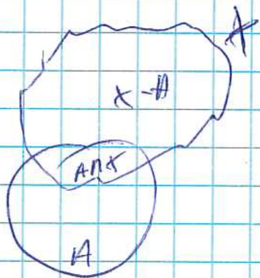
Kolgorov:

$\overline{Vol}_K(B) = \overline{Vol}_K(A)$

B határolja A-t.

Vol_K mérhető $\iff \forall X \subset \mathbb{R}^N$

$\overline{Vol}_K(A) = \overline{Vol}_K(A \cap X) + \overline{Vol}_K(X - A)$



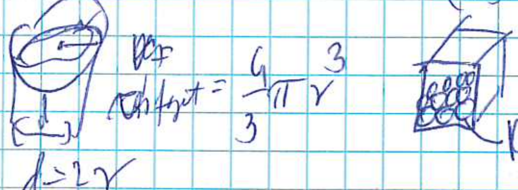
T 1, A zárt halmaz $\Rightarrow$ A $\overline{Vol}_K$ mérhető tetszőleges K-ra.

2, $A = \bigcup_{n=1}^{\infty} A_n$ $\quad A_1, A_2 \dots \overline{Vol}_K$ mérhető $\Rightarrow$ ~~$\overline{Vol}(A) = \sum_{n=1}^{\infty} \overline{Vol}_K(A_n)$~~

3, A, B V_K mérhető $\Rightarrow A \setminus B$ is $\overline{Vol}_K$ mérhető $\quad Vol_K(A) = \sum_{n=1}^{\infty} \overline{Vol}_K(A_n)$

4, $k = N$

$\overline{Vol}_K \, (0,1)^N =$



térfogat $= \frac{4}{3}\pi r^3$

$d = 2r$

B_n

δ_n átmérőjű gömb

Térf: $\frac{4}{3}\pi\left(\frac{d_n}{2}\right)^3$

Felsorolható gömbökkel $\sum_{i=1}^{\infty} \frac{4\pi}{3}\left(\frac{\delta_i}{2}\right)^3 = 1$ és a maradék tetszőlegesen kis gömbökkel lefedhető.

$B_1, B_2 \dots B_N$

SAJT CEMMA

$$\mathrm{Vol}_3\,[0,1]^3 = \sum_{n=1}^{\infty} \delta_n^3 = \frac{1}{\left[\frac{4\pi}{3}\,\frac{1}{8}\right]} \sum_{n=1}^{\infty} \frac{\pi}{3}\,\frac{\delta_n^3}{8} = \frac{8}{\pi/3}$$

Ált is N-dim $\overline{\mathrm{Vol}}_N\,[0,1]^N = \frac{2^N}{\omega_N}$ → N dim egységsugarú gömb térfogata

~~90 - 31 - 31 - 31~~

Pl: $N=2$ $\mathrm{Vol}_2[0,1]^2 = \frac{2^2}{\omega_2} = \frac{4}{\omega_2} = \frac{4}{\pi}$

$\omega_N = \pi^{N/2}\,\Gamma\left(1+\frac{n}{2}\right)$

Γ egységgömb térfogata.

↓ gamma

$\Gamma(M) = (M-1)!$

↓ egész

$\omega_2 = \pi^{2/2}\,\Gamma\left(1+\frac{2}{2}\right) = \pi\,\Gamma(2) = \pi\,1! = \pi$

Def: Normált K-dim HAUSDORFF-térfogat:

$\underbrace{\overline{\mathrm{Vol}}_K(A)\cdot\frac{2^K}{\omega_K}}_{\mathrm{Vol}_K(A)}$ HA K egész. $\omega_K = \pi^{K/2}\,\Gamma\left(1+\frac{K}{2}\right)$

Alakzat dimenziója:

T nem def-lett.

HA $\mathrm{Vol}_K(A)=0$ akkor $\overline{\mathrm{Vol}}_\lambda(A)=0$ $\lambda > K$

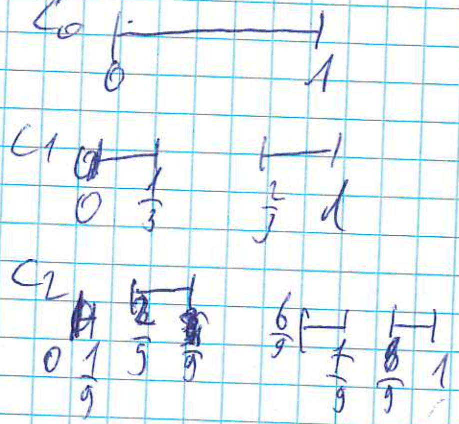
HA $\mathrm{Vol}_K(A) < \infty$ akkor $\overline{\mathrm{Vol}}_\lambda(A) = \infty$ ha $\lambda < K$

ha $\mathrm{Vol}_K(A) = \infty$ akkor $\mathrm{Vol}_\lambda(A) > \infty$ ha $\lambda < K$

$d = \dim(A)$

Egy nem egész dim alakzat:

Cantor halmaz (első fraktál)



1, LU 3x3 $\mathbb{Z}_5$-ön ✓

2, QR 3x3 ✓

3, főtengely trafó 2x2 szimmetrikus ✓

4, SVD felbontás 3x2

5, ált. inverz 3x3 (det=0)

$C_{i+1} = C_i$-ből kihagyjuk a középső harmadat

$$C = \bigcap_{n=1}^{\infty} C_n \quad (\neq \emptyset)$$

$$\dim C = \frac{\log 2}{\log 3}$$

PARALELLEPIPEDON térfogata

$P = \{p + [0,1)v_1 + [0,1)v_2 + \dots + [0,1)v_n\}$ $p, v_1, v_n \in \mathbb{R}^n$

$Vol_N(P) = ?$

$Vol_N(P) = Vol_N([0,1]v_1 + \dots + [0,1]v_n)$ eltolás nem Vol_N változtató.

$V = [v_1 \dots v_K] = \underbrace{\square}_{K}{}_N$

oszlopok

OKT + ReF nem vált. a térfogatot

(mb OKT szorzódik egyenesen ~~ua~~ ua a térfogat)

$SVD: V = Q \Lambda R^T_{K\times K}$, $\Lambda = \begin{bmatrix} \Lambda' \\ 0 \end{bmatrix}_{N\times K}$, $Q_{N\times N}$

$\mathrm{Vol}_K(Q^T V R) = \mathrm{Vol}_K(V)$ $20+10=30$

$120-30=90$ $37+34=$

$\|$ základní paralelepiped $\downarrow$ základní paralelepiped $90-71=19$

$\mathrm{Vol}_K(\Lambda) = \lambda_1 \cdots \lambda_K$ Λ základní paralelogram

kvádr λ_1, λ_2 e_1, e_2

$V = \ldots$

$\lambda_1 \ldots \lambda_k$ ...

-2
-3
14
-5
$9-5$
4
2
3
3
3
2

$$V^T V = R \Lambda^T \underbrace{Q^T Q}_{1} \Lambda R^T = R \Lambda^T \Lambda R^T = R \begin{bmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \ddots \lambda_k^2 \end{bmatrix} R^T$$

$\mathrm{diag}(\lambda_1 \ldots \lambda_k)$

$$\det(V^T V) = \det R (\lambda_1 \ldots \lambda_k) \cdot R^T = (\det R) \cdot \det \mathrm{diag}(\lambda_1^2 \ldots \lambda_k^2)(\det R^T) =$$

$$(\pm 1)(\lambda_1^2 \lambda_2^2 \cdots \lambda_k^2)(\pm 1) = \lambda_1^2 \lambda_2^2 \cdots \lambda_k^2$$

$$\mathrm{Vol}_K(P) = \sqrt{\det(V^T V)} = \lambda_1 \cdots \lambda_k$$

$$\mathrm{Vol}_2\left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \text{ rovnoběžník}\right) = \left[\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\right]^{\frac{1}{2}} =$$

$$\left(\det \begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}\right)^{\frac{1}{2}} = \sqrt{14 \cdot 77 - 32^2} = \sqrt{54} = 7{,}34$$

$$\det \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}^2 + \det \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}^2 + \det \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}^2 = \sqrt{(-3)^2 + 6^2 + (-3)^2} = \sqrt{54}$$

$9 \quad 36$

inverz: ha $\det \neq 0$

$$\frac{1}{(7\cdot 14)-(9\cdot 9)}\begin{bmatrix}7 & 9\\ 9 & 14\end{bmatrix} = \frac{1}{17}\begin{bmatrix}14 & -9\\ -9 & 7\end{bmatrix}$$

$A = (3\times 2)$

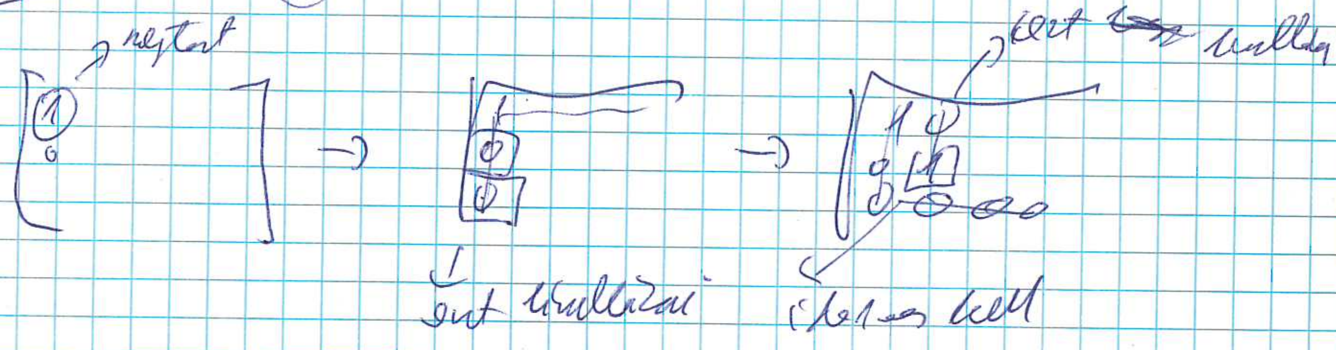
$A^TA \to \underbrace{[\ \ ]}_{2\times 2}$ négyzetes

↓

négyzetes mátrix

go

Alt eset: ha lehet Gauss



és $\emptyset$ kell .... Z_1^+ Z_2^+

Ha nem lehet Gauss

$A^T(AA^T)^{-1}$ ↓ rank $= n$

-1 ↓ $\frac{1}{\det}\,\mathrm{adj}(AA)$

$\boxed{5 \text{ dim}}$

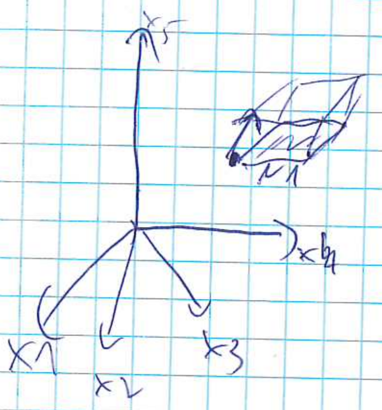
intervallum

$$P = p + [0,1]\underbrace{\begin{bmatrix}1\\1\\2\\3\\4\end{bmatrix}}_{v_1} + [0,1]\underbrace{\begin{bmatrix}0\\0\\1\\0\\2\end{bmatrix}}_{v_2} + [0,1]\underbrace{\begin{bmatrix}0\\1\\1\\0\\0\end{bmatrix}}_{v_3}$$

5-ant $\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \rightarrow$ vektor $\mathbb{R}^5$-ból

$P \subset \mathbb{R}^5$ $\quad$ P 3 dim parallelepipedon $\mathbb{R}^5$-ben

$\rightarrow$ mert 3 vektor van.



$Vol_3 P = ?$

$Vol_{2,5} = \infty$

$Vol_{4,1} = 0$

$$Vol_k P = \begin{cases} 0 & \text{ha } k > 3 \\ \infty & \text{ha } k < 3 \end{cases}$$

$\sqrt{\det(V^T V)} =$ $\quad a \cdot b =$ [illegible] skalárszorzata

$$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

$$\underbrace{V^T V}_{3\times 3} = \begin{bmatrix} \langle v_1|v_1\rangle & \langle v_1|v_2\rangle & \langle v_1|v_3\rangle \\ \langle v_2|v_1\rangle & \langle v_2|v_2\rangle & \langle v_2|v_3\rangle \\ \langle v_3|v_1\rangle & \langle v_3|v_2\rangle & \langle v_3|v_3\rangle \end{bmatrix}$$

$v_1|v_2 = v_2|v_1$

$$\begin{bmatrix} 31 & 10 & 3 \\ 10 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix} \quad \det(V^T V)$$

$31 \cdot 5 \cdot 2 + 10 \cdot 1 \cdot 3 + 3 \cdot 10 \cdot 1 =$

$- 9 \cdot 5 - 10 \cdot 10 \cdot 2 - 31$

$310 \quad 30 + 30$

$-45 - 200 - 31 = 94$

$$\underbrace{\det(V^TV)}_{\geq 0} = \det\begin{bmatrix} 1 & -5 & 0 \\ 10 & 5 & 1 \\ -17 & -9 & 0 \end{bmatrix}$$

kiépsável lett sz... v mat

szorzese miatt ⊖ előj... el lett ... a det-et

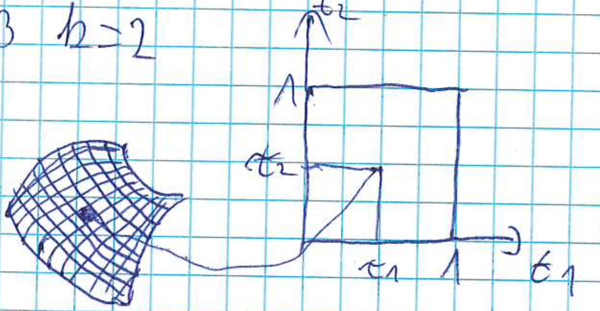
$$\begin{bmatrix} 1 & -5 & 0 \\ 0 & 55 & 1 \\ -17 & -9 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 55 & 1 \\ 0 & -94 & 0 \end{bmatrix} = -\det\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 55 \\ 0 & 0 & -94 \end{pmatrix}$$

$\sqrt{94}$

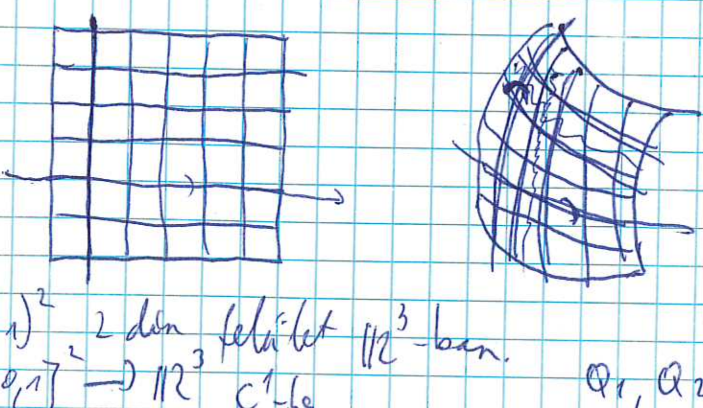
Felszín formula:

C^1-kockoc [k-dim C^1 kocka $\mathbb{R}^n$-ben]

$n=3$ $k=2$



$$\begin{bmatrix} Q_1(t_1,t_2) \\ Q_2(t_1,t_2) \\ Q_3(t_1,t_2) \end{bmatrix} = Q(t_1,t_2)$$



$Q(0,1)^2$ 2 dim felület $\mathbb{R}^3$-ban.

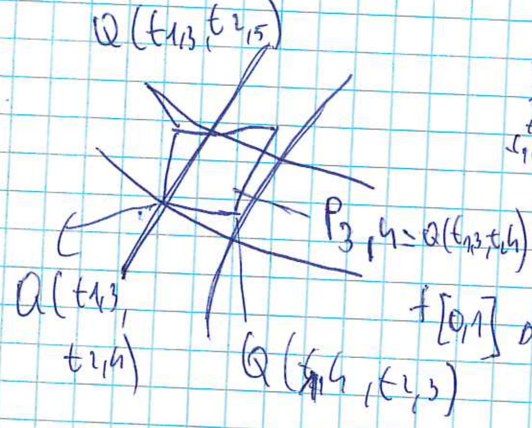
$Q: [0,1]^2 \to \mathbb{R}^3$ C^1-ben

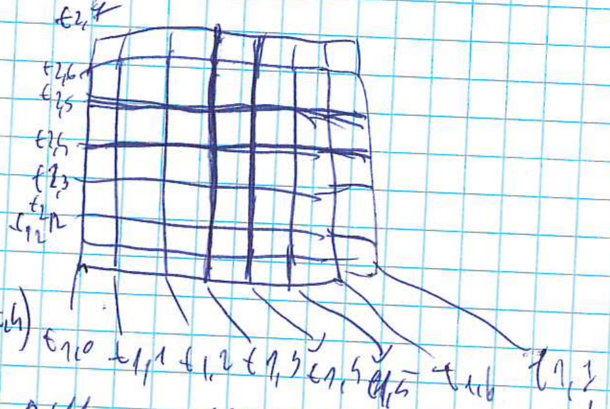
Q_1, Q_2, Q_3 folyt. Diff. értelmű...

$(\tau_1, \tau_2) \neq (\sigma_1, \sigma_2) \Rightarrow Q(\tau_1, \tau_2) \neq Q(\sigma_1, \sigma_2)$

Vol_K $\quad Vol_K(Q[0,1]^K)$

Euler-PIKKELYEK





$f[0,1]$ deff. közelítés $+ [0,1]$ (deff. közelítés) ≈ 1

$Q(t_{1,4}\ t_{2,4}) - Q(t_{1,3}\ t_{2,4})$

deff. közelítés 1. tagja.

ALT is $\quad Q(t_{1,3}\ t_{2,5}) - Q(t_{1,3}\ t_{1,5}) - Q(t_{1,5}\ t_{2,4})$

$P_{i_1 \dots i_k} = Q(t_{1,i_1} \dots t_{k,i_k}) + \sum_{d=1}^{K} [0,1] \cdot [\text{deff közelítés } Q(t_{1,i_1} \dots t_{k,i_k}) -$

dimenzió

t_{d,i_d+1} helyett $t_{d,\ i_d+1}$

$- [\text{deff } Q(t_{1,i_1} \dots, t_{d,i_d+1}, \dots t_{k,i_k}) - Q(t_{1,i_1}, \dots, t_{k,i_k})] =$

$\frac{\partial Q}{\partial t_d}(t_{1,i_1} \dots t_{k,i_k}) \cdot (t_{d,i_d+1} - t_{d,i_d}) = \frac{\partial Q}{\partial t_1}(t_{i_1}, t_{i_2} \dots i_k)$

származtatás.

$Vol_K[\text{PIKKELYFELÜLET}] = \sum_{i_1 \dots i_k} Vol_K\left(\left\langle \frac{\partial Q}{\partial t_1} b_{i_1} t_1, \dots, \frac{\partial Q}{\partial t_k} b_{i_k} t_k \right\rangle\right)$

$$V_{0,ik} = \left(\frac{\partial Q}{\partial t_1}(t_{i1} \ldots i_k)\Delta_{i1} t_1, \; \frac{\partial Q}{\partial t_2}(t_{i1} \ldots t_{ik})\Delta_{i2} t_2, \; \ldots, \; \frac{\partial Q}{\partial t_k}(t_{i1} \ldots t_{ik}) \cdot \Delta_{ik} t_k \right)$$

n oldalvektor

$$\mathrm{Vol}_k([t_{i1}, i_k]) = \sqrt{\det\left(V_{01}^T \ldots {}_{ik} V_{01} \ldots V_{ik}\right)} =$$

közelítés

$$\sqrt{\det\left(\frac{\partial Q}{\partial t_1}, \ldots, \frac{\partial Q}{\partial t_k}\right)^T \left(\frac{\partial Q}{\partial t_1}, \ldots, \frac{\partial Q}{\partial t_k}\right)} \cdot \Delta_1 t_{i1} \ldots \Delta_k t_{ik} \; t_{i1} \ldots i_k$$

beosztás sűrítés

$$\int_{t_1=0}^{1}\int_{t_2=0}^{1} \ldots \int_{t_k=0}^{1} \sqrt{\det\left(\left[\frac{\partial}{\partial t} Q\right]^T \left[\frac{\partial Q}{\partial t}\right]\right)} \, dt_1 \ldots dt_k$$

$(\Sigma \to \int)$

$$\underline{T} \text{ (felszín formula)} = \mathrm{Vol}_k\left(Q[0,1]^k\right) = \int_{t_1=0}^{1} \ldots \int_{t_k=0}^{1} \sqrt{\det (Q')^T Q'} \, dt_1 \ldots dt_k$$

Pl: $Q(t_1, t_2) = \begin{bmatrix} t_1 \cos(2\pi t_2) \\ t_1 \sin(2\pi t_2) \\ t_1^2 \end{bmatrix}$ ahol $(t_1, t_2) \in [0,1]^2$

$$Q(t_1,t_2) = \begin{pmatrix} Q_1(t_1,t_2) \\ Q_2(t_1,t_2) \\ Q_3(t_1,t_2) \end{pmatrix} \quad Q' = \begin{bmatrix} \frac{\partial Q}{\partial t_1} & \frac{\partial Q}{\partial t_2} \\ | & | \end{bmatrix} = \begin{bmatrix} \cos(2\pi t_2) & 2\pi t_1 - \sin(2\pi t_2) \\ \sin(2\pi t_2) & 2\pi t_1 \cos(2\pi t_2) \\ 2t_1 & 0 \end{bmatrix}$$

3-as vektor

$\frac{dQ}{dt_1}$

$\boxed{\cdot \; \Delta_i d \, t_d}$

$$Q' = \begin{bmatrix} \partial Q_1/\partial t_1 & \partial Q_1/\partial t_2 \\ \partial Q_2/\partial t_1 & \partial Q_2/\partial t_2 \\ \partial Q_3/\partial t_2 & \partial Q_3/\partial t_2 \end{bmatrix} \Rightarrow \frac{\partial Q}{\partial T} = \begin{bmatrix} \cos(2\pi t_2) & -2\pi t_1 \sin(2\pi t_2) \\ \sin 2\pi t_2 & 2\pi t_1 \cos(2\pi t_2) \\ 2t_1 & 0 \end{bmatrix}$$

$t_1 \quad t_2$

Newton

Leibniz

2

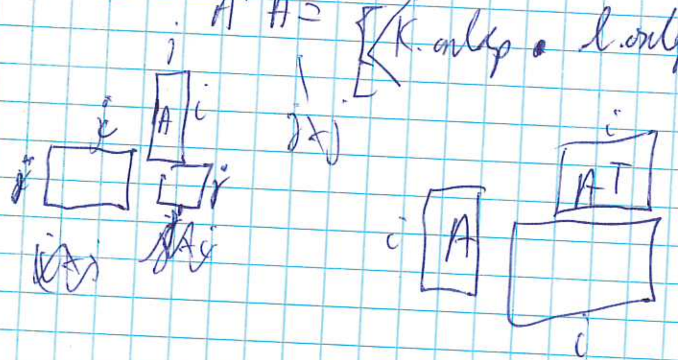
$$= \int_{t_1=0}^{1}\int_{t_2=0}^{1} \cancel{[\;]} \, dt$$

$$Q'^T Q' = \begin{bmatrix} 1+4t_1^2 & 0 \\ 0 & 4\pi^2 t_1^2 \end{bmatrix} = \begin{bmatrix} \langle i.\text{oszlop} \cdot j.\text{oszlop} \rangle \\ \end{bmatrix}_{i,j=1}$$

$A \in \text{Matrix}(i,j)$

$$A^T A = \Big[\langle k.\text{oszlop} \cdot l.\text{oszlop} \rangle \Big]_{k,l=1}^{j}$$

$$A A^T = \Big[\langle k.\text{sor} \mid l.\text{sor} \rangle \Big]_{k,l=1}^{i}$$



$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \int_{t_1=0}^{1}\int_{t_1=0}^{1} \left[\det \text{diag}\left(1+4t_1^2, 4\pi^2 t_1^2\right) \right]^{\frac{1}{2}} dt_1, t_2 =$$

$$\int_{t_1=0}^{1}\int_{t_2=0}^{1} \left[(1+4t_1^2) \cdot (4\pi^2 t_1^2)\right]^{\frac{1}{2}} dt_1\, dt_2 = 2\pi \int_{t_1=0}^{1}\int_{t_2=0}^{1} \sqrt{1+4t_1^2}\, t_1 \, dt_2\, dt_1 =$$

$$2\pi \int_{t_1=0}^{1} \sqrt{1+4t_1^2}\, \underbrace{t_1\, dt_1}_{d} = 2\pi \int_{u=\sqrt{1+4\cdot 0^2}}^{\sqrt{1+4\cdot 1^2}} \sqrt{u}\, \frac{1}{8}\, du = 2\pi \cdot \frac{1}{8} \int_{u=1}^{u=\sqrt{5}} u^{\frac{1}{2}}\, du =$$

helyettesítéses integrál.

$u = 1+4t_1^2$

$$\frac{\pi}{4} \int_{u=1}^{u=\sqrt{5}} u^{\frac{1}{2}}\, du =$$

$$\int_{t_2=0}^{1} x = \left[x\, t_2 \right]_0^1 = x$$

$$\frac{du}{dt_1} = 8t_1$$

$$du = 8t_1 \cdot dt_1$$

$$\left[\frac{\pi}{4}\cdot\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{u=1}^{\sqrt{5}} = \frac{\pi}{6}\left[\sqrt{5}^{3/2}-1\right] = \frac{\pi}{6}\sqrt[4]{125}-1$$

$$\to 5^{3\cdot\frac{1}{2}} = \sqrt{125}^{\frac{1}{2}} = \sqrt[2]{125}^{\frac{1}{2}} = \sqrt[4]{125}$$

$$Vol_2\, Q[0,1]^2 = \frac{\pi}{6}\sqrt[4]{125}-1$$

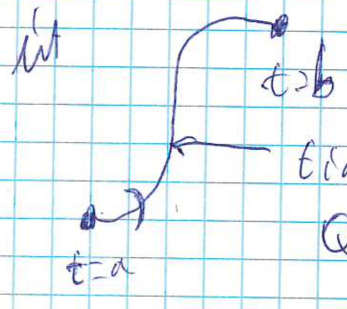
erőtér

Vonalintegrál

Fizi: erő munkája úton

[idő ↦ térbeli pontok]

út

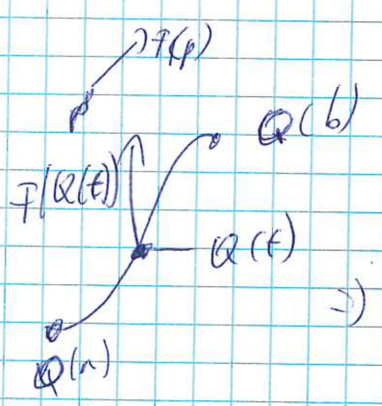


$Q:[a,b]\to\mathbb{R}^3$

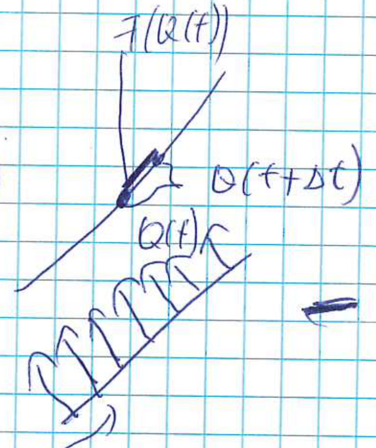
$$Q(t)=\begin{bmatrix}Q_1(t)\\Q_2(t)\\Q_3(t)\end{bmatrix}\in\mathbb{R}^3$$

$F:\mathbb{R}^3\to\mathbb{R}^3$

F erő $p=\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}$ pontban



⇒



Manda ...

≈ egyenes vonalú egyenletes mozgás $\frac{\partial Q}{\partial t}$ sebességgel dt ideig.

Ezen $F(Q(t))$ állandó ideig hat.

$$\Delta W = \left\langle F(Q(t)) \,\middle|\, \frac{\partial Q}{\partial t}\Delta t\right\rangle$$

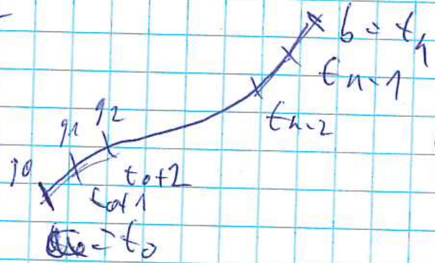
d, kicsi nagyot

$$\hat{=}\ \left\langle F(Q(t)) \,\middle|\, Q(t+\Delta t)-Q(t)\right\rangle$$

$$F(Q(t)) = \begin{bmatrix}F_1(Q(t))\\F_2(Q(t))\\F_3(Q(t))\end{bmatrix}\begin{matrix}\to x \text{ of } F(Q(t))\\ \to y \text{ of}\\ \to z \text{ of}\end{matrix}$$

$$= F_1(Q(t))\,(Q_1(t+\Delta t)-Q_1(t))^{\to x} + F_2(Q(t))\,(Q_2(t+\Delta t)-Q_2(t))^{\to y} + F_3(Q(t))\,(Q_3(t+\Delta t)-Q_3(t))^{\to z \text{ irány}}$$

$W \simeq$



$g_k = \cancel{Q(t_k)}$ $Q(t_k) = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} \begin{matrix} \to Q_1(t_1) \\ \to Q_2(t_2) \\ \to Q_3(t_3) \end{matrix}$

$$W = \sum_{i=1}^{n} x \circ F(g_i) \underbrace{(x_i - x_{i-1})}_{\Delta_i x} + \sum_{i=1}^{n} y \circ F(g_i) \cdot \underbrace{(y_i - y_{i-1})}_{\Delta_i y} +$$

$$\sum_{i=1}^{n} z \circ F(g_i) \cdot \overbrace{(z_i - z_{i-1})}^{\Delta_i z}$$

Emlékeztető: $x \circ F$ $\quad x : \mathbb{R}^3 \to \mathbb{R}$ $\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = g_1$ $\quad$ Szuperpozíció tétele.

$y : \mathbb{R}^3 \to \mathbb{R}$ $\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = g_2$ $\quad$ Határ után változtatás

$z : \mathbb{R}^3 \to \mathbb{R}$ $\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = g_3$

Beadás részletezve:

Van-e határértéke, limesze? $\to$ IGEN

$\Delta_i x = \frac{\partial Q}{\partial t}(g_i) \Delta_i t$ $\qquad W \simeq \sum_{i=1}^{n} x \circ F(g_i) \cdot \frac{\partial Q}{\partial t}(g_i) \cdot \Delta_i t +$

$$\sum_{i=1}^{n} y \circ F(g_i) \frac{\partial Q}{\partial t}(g_i) \cdot \Delta_i t +$$

$$\int_{t=a}^{b} \left[x \circ F(Q(t)) \frac{\partial Q}{\partial t} \cdot x \circ dt \right] + \int_{t=a}^{b} \left[y \circ F(Q(t)) \frac{\partial Q}{\partial t} \cdot y \circ dt \right] \qquad \sum_{i=1}^{n} z \circ F(g_i) \frac{\partial Q}{\partial t}(g_i) \Delta_i t$$

$$+ \int_{t=a}^{b} z \circ F(Q(t)) \frac{\partial Q}{\partial t} \cdot z \circ dt$$

$$\cancel{\frac{\Delta_i x}{\frac{\partial Q}{\partial t}}} \qquad 1 - W\Big|_{\text{maximum}}^{\text{benti}} \leq \frac{\Delta_i x}{\frac{\partial Q}{\partial t}\cdot \Delta t} \leq 1 + W\Big|_{\text{maximum}}^{\text{benti}}$$

$$\text{rendszerele} \longleftarrow \quad \underset{0}{\downarrow} \qquad\qquad\qquad \underset{0}{\downarrow}$$

Közelít $\sum_{i=0}^{n} \left[x \circ F(g_i) \cdot \Delta_i x + y \circ F(g_i) \cdot \Delta_i y + z \circ F(g_i) \cdot \Delta_i z \right]$

$$\to \int_{t=a}^{b} x \circ F(Q(t)) \cdot \frac{x \circ \partial Q}{\partial t} + \int_{t=a}^{b} y \circ F(Q(t)) \cdot \frac{y \circ \partial Q}{\partial t} +$$

$$\int_{t=a}^{b} x \circ F(Q(t)) \cdot \frac{z \circ \partial Q}{\partial t}$$

$$\int_Q \left[(x \circ F)\, dx + (y \circ F)\, dy + (z \circ F)\, dz \right] \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\int_Q \left[f_1\, dx + f_2\, dy + f_3\, dz \right]$$

Def: $Q : [a, b] \to \mathbb{R}^N$ 1 dim C^1-hacka

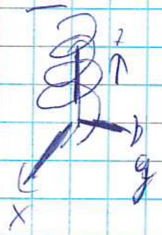
$t \to Q(t) = \begin{bmatrix} Q_1(t) \\ \vdots \\ Q_N(t) \end{bmatrix}$ $\qquad Q_i(t) = x_i \circ Q(t)$

FOLYT DIFF.

$f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}$ $\qquad \int_Q \left[f_1 dx_1 + f_2 dx_2 + \dots + f_N dx_N \right] =$

$$\int_{t=a}^{b} f_1(Q(t)) \cdot \frac{\partial Q_1}{\partial t}\, dt + \int_{t=a}^{b} f_2(Q(t)) \frac{\partial Q_2}{\partial t}\, dt + \dots + \int_{t=a}^{b} f_n(Q(t)) \frac{\partial Q_n}{\partial t}\, dt$$

P1: $F(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$



$Q(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ -\cos(t) \end{bmatrix}$

$t \in [0, 2\pi]$

$\int_Q [(x \circ F)\,dx + (y \circ F)\,dy + (z \circ F)\,dz] = ?$

$\int_Q [-y\,dx + x\,dy + dz]$

$x = Q_1(t)$ $\quad dx = \frac{\partial Q_1}{dt}dt = \frac{\partial \cos t}{dt}dt = -\sin t\,dt$

$y = Q_2(t)$ $\quad dy = \frac{\partial Q_2}{dt}dt = \frac{\partial \sin t}{dt}dt = \cos\,dt$

$z = Q_3(t)$ $\quad dz = \frac{\partial Q_3}{dt}dt = \frac{\partial -\cos dt}{dt} = -\sin t\,dt$

$\int_{t=0}^{2\pi} [-\sin t\,(-\sin t)\,dt + \cos t \cdot \cos t\,dt + \sin t\,dt] = \int_{t=0}^{2\pi} \sin^2 t + \cos^2 t + \sin t = \int_{t=0}^{2\pi} (1 + \sin t)\,dt = 2\pi + 0 = 2\pi$

$Q(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^4 \end{bmatrix}$ $\quad f_1 = x_4^2 \quad f_2 = x_3^2 \quad f_3 = x_2^2 \quad f_4 = x_1^2$

$t \in [-1, 1]$

$\int_Q [f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3 + f_4\,dx_4] =$

Nullstellen: $x_1 = 1 \quad x_2 = t \quad x_3 = t^2 \quad x_4 = t^4$

$dx_1 = 0\,dt \quad dx_2 = 1 \cdot dt \quad dx_3 = 2t\,dt \quad dx_4 = 4t^3\,dt$

$f_1 = t^8 \quad f_2 = t^4 \quad f_3 = t^2 \quad f_4 = 1$

$$\int_{t=-1}^{1} \Big[t^8 \cdot 0\,dt + t^4 \cdot dt + t^2 \cdot 2t\,dt + 1 \cdot 4t^3\,dt =$$

$$0 + \frac{2}{5} + 0 + 0 = \frac{2}{5}$$

$0 - 1 \Rightarrow \frac{1}{5}$ +

$-1 - 0 \to \frac{1}{5}$

$Q: t \begin{bmatrix} t^2 \\ t \\ t^3 \end{bmatrix}$ $\quad f = (x, y, z) \mapsto \begin{bmatrix} x^6 y^3 \\ z^2 x^4 \\ y^5 z^6 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

$t : [0, 1] \to \mathbb{R}^3$

$$\int_Q f_1\,dx + f_2\,dy + f_3\,dz =$$

$f_1 = x^6 y^3 \quad f_2 = z^2 x^4$

$f_3 = y^5 z^6$

$$\int_{t=0}^{1} (t^{15} \cdot 2t\,dt) + (t^{14} \cdot dt) + (t^{23} \cdot 3t^2\,dt)$$

$x = t^2$ → pot & [illegible]

$y = t$

$z = t^3$

$dx = 2t\,dt$

$dy = dt$

$dz = 3t^2\,dt$

$$\int_0^1 \Big[2t^{16}\,dt + t^{14} + 3t^{25} \Big] dt =$$

$$2 \cdot \frac{1}{17} + \frac{1}{15} + 3 \cdot \frac{1}{26}$$

$f_1 = x^6 y^3 = (t^2)^6 t^3 = t^{15}$

$f_2 = z^2 x^4 = (t^3)^2 (t^2)^3 = t^{14}$

$f_3 = y^5 z^6 = (t^1)^5 \cdot (t^3)^6 = t^{23}$

ZH-ban pl.:

ellipszis

x, y síkban

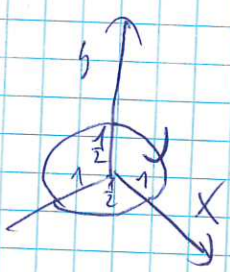
x tengely irányú 2 hossz

y tengely irányú 1 hossz

tehát (+) elforgatva

F erő mentén.

$$F(x,y,z) = \begin{bmatrix} 4z \\ 3x \\ x^2y^2 \end{bmatrix}$$



$$Q: \varphi \mapsto \begin{bmatrix} \sin\varphi \\ \frac{1}{2}\cos\varphi \\ 0 \end{bmatrix}$$

$\varphi \in [0;2\pi]$

$x = \sin\varphi \quad dx = \cos\varphi \, d\varphi$

$y = \frac{1}{2}\cos\varphi \quad dy = -\frac{1}{2}\sin\varphi \, d\varphi$

$z = 0 \quad dz = 0\, d\varphi$

$$\int_Q \left(4z\,dx + 3x\,dy + x^2y^2\,dz\right) =$$

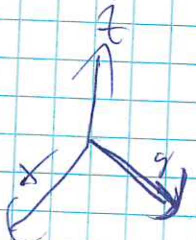
$$\int_{\varphi=0}^{2\pi} \left[0 + 3\cdot\sin\varphi\cos\varphi\,d\varphi + 0\right] = \frac{3}{2}\int_{\varphi=0}^{2\pi} \sin 2\varphi\,d\varphi = 0$$

egész periódus nulla

$2\sin\varphi\cos\varphi = \sin 2\varphi$

ÁTFOLYÁS (FLUXUS)

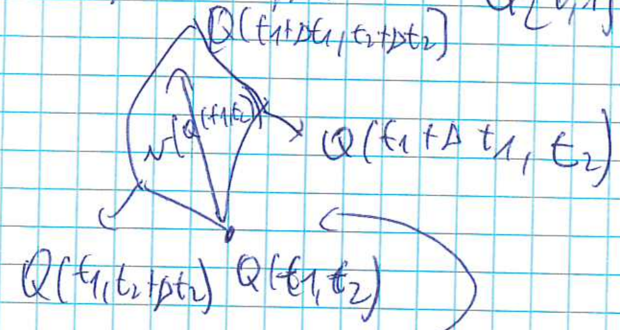
kihelyezett lapok.



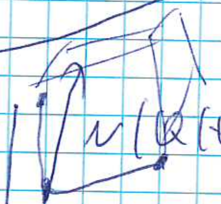
v sebességű folyadék

$$Q \quad [0,1]^2 \mapsto \mathbb{R}^3 \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) \end{bmatrix} : \mathbb{R}^3 \mapsto \mathbb{R}^3$$

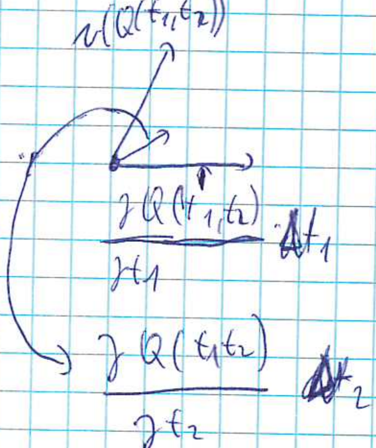
Mennyi folyadék megy át időegységenként $Q[0,1]^2$-en



Ezen a darabon $\frac{\text{átfolyó}}{\sec} \approx$



$v(Q(t_1,t_2))$ euler - pillanat



$$\approx Vol_3\left[v(Q(t_1,t_2)), \frac{\partial Q(t_1,t_2)}{\partial t_1}\Delta t_1, \frac{\partial Q(t_1,t_2)}{\partial t_2}\Delta t_2\right]$$

térfogata $\approx$

$$v_1 \cdot \frac{\partial Q_2}{\partial t_1}\Delta t_1 \cdot \frac{\partial Q_3}{\partial t_2}\Delta t_2 +$$

$$v_2 \cdot \frac{\partial Q_3}{\partial t_1}\Delta t_1 \cdot \frac{\partial Q_1}{\partial t_2}\Delta t_2 +$$

$$v_3 \cdot \frac{\partial Q_1}{\partial t_1}\Delta t_1 \cdot \frac{\partial Q_2}{\partial t_2}\Delta t_2 \qquad \partial x \; \partial y \; \partial z$$

Szuperpozíció elve:

v, F felületdb

[ÁTFOLYÁS] =

v_1

x-re

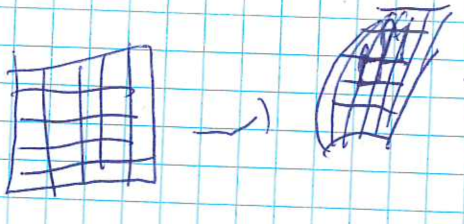
való vetülete

az F-nek

$$F = \begin{bmatrix}\Delta^2 x\\ \Delta^2 y\\ \Delta^2 z\end{bmatrix} \quad \begin{bmatrix}\Delta^1 x\\ \Delta^1 y\\ \Delta^1 z\end{bmatrix} \quad \begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix}$$

$$x \text{ or } v \cdot Vol_2(P_{yz}F) + y \text{ or } v\, Vol_2(P_{zx}F) + z \text{ or } v\, Vol_2(P_{xy}F)$$

$$f_x\left[\Delta^1 y\,\Delta^2 z - \Delta^2 y\,\Delta^1 z\right] + f_y\left[\Delta^1 z\,\Delta^2 x - \Delta^2 z\,\Delta^1 x\right] + f_z\left[\Delta^1 x\,\Delta^2 y - \Delta^2 x\,\Delta^1 y\right]$$



$$\rightsquigarrow \int_{t_1,t_2=0}^{1} v_x(Q(t_1,t_2)) \cdot \det\begin{pmatrix}\Delta^1 y & \Delta^2 y\\ \Delta^1 z & \Delta^2 z\end{pmatrix}$$

$$\frac{\partial Q_2}{\partial t_1}\Delta t_1 \qquad \frac{\partial Q_3}{\partial t_1}\Delta t_1$$

$$\int_{t_1,t_2=0}^{1} \Big\{ v_1 \det\begin{pmatrix}\frac{\partial Q_2}{\partial t_1} & \frac{\partial Q_2}{\partial t_2}\\ \frac{\partial Q_3}{\partial t_1} & \frac{\partial Q_3}{\partial t_2}\end{pmatrix} dt_1\,dt_2 + v_2 \det\begin{pmatrix}\frac{\partial Q_3}{\partial t_1} & \frac{\partial Q_3}{\partial t_2}\\ \frac{\partial Q_1}{\partial t_1} & \frac{\partial Q_1}{\partial t_2}\end{pmatrix} dt_1\,dt_2 +$$

$$v_3 \cdot \det\begin{pmatrix}\frac{\partial Q_1}{\partial t_1} & \frac{\partial Q_1}{\partial t_2}\\ \frac{\partial Q_2}{\partial t_1} & \frac{\partial Q_2}{\partial t_2}\end{pmatrix} dt_1 + dt_2 \Big\}$$

$$\det\begin{pmatrix}d_1 z & d_2 z\\ d_1 x & d_2 x\end{pmatrix}$$

$$\rightarrow \det\begin{pmatrix}d_1 x & d_2 x\\ d_1 y & d_2 y\end{pmatrix}$$

$$\rightarrow \det\begin{pmatrix}d_1 y & d_2 y\\ d_1 z & d_2 z\end{pmatrix}$$

$y \sim Q_2$

$x \sim Q_1$

$z \rightarrow Q_3$

MEGY: $dx \rightsquigarrow \frac{\partial Q_1}{\partial t_1} dt_1 + \frac{\partial Q_1}{\partial t_2} dt_2$

$dy \rightsquigarrow \frac{\partial Q_2}{\partial t_1} dt_1 + \frac{\partial Q_2}{\partial t_2} dt_2$

$dz \rightsquigarrow \frac{\partial Q_3}{\partial t_1} dt_1 + \frac{\partial Q_3}{\partial t_2} dt_2$

$Q: [0,1]^k \to \mathbb{R}^n$

$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ ~~$\mathbb{R}^n \to \mathbb{R}^n$~~

$$\int_{t_1, t_k = 0}^{1} v(x_1 \ldots x_n) \det \begin{pmatrix} d_1 x_{i,1} & d_2 x_{i,1} & \ldots & d_{k,} x_{i1} \\ d_1 x_{i,1} & d_2 x_{i,2} & \ldots & d_{k,} x_{i2} \\ \vdots & & & \vdots \\ d_1 x_{i,k} & \ldots & & d_{k,} x_{ik} \end{pmatrix}$$

Külső algebra jelöléssel

külső szorzat.

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{1k} \\ a_{21} & a_{22} & a_{2k} \\ \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} \ldots & a_{k,k} \end{vmatrix} = \det \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} = a_1 \wedge a_2 \wedge \ldots \wedge a_k$$

a 1. sor

$$\int_Q v(x_1 \ldots x_n)\, dx_{i1} \wedge dx_{i2} \wedge \ldots \wedge dx_{ik}$$

$a_1 \wedge a_2 \wedge \ldots \wedge a_k =$ ~~$a_1 \wedge \ldots \wedge a_{i-1} \wedge a_i$~~

$a_1 \wedge \ldots \wedge a_i \wedge a_{i-1} \wedge a_{i+1} \wedge \ldots \wedge a_k$

mert lineáris sorcsere $\to \ominus$

$a_1 \wedge \ldots \wedge a_i \wedge b_1 \wedge \ldots \wedge b_1 \wedge \ldots = 0$

mert lesz a 2 darab sor.

Disztributív kifejtés.

Példa: $v = x_1^2 \cdot x_2^3\, x_3^5\, x_4^2\, x_5^3 \qquad Q: [0,1]^3 \to \mathbb{R}^5$

$Q_i = \cancel{x_1^{2i}\, x_2^{i}\, x_3^{4i}\, x_4^{5i}\, x_5^{6i}} \quad i \in [1 \ldots 5]$

$t_1^{2i}\; t_2^{i}\; t_3^{i+1}$

$$\int_Q v\, dx_1 \wedge dx_3 \wedge dx_4$$

$x_1 = Q_1(t_1,t_2,t_3) = t_1^2 t_2 t_3^2 \qquad dx_1 = 2t_1 t_2 t_3^2\, dt_1 +$

$t_1^2 t_3^2\, dt_2 +$

$2t_1^2 t_2 t_3\, dt_3$

$x_2 = t_1^4 t_2^2 t_3^3 \qquad dx_2 = 4t_1^3 t_2^2 t_3^3\, dt_1 +$

$2t_1^4 t_2 t_3^3\, dt_2 +$

$3t_1^4 t_2^2 t_3^2$

$x_3 = t_1^6 t_2^3 t_3^4 \qquad dx_3 = 6t_1^5 t_2^3 t_3^4\, \cancel{dt_1}$

$+ 3t_1^6 t_2^2 t_3^4\, dt_2 +$

$4t_1^6 t_2^3 t_3^3\, dt_3$

$dx_1 \wedge dx_3 = [2t_1 t_2 t_3^2 + \dots] \wedge [6t_1^5 t_2^3 t_3^4\, dt_1 \dots]$

$() \underbrace{dt_1 \wedge dt_1}_{0} + ()\, dt_1 \wedge dt_2 + ()\, dt_1 \wedge dt_3$

$() \underbrace{dt_2 \wedge dt_1}_{-dt_1 dt_2} + () \underbrace{dt_2 \wedge dt_2}_{0} + ()\, dt_2 \wedge dt_3$

$() \underbrace{dt_3 \wedge dt_1}_{-dt_1 \wedge dt_3} + () \underbrace{dt_3 \wedge dt_2}_{-dt_2 \wedge dt_3} + () \underbrace{dt_3 \wedge dt_3}_{0}$

$= ()\, dt_1 \wedge dt_2 + ()\, dt_1 \wedge dt_3 + ()\, dt_2 \wedge dt_3$

$\boxed{1.\text{dön}}$

$x_1 = t_1^2 t_2 t_3^2$

$x_2 = t^4 t_2^2 t_3^3$

$x_3 = \cancel{t_1}^6 t_2^3 t_3^4$

$x_4 = t_1^8 t_2^4 t_3^5$

$x_5 = t_1^{10} t_2^5 t_3^6$

$(5 + 3x^2 + 2x^3 + 9x^7) \cdot (3 + x + 4x^2 + 6x^4)$ > [illegible]

$\boxed{5\cdot3} + \boxed{5}x + \boxed{9+20}x^2 + \dots +$

$\boxed{6+3}x^3 + \boxed{4}x^4 + \dots$

$30+12+\dots$

$dx_1 = 2t_1 t_2 t_3^2\, dt_1 + t_1^2 t_3^2\, dt_2 + 2t_1^2 t_2 t_3\, dt_3$

$dx_2 = 4t^3 t_2^2 t_3^3\, dt_1 + 2t^4 t_2 t_3^3\, dt_2 + 3t^4 t_2^2 t_3^2\, dt_3$

$dx_3 = 6t_1^5 t_2^3 t_3^4\, dt_1 + 3t_1^6 t_2^2 t_3^4\, dt_2 + 4t_1^6 t_2^3 t_3^3\, dt_3$

$dx_4 = 8t_1^7 t_2^4 t_3^5\, dt_1 + 4t_1^8 t_2^3 t_3^5\, dt_2 + 5t_1^8 t_2^4 t_3^4\, dt_3$

$dx_5 = 10t_1^9 t_2^5 t_3^6\, dt_1 + 5t_1^{10} t_2^4 t_3^6\, dt_2 + 6t_1^{10} t_2^5 t_3^5\, dt_3$

$dx_3 \wedge dx_5 = (\quad)\, dt_1 \wedge dt_2 + (\quad)\, dt_1 \wedge dt_3 + (\quad)\, dt_2 \wedge dt_3$

$(30t_1^{15} t_2^7 t_3^{10} - 30t_1^{15} t_2^7 t_3^{10})\, dt_1 \wedge dt_2 +$

$(36t_1^{15} t_2^8 t_3^9 - 40t_1^{15} t_2^8 t_3^9)\, dt_1 \wedge dt_3 +$

$(18t_1^{16} t_2^7 t_3^9 - 20t_1^{16} t_2^7 t_3^9)\, dt_2 \wedge dt_3$

~~[illegible]~~ $= -4t_1^{15} t_2^8 t_3^9\, dt_1 \wedge dt_3 + -2t_1^{16} t_2^7 t_3^9\, dt_2 \wedge dt_3$

$dx_1 \wedge dx_3 \wedge dx_5 = \cancel{t}$

$) \cancel{dt_1 \wedge dt_2}$

$) \cancel{dt_1 \wedge} dt_3$

$) \cancel{dt_2 \wedge dt_3}$

$= \left(-4\, t_1^{17} t_2^{8} t_3^{11} + 4\, t_1^{17} t_2^{8} t_3^{11}\right) = 0$

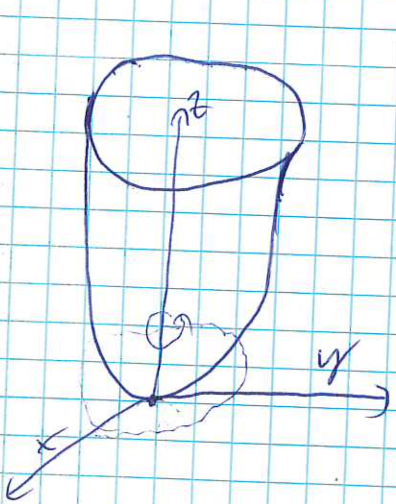
$) \, dt_1 \wedge dt_2 \wedge dt_3$

$dt_1 \wedge$ $dt_2 \wedge dt_3$ elsőnél jó de

→ ez pedig $dt_2 \wedge (dt_1 \wedge dt_3)$

$t_1^2 t_3^2\, dt_2 \wedge (-4) \cdot t_1^5 t_2^8 t_3^9\, dt_1 \wedge dt_3$

$dt_2 \wedge dt_1 \wedge dt_3$ → csere ⊖ előjelet vált ki.



$(z = x^2 + y^2 \le 1) = \{ p \in \mathbb{R}^3 : z(p) = x(p)^2 + y(p)^2 \le 1 \}$

$= P$

$Vol_3\, P = ?$ → 1. módszer

1) PARAMÉTEREZÉS

$x = r \cdot \cos\varphi$

$y = r \cdot \sin\varphi$

$x^2 + y^2 = r^2$

$r \cdot \cos\varphi + r \cdot \sin\varphi$

$0 \le r \le 1$

$0 \le \varphi \le 2\pi$

1) Számolás (2) lépés

$Q\ [0,1] \times [0, 2\pi] \to \mathbb{R}^3$

...ktor.

$$Vol_2\, P = \int_0^1 \int_0^{2\pi} \sqrt{\det(Q'^T Q')}\, dr\, d\varphi \qquad (r, \varphi) \to \begin{bmatrix} r \cos\varphi \\ r \sin\varphi \\ r^2 \end{bmatrix}$$

2. módszer

$$Vol_2\, P = \int_Q \sqrt{\|dx \wedge dy\|^2 + \|dx \wedge dz\|^2 + \|dy \wedge dz\|^2}$$

paraméter

$$= \int_{r=0}^{1} \int_{\varphi=0}^{2\pi}$$

$x = r \cdot \cos\varphi$ (Q által leképezve)

$y = r \cdot \sin\varphi$

$z = r^2$

$dx \wedge dy = (\cos\varphi\, dr - r \cdot \sin\varphi\, d\varphi) \wedge (\sin\varphi\, dr + r \cdot \cos\varphi\, d\varphi) =$

$\cos^2\varphi + r \sin^2\varphi \,|\, dr \wedge d\varphi$ $\quad \|dr \wedge d\varphi\|$

$\ominus \cdot \ominus$

$r^2 \|dr \wedge d\varphi\|^2 = r^2 (dr \cdot d\varphi)^2$

$\sin\varphi\, d\varphi + \sin\varphi\, dr$

jelenleg nagy $\Rightarrow$ ezt $\ldots = \oplus$

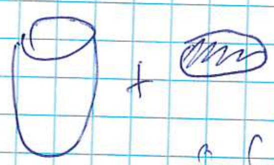
$dx \wedge dz = (\cos\varphi\, dr - r \cdot \sin\varphi\, d\varphi) \wedge (2r \cdot dr) =$

$dy \wedge dz = (-2r^2\cos\varphi \; dr \wedge d\varphi) \to \|\wedge\| = 4r^4\cos^2\varphi\,(dr\cdot d\varphi)^2$

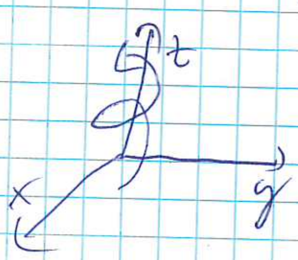
$\sqrt{\|dx\wedge dy\|+\|dx\wedge dz\|+\|dy\wedge dz\|} = \Big(\sqrt{\underbrace{4r^4\sin^2\varphi + 4r^4\cos^2\varphi}_{1 = \sin^2+\cos^2} + r^2}\Big)\, dr\cdot d\varphi$

$= \sqrt{1+4r^2}\cdot r\, dr\, d\varphi$

$$\int_{r=0}^{1}\int_{\varphi=0}^{2\pi} \sqrt{1+4r^2}\, r\, dr\, d\varphi = \frac{\pi}{6}\left(\sqrt{125}-1\right)$$

$\tilde{P} =$  befedett palást

ömlő víz. Mennyi víz megy át $\tilde{P}$ határán?



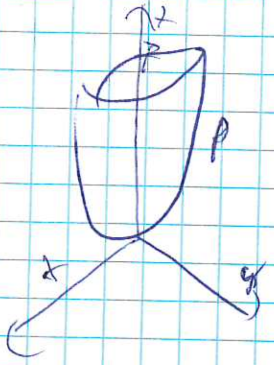
$v(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$ (0 mert a víz összenyomhatatlan)

[$\tilde{P}$ határa] = [P + felső körlap]

 P bemegy a víz  F kimegy a víz

[P-be menő víz/sec] $= \int_Q z\cdot v\; dx\wedge dy + x\circ v\; dy\, dz + y\circ v\; dz\, dx$]

x, y szerint
y, z szerinti



$Q: (r,\varphi) \mapsto \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \\ r^2 \end{bmatrix}$ $\quad [0,1]\times[0,2\pi] \to \mathbb{R}^3$

P

$R: (r,\varphi) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \\ 1 \end{bmatrix}$

F

$v: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad v(x,y,z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix}$

$[v \text{ áthaladása } P\text{-n}] = \int_Q (x\circ v \, dy\wedge dz + y\circ v \, dz\wedge dx + z\circ v \, dx\wedge dy)$

$[v \text{ áthaladása } F_0] = \int_R \ldots$

$x\circ v = -y = (r\cdot\sin\varphi)$

$[v \text{ áthaladása } P\text{-n}] = \int_{r=0} \int_{\varphi=0}$

$x = r\cos\varphi$ $\quad$ ~~$dx = \cos\varphi + r\sin\varphi \, d\varphi$~~

$y = r\sin\varphi$ $\quad$ ~~$dy =$~~

$z = r^2$ $\quad$ $dz = \cancel{2r} \, dr\, d\varphi$

~~$\sin\varphi \; r\cos\varphi \; dr \, d\varphi$~~

~~$\int_{r=0}^{1}\int_{\varphi=0}^{2\pi} r\sin\varphi \, (\sin\varphi\, dr)\wedge(2r\,dr) +$~~

~~$r\cos\varphi \cdot (2r\,dr)\wedge(\cos\varphi\, dr) +$~~

~~$1(\cos\varphi\, dr)\wedge(-\sin$~~

$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi$

$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$

$dz = 2r\,dr$

$y\circ v = r\cos\varphi$

$z\circ v = 1$

$x\circ v = -r\sin\varphi$

$$\int_{r=0}^{1}\int_{\varphi=0}^{2\pi} -r\sin\varphi\,(\sin\varphi\,d\varphi + r\cos\varphi\,d\varphi) \wedge (2r\,dr) +$$

$$r\cos\varphi\,(2r\,dr) \wedge (\cos\varphi\,dr - r\sin\varphi\,d\varphi) +$$

$\boxed{dr \cdot dr = 0}$

$$1 \cdot (\cos\varphi\,dr - r\sin\varphi\,d\varphi) \wedge (\sin\varphi\,dr + r\cos\varphi\,d\varphi)$$

$$= \int_{r=0}^{1}\int_{\varphi=0}^{2\pi} \left(-r\sin\varphi + 2r^2\cos\varphi\right) d\varphi\,dr +$$

$$\cancel{r}\cos\varphi\,(2r^2\sin\varphi)\,d\varphi\,dr + -r\,d\varphi\,d\varphi$$

$$r\cos^2 \cancel{d\varphi}\,d\varphi - r\sin^2\varphi\,d\varphi\,dr$$

$$-d\varphi\,dr = -r\,d\varphi\,dr$$

$$-r\cdot\cos^2 d\varphi\,dr$$

$$-r\,(\cos^2 \underbrace{d\varphi\,dr + \sin^2\varphi}_{=1}\,d\varphi\,dr) = -r$$

$$= -\int_{r=0}^{1}\int_{\varphi=0}^{2\pi} r\,d\varphi\,dr = -\int_{r=0}^{1} 2\pi r\,dr = -(\pi\cdot 1)$$

$$\int_{r=0}^{1}\int_{\varphi=0}^{2\pi} \left[0 + 0 - r\right] = -\int_{r=0}^{1} 2\pi r = -2\pi\cdot\frac{r^2}{2}\Big|_0^1 = -\pi$$

$dz = t$ konstans

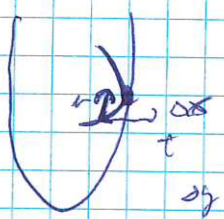
$dz = 0$

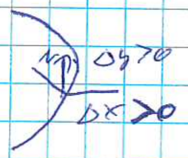
⊕ ~ jobbról körülnézz a víz

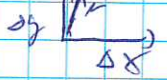
⊖ Mikor körülnéz

ÁTFOLYÁS F-en : kifelé megy a víz

ÁTFOLYÁS F-n : bemegy a víz

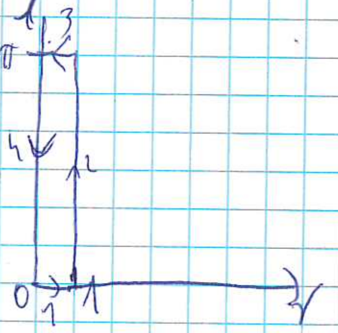


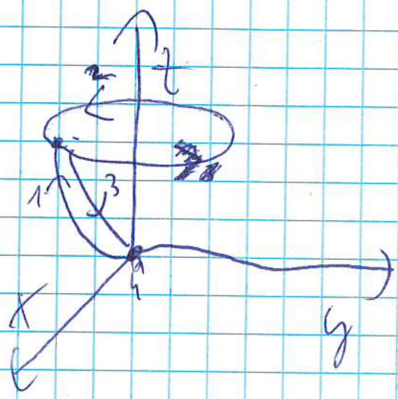




jobb sodrású

$$\int_{r=0}^{1}\int_{\varphi=2\pi}^{0}(\dots)$$





itt adott jobb kéz a pozitív oldalt.

1) $\varphi = 0 \quad r : 0 \to 1$

$$Q = \begin{bmatrix} r \\ 0 \\ r^2 \end{bmatrix}_{(\varphi = 0)}$$

$$Q = \begin{bmatrix} r \cos\varphi \\ r \sin\varphi \\ r^2 \end{bmatrix}$$

2) $\varphi = 0 \to 2\pi \quad r = 1$

$$\begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{bmatrix}$$

3) $r = 1 \to 0 \quad \varphi = 2\pi$

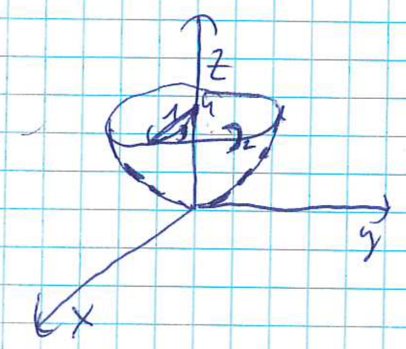
$$Q = \begin{bmatrix} r \\ 0 \\ r^2 \end{bmatrix}$$

4) $\varphi \to 2\pi \to 0$

$r = 0$

$$Q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ÁTFOLYÁS F-en



1) $Q = \begin{bmatrix} r\cdot\cos\varphi \\ r\cdot\sin\varphi \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2) $\varphi = 0 \to 2\pi \quad r = 1$

$Q = \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{bmatrix}$

3)

Meg kell fordítani valamelyik ~~irányát~~.

p-nek vagy F-nek az irányítását.

(p-F)

(átfolyás p-n helyett -átfolyás p-n)

A vató kiegészítései:

Víz

→ víz 1db

Tudjuk, nem mozdul. nem süllyed ⇒ rá ható erők összege 0

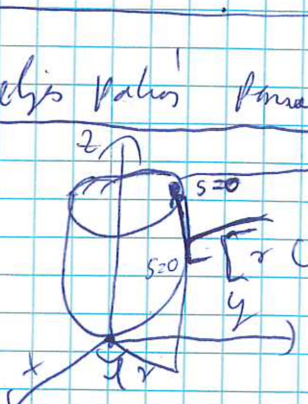
Felhajtja

Tehát = dual víz súlya

→ ugyanakkora db más anyagból

erő u. az mint vízzel.

A teljes palást paraméterezése



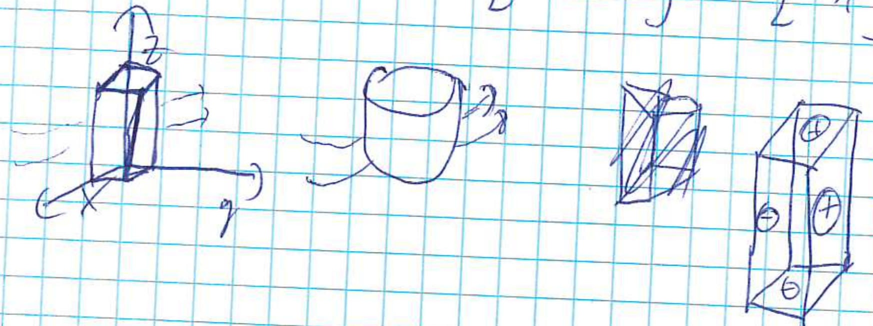
$\begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{bmatrix} \to (1-s)\begin{bmatrix} r\cos\varphi \\ r\sin\varphi \\ r^2 \end{bmatrix} + s\begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{bmatrix}$

$[r\cos\varphi, r\cdot\sin\varphi, r^2]$

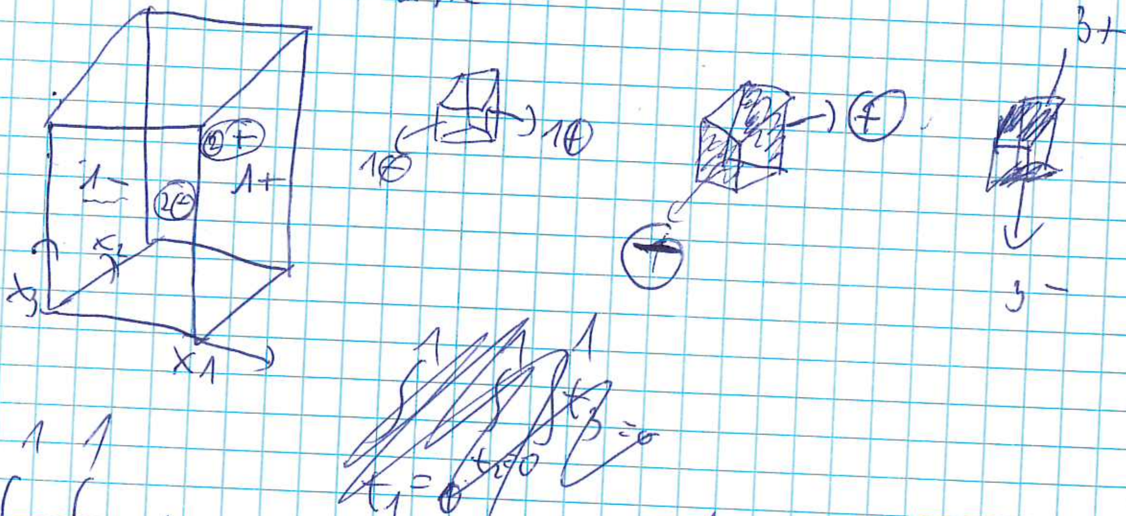
φ

$Q: [0;1]\times[0;1]\times[0;2\pi] \to \mathbb{R}^3$

$$\bar{a}(r,s,1) = (1-s)\begin{bmatrix} r-\cos 1 \\ r-\sin 1 \\ 2\cdot 2 \end{bmatrix} + s\begin{bmatrix} \cos 1 \\ \sin 1 \\ 1 \end{bmatrix}$$



Kocka irányított határa

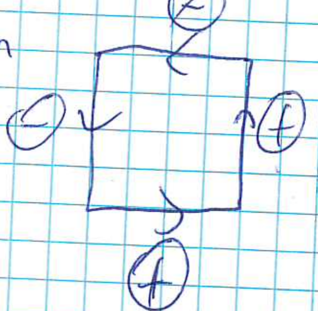


$$\int_{t_1=0}^{1}\int_{t_2=0}^{1}\int_{t_3=0}^{1} f(Q)\, dx_i \wedge Q \; \ldots = \int_{t_{1,2,3}=0}^{1} [\ldots]\, dt_3\, dt_2\, dt_1$$

$k+$ oldal fölött előjel $(-1)^{k-1}$

$1-$ oldal fölött előjel $(-1)^k$

2D-ben



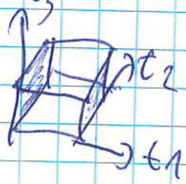
8. hét

ÁTFOLYÁS

~~hatdor~~

$Q : [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$

$v : \mathbb{R}^3 \to \mathbb{R}^3$ víz sebesség



$Q_1+ = (t_2, t_3) \to Q(b_1, t_2, t_3)$

$Q_1- = (t_2, t_3) \to Q(a_1, t_2, t_3)$

hátsó $Q_2+ = (t_1, t_3) \to Q(t_1, b_2, t_3)$

első $Q_2- = (t_1, t_3) \to Q(t_1, a_2, t_3)$

felső $Q_3+ = (t_1, t_2) \to Q(t_1, t_2, b_3)$

alsó $Q_3- = (t_1, t_2) \to Q(t_1, t_2, a_3)$

kifelé $\left(\int_{Q_1+} - \int_{Q_1-}\right) - \left(\int_{Q_2+} - \int_{Q_2-}\right) + \left(\int_{Q_3+} - \int_{Q_3-}\right)$ $v_1 \cdot dx_2 \wedge dx_3 +$

$v_2 \cdot dx_3 \wedge dx_1 +$

$v_3 \, dx_1 \wedge dx_2$ ω

Szemben levő oldalakat előjellel

ráadásul

$$\int_{\partial Q} \qquad w = \sum_{i<j} f_{ij}\, dx_i \wedge dx_j$$

∂Q határán vett integrálás ... $\mathbb{R}^n$

Általános eset k dim C_1 kocka $\mathbb{R}^n$-ben $(k \leq n)$

$Q: [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k] \to \mathbb{R}^n$

$Q_{1+}: (t_2, t_3, \dots t_k) \to Q(t_1, t_2, \dots t_k)$

$$\int_{\partial Q} = \left(\int_{Q_{1+}} - \int_{Q_{1-}}\right) - \left(\int_{Q_{2+}} - \int_{Q_{2-}}\right) + \dots + (-1)^{k-1}$$

$Q_{1-}: (t_2, t_3, \dots t_k) \to Q(a_1, t_2, \dots t_k)$

$Q_{2+}: (t_1, t_3, \dots t_k) \to Q(t_1, b_1, t_3, \dots t_k)$

$$\left(\int_{Q_{k+}} - \int_{Q_{k-}}\right) = \sum_{d=1}^{k} (-1)^{d-1} \left(\int_{Q_{d+}} - \int_{Q_{d-}}\right)$$

...

$$w = \sum_{i_1 < \dots < i_{k-1}} f_{i_1 i_2}\, dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_{k-1}}$$

~~Q_2~~

$Q_{d+}: (t_1 \dots t_{d-1}, t_{d+1}) \to Q(t_1, \dots t_{d-1}, b_d, t_{d+1} \dots t_k)$

$Q_{d-}: (t_1 \dots t_{d-1}, t_{d+1} \dots t_k) \to Q(t_1 \dots t_{d-1}, b_d, t_{d+1} \dots t_k)$

$N = 4$

d-KALKULUS

$f: \mathbb{R}^n \to \mathbb{R} \qquad df =$

Pl $f = x_1^2 + x_2^4 + x_3^3 + x_4$

$df =$ parciális deriválás x_1, x_2, x_3, x_4 szerint

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j$$

$$d\left[f \cdot dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}\right] = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} =$$

$$\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}$$

pl: $N = 3$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) = \text{gradiense az } f \text{ fgv-nek.}$$

$$d(v_1\, dx_2 \wedge dx_3 + v_2\, dx_3 \wedge dx_1 + v_3\, dx_1 dx_2) =$$

$$\underbrace{\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)}_{\text{divergenciája } v\text{-nek}} dx_1 \wedge dx_2 \wedge dx_3$$

$$d(u_1 x_1 + u_2\, dx_2 + u_3\, dx_3) = \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) dx_1 \wedge dx_2 +$$

$$\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial u_3}\right) dx_2 \wedge dx_3 +$$

$$\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) dx_3 \wedge dx_1$$

$$[\ ,\ ,\ ] = \mathrm{rot}(u)$$

TÉTEL (Stokes tétele)

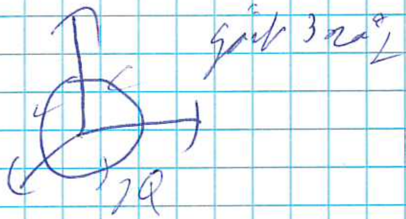
$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

TÉTEL $d(d\omega) = 0$

TÉTEL Poincaré lemma

$$d\omega = 0 \Rightarrow \exists\, \bar{\omega} \quad \omega = d\bar{\omega}$$

Pl. $Q : [0, \pi/2] \times [0, \frac{\pi}{2}] \to \mathbb{R}^3$



$$Q(\psi, \varphi) = \begin{bmatrix} \cos\psi \cdot \cos\varphi \\ \cos\psi \ \sin\varphi \\ \sin\psi \end{bmatrix}$$

$u : \mathbb{R}^3 \to \mathbb{R}^3$

$$u(x, y, z) = \begin{bmatrix} -y \\ x \\ 1 \end{bmatrix} \qquad \int_{\partial Q} u_1\, dx + u_2\, dy + u_3\, dz = ?$$

kinámali: $\int\limits_{\partial Q} (\int_{Q_{1+}} - \int_{Q_{1-}}) - (\int_{Q_{2+}} - \int_{Q_{2-}}) w$

$x=0$
$y=0$
$z=1$

$dx = dy = dz = 0$

$$\int_{Q_{1+}} w = \int_{1=0}^{\frac{\pi}{2}} 0$$

Q_{1+}: $1 \mapsto Q(\frac{\pi}{2}, 1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Q_{1-}: $1 \mapsto Q(0, 1) = \begin{bmatrix} \cos 1 \\ \sin 1 \\ 0 \end{bmatrix}$

ψ lileli $= \frac{\pi}{2}$

1: mllhau

Q_{2+}: $\psi \mapsto \begin{bmatrix} 0 \\ \cos\psi \\ \sin\psi \end{bmatrix}$

ψ: valt

1: kostans $= \frac{\pi}{2}$

Q_{2-}: $\psi \mapsto \begin{bmatrix} \cos\psi \\ 0 \\ \sin\psi \end{bmatrix}$

$$\int_{Q_{1-}} w = \int_0^{\frac{\pi}{2}} \Big(\underbrace{(-\sin 1)^2}_{u_1 dx} + \underbrace{\cos^2 1}_{u_2 dy} + \underbrace{0}_{u_3 dz} \Big) d1 = \int_0^{\frac{\pi}{2}} 1 \, d1 = \frac{\pi}{2}$$

$x = \cos 1$ $\quad dx = -\sin 1 \, d1$ $\quad u_1 = -y = -\sin 1$

$y = \sin 1$ $\quad dy = \cos 1 \, d1$ $\quad u_2 = x = \cos 1$

$z = 0$ $\quad dz = 0 \, d1$ $\quad u_3 = z = 1$

$\int_{Q_{2+}}$ $\quad \int_{Q_{2-}}$

HF(((

2, 1db $\int$ plaha tételével.

$$\int_{\partial Q} w = \int_Q dw$$

$$w = u_1 dx + u_2 dy + u_3 dz = -y\,dx + x\,dy + dz$$

$$dw = \underbrace{d(-y\,dx) + d(x\,dy) + d(dz)}_{0}$$

$$d-y \wedge dx + dx \wedge dy = -dy \wedge dx + dx \wedge dy = 2\,dx \wedge dy$$

$$\int_{\partial Q} w = \int_Q dw = \int_Q 2\,dx \wedge dy = \int_{\varphi=0}^{\frac{\pi}{2}} \int_{t=0}^{\frac{\pi}{2}}$$

$x = \cos\varphi \cos t$

$dx = -\sin\varphi \cos t \, d\varphi + -\cos\varphi \cdot \sin t \, dt$

$dy = \cancel{} -\sin\varphi \sin t \, d\varphi + \cos\varphi \cdot \cos t \, dt$

$y = \cos\varphi \cdot \sin t$

$dx \wedge dy = \cos\varphi \sin t \cdot \sin^2 t + \cos\varphi \sin t \cos^2 t \quad dt\,d\varphi$

$$4 \int_{\varphi=0}^{\frac{\pi}{2}} \int_{t=0}^{\frac{\pi}{2}} \cos t \sin\varphi \, dt\,d\varphi = \frac{\pi}{2} \int_{\varphi=0}^{\frac{\pi}{2}} \cos\varphi \sin\varphi \, d\varphi =$$

$$\frac{\pi}{2} \int_{\varphi=0}^{\pi/2} \sin 2\varphi \, d\varphi = \frac{\pi}{8} \left(-\cos 2\varphi\right)\Big|_0^{\pi/2}$$

$\boxed{9\text{ hét}}$

KOORDINÁTA CSERE

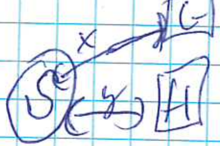
S halmaz $x_1, \dots x_N : S \to \mathbb{R}$ N dim koordinátarendszer S-nek, ha megkülönbözteti a pontjait.

$p \neq q \in S$ $\exists j$ $x_j(p) \neq x_j(q)$

NYITOTT ha $\exists G$ nyitott $\subset \mathbb{R}^N$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} : p \mapsto \begin{bmatrix} x_1(p) \\ \vdots \\ x_N(p) \end{bmatrix} \quad X : S \to G$$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ nyitott koordinátarendszerek S-en



félgömb

tér

Θ-val látjuk

Tudjuk y alapján hogyan számolható ki X

$$x_1 = x_1(y_1, \dots y_N), \dots, x_N(y_1, \dots y_N)$$

Kérdés: $f : S \to \mathbb{R}$

$\frac{df}{dy_1} \dots \frac{df}{dy_N}$ alapján hogyan számoljuk ki $\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_N}$-et.

T/

$$\left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right] = \left[\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}\right] \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n} \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & & \frac{\partial y_n}{\partial x_n} \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \\ \vdots & & \\ & & \frac{\partial y_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \vdots & \dots & \frac{\partial x_n}{\partial y_n} \end{bmatrix}^{-1}$$

$$= \left[\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}\right] \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial x_n}{\partial y_1} & \dots & \frac{\partial x_n}{\partial y_n} \end{bmatrix} \Rightarrow \frac{\partial x}{\partial y} = \begin{bmatrix} \frac{\partial x_1}{\partial y} \\ \frac{\partial x_2}{\partial y} \\ \frac{\partial x_n}{\partial y} \end{bmatrix}$$

<u>résultats</u> : $\left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right] = \frac{\partial f}{\partial X}$

<u>Bcl</u> : $df = \frac{\partial f}{\partial y_1} dy_1 + \dots + \frac{\partial f}{\partial y_n} dy_n$

$$= \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

$$= \frac{\partial f}{\partial x_1}\left[\frac{\partial x_1}{\partial y_1}dy_1 + \frac{\partial x_1}{\partial y_2}dy_2 + \dots + \frac{\partial x_1}{\partial y_n}dy_n\right] + \dots +$$

$$\frac{\partial f}{\partial x_n}\left[\frac{\partial x_n}{\partial y_1}dy_1 + \dots + \frac{\partial x_n}{\partial y_n}dy_n\right]$$

$$= [\,?\,]dy_1 + \dots + \left[\frac{\partial f}{\partial y_n}\right]dy_n = \left[\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_1}\right]dy_1 +$$

$$\left[\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_2}\right]dy_2 + \dots +$$

$$\left[\frac{\partial f}{\partial x_1}\frac{\partial x}{\partial y_n} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_n}\right]dy_n$$

$$\frac{\partial f}{\partial y_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_n}{\partial y_1} + \dots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_1} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right]\begin{bmatrix}\frac{\partial x_1}{\partial y_1}\\ \frac{\partial x_2}{\partial y_1}\\ \vdots \\ \frac{\partial x_n}{\partial y_1}\end{bmatrix}$$

$$\frac{\partial f}{\partial y_2} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right]\begin{bmatrix}\frac{\partial x_1}{\partial y_2}\\ \vdots \\ \frac{\partial x_n}{\partial y_2}\end{bmatrix}$$

$$\vdots$$

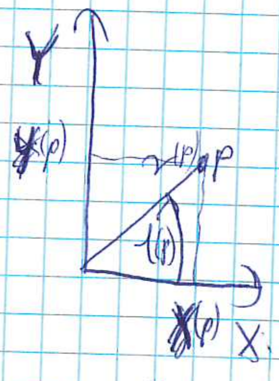
$$\frac{\partial f}{\partial y_n} = \frac{\partial f}{\partial x}\begin{bmatrix}\frac{\partial x_1}{\partial y_n}\\ \vdots \\ \frac{\partial x_n}{\partial y_n}\end{bmatrix}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} = \frac{df}{\cancel{dx}}\frac{\partial x}{\partial y} = \frac{\partial f}{\partial y}$$

Δ Polárkoordinátás alakja

→ Laplace

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \qquad \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \qquad f: \mathbb{R}^2 \to \mathbb{R} \quad C^2\text{-osztály}$$



$$x(p) = r(p)\cos\varphi(p)$$
$$y(p) = r(p)\sin\varphi(p)$$

$$x = \begin{bmatrix} x \\ y \end{bmatrix} \qquad p = \begin{bmatrix} r \\ \varphi \end{bmatrix}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial \cancel{x}} \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{bmatrix}$$

$$\left.\begin{aligned} x &= r\cdot\cos\varphi \\ y &= r\cdot\sin\varphi \end{aligned}\right\} \quad \frac{\partial x}{\partial r} = \cos\varphi$$

$$\frac{\partial x}{\partial \varphi} = -r\sin\varphi$$

$$\frac{\partial y}{\partial r} = \sin\varphi \qquad \frac{\partial y}{\partial \varphi} = r\cdot\cos\varphi$$

De ehh → $\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \end{bmatrix}$ → ez ROSSZ.

$$\cancel{\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}} \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cdot\cos\varphi \end{bmatrix}^{-1}$$

$$\begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \varphi} \end{bmatrix} \frac{1}{r} \begin{bmatrix} r\cdot\cos\varphi & r\cdot\sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{1}{r} \cdot r \cdot \cos\varphi + \frac{\partial f}{\partial \varphi} \cdot \frac{1}{r} \cdot (-\sin\varphi) =$$
$$\cos\varphi \frac{\partial f}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial f}{\partial \varphi}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial r}\frac{1}{r} r\sin\varphi + \frac{\partial f}{\partial \varphi}\frac{1}{r}\cos\varphi = \sin\varphi\frac{\partial f}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial f}{\partial \varphi}$$

илл $\frac{\partial^2 f}{\partial x^2} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x}$, $\frac{\partial^2 f}{\partial y^2} = \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial\left(\overset{g}{\frac{\partial f}{\partial x}}\right)}{\partial x} = \frac{\partial g}{\partial x} = \cos\varphi\frac{dg}{dr} - \frac{1}{r}\sin\varphi\frac{\partial g}{d\varphi}$$

$$\cos\varphi\frac{\partial}{\partial r}\left[\cos\varphi\cdot\frac{\partial f}{dr} - \frac{1}{r}\cdot\sin\varphi\frac{\partial f}{d\varphi}\right] - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\left[\cos\varphi\frac{df}{dr} - \frac{1}{r}\sin\varphi\frac{\partial f}{\partial\varphi}\right]$$

$$= \cos\varphi\left[\cos\varphi\frac{d^2 f}{\partial r^2} - \frac{1}{r^2}\sin\varphi\frac{df}{\partial\varphi} - \frac{1}{r}\sin\varphi\frac{\partial^2 f}{\partial\varphi\partial r}\right] -$$

$$\frac{1}{r}\sin\varphi\left[-\sin\varphi\frac{\partial f}{\partial r} + \cos\varphi\frac{\partial^2 f}{\partial r\partial\varphi} - \frac{1}{r}\cos\varphi\frac{\partial f}{\partial\varphi} - \frac{1}{r}\sin\varphi\frac{\partial^2 f}{\partial\varphi^2}\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \sin\varphi\left[\sin\varphi\frac{\partial^2 f}{\partial r^2} - \frac{1}{r^2}\cos\varphi\frac{\partial f}{\partial\varphi} + \frac{1}{r}\cos\varphi\frac{\partial f}{\partial\varphi\partial r}\right] +$$

$$\frac{1}{r}\cos\varphi\left[\cos\varphi\frac{\partial f}{\partial r} + \sin\varphi\frac{\partial^2 f}{\partial r\partial\varphi} - \frac{1}{r}\sin\varphi\frac{\partial f}{\partial\varphi} + \frac{1}{r}\sin\varphi\frac{\partial f^2}{\partial\varphi^2}\right]$$

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2} = \Delta f$$

$$= (\cos^2\varphi + \sin^2\varphi) \cdot \frac{\partial^2 f}{\partial r^2} + 0 \frac{\partial f^2}{\partial r \partial \varphi} + \frac{1}{r}(\cos^2\varphi + \sin^2\varphi)\frac{\partial f}{\partial r} + 0 \cdot \frac{\partial f}{\partial \varphi} +$$

$$\frac{1}{r^2}(\sin^2\varphi + \cos^2\varphi)\frac{\partial f^2}{\partial \varphi^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \underbrace{\frac{1}{r}\frac{\partial f^2}{\partial \varphi^2}}_{}$$

tócós ugyanaz az

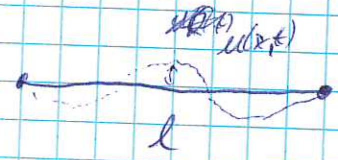
Megj: MEGFORDÍTHET $x = \begin{bmatrix} x \\ \varrho \end{bmatrix}$ $u = \begin{bmatrix} x \\ u \end{bmatrix}$

egyik koordináta kicsés

Sokaság módja: $u = \begin{bmatrix} \xi \\ u \end{bmatrix}$ jelöljük ezt

$\boxed{10.\text{hét}}$

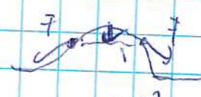
Hullámegyenlet

10)  Felfeszített húr mozgása

T feszítő erő

ϱ sűrűségű (kg/m)

$F = m \cdot a$ $u(x,t)$ kitérés

 x helyen

t időben

$\approx \frac{\partial^2 u}{\partial x^2}$ görbület ??

$$T \frac{\partial^2 u}{\partial x^2} = \varrho \frac{\partial^2 u}{\partial t^2}$$

$T \sim kg \frac{m}{s^2}$ erő, $\varrho \sim kg/m$ $\Big\rangle$ $T/\varrho = \frac{m^2}{s^2} = [\text{sebesség}]^2$

$c = \sqrt{T\varrho}$ sebesség $\qquad$ $c \Rightarrow$ fénysebesség

$$\frac{\partial^2 u}{\partial t^2} - c^2 \cdot \frac{\partial^2 u}{\partial x^2} = 0$$

y: egyenértékű változó!
(skalár)

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad y = c \cdot t$$

$$\begin{bmatrix} \frac{\partial f}{\partial t} & \frac{\partial f}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial \bar{x}} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \bar{x}} \\ \frac{\partial \bar{x}}{\partial t} & \frac{\partial \bar{x}}{\partial \bar{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial \bar{x}} \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{\partial f}{\partial t} & \frac{\partial f}{\partial \bar{x}} \end{bmatrix} \qquad \begin{bmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} \end{bmatrix} = \begin{bmatrix} c \cdot \frac{\partial}{\partial y} & 1 \frac{\partial}{\partial x} \end{bmatrix}$$

$$\frac{\partial f}{\partial t} = c \cdot \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}$$

$$\partial^2 f = \frac{c^2 \partial^2 f}{\partial y^2}$$

$$c^2 \left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \right) = 0$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0}$$

$$\Delta u = 0$$

$$\Delta = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

D'Alembert helyettesítése

$$x = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x+y \\ x-y \end{bmatrix} = \begin{bmatrix} w \\ v \end{bmatrix}$$

$$w = x + y \qquad v = x - y$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \quad \text{Kif } w, v$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{bmatrix} \frac{\partial f}{\partial w} & \frac{\partial f}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$\begin{pmatrix} \frac{\partial f}{\partial w} & \frac{\partial f}{\partial v} \end{pmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial w} + \frac{\partial f}{\partial v}, & \frac{\partial f}{\partial w} - \frac{\partial f}{\partial v} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x} = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} = \frac{\partial}{\partial u}\left[\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right] + \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right] =$$

$$g = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

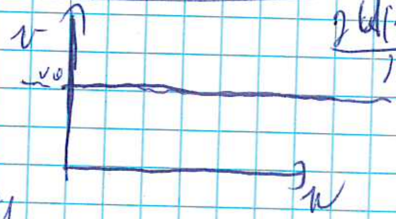
$$\frac{\partial^2 f}{\partial u^2} + 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} - 2\frac{\partial^2 f}{\partial u \partial v} + \frac{\partial^2 f}{\partial v^2}$$

tar bort varandra

$$\Delta f = 4\frac{\partial^2 f}{\partial u \partial v} \qquad \Delta u = 0 \iff \frac{\partial^2 u}{\partial u \partial v} = 0$$

$$u = \frac{\partial u}{\partial v} \qquad \frac{\partial u}{\partial w} = 0$$



$$\frac{\partial u(v_0, w)}{\partial w} = 0$$

$u(v_0, w) =$ Constant

fångster ~~[illegible]~~ w-tal

som fångs u av w-tal

$$\boxed{\text{De fångs } v_0\text{-tal}}$$

$$u = u_1(v) + u_2(w)$$

$$\frac{\partial u_1(v)}{\partial w \partial v} + \frac{\partial u_2(w)}{\partial w \partial w} = 0 + 0 = 0 = \frac{\partial u_1'(v)}{\partial w} + \frac{\partial u_2'(w)}{\partial v}$$

ser v-tal fångs

ser w-tal fångs

$u(x,t) =$

u_1, u_2 tetszőleges

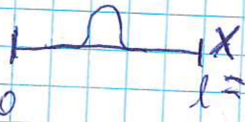
$$u_1(v) + u_2(w) = u_1(x-y) + u_2(x+y) = \boxed{u_1(x-ct) + u_2(x+ct)}$$

Speciális esetek:

1) $u_2 = 0$

„~~[illegible]~~ vidor felület" =

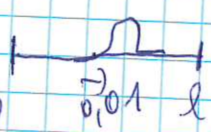
$t = 0$



$u(x,0) = u_1(x)$

$t = 0{,}0001 \to 10000$ kép/s

$c = 100 \frac{m}{s}$

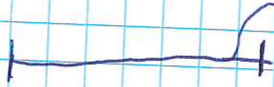


$u(x, 0.0001) = u_1(x - 0{,}01)$

$(100 \cdot 0{,}0001)$

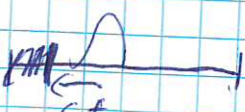
$t = 0{,}0002$

t esetén $\to$ ct

De  ilyen nem lehet most

$\to$ ez rögzített

2) $u_1 = 0$ $\quad u(x,0) = u_2(x)$



 ellenes esetben

ZH !!!

Húr szét kezelése

$u_1, u_2 : \mathbb{R} \to \mathbb{R}$

$u(x,t) = u_1(x-ct) + u_2(x+ct)$

$u(0,t) = 0 \quad \forall t$ (rögzített bal oldalon)

$u(l,t) = 0 \quad \forall t$ (rögzített jobbról)

$u(0,t) = u_1(-ct) + u_2(ct) = 0$

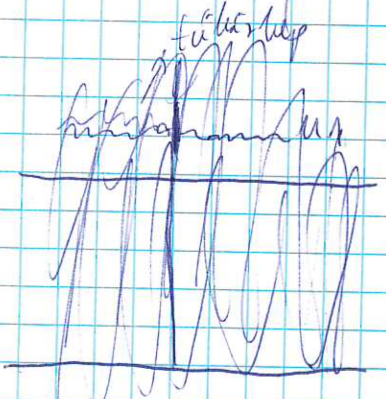
Képzeletbeli

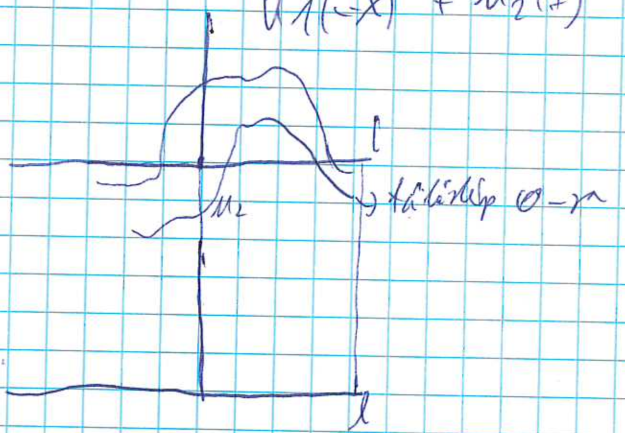
$0 = u(l,t) = u_1(l-ct) + u_2(l+ct)$

$u_1(-ct) + u_2(ct) = 0$

$u_1(-x) + u_2(x)$

[x = ct most csak ez lehet]





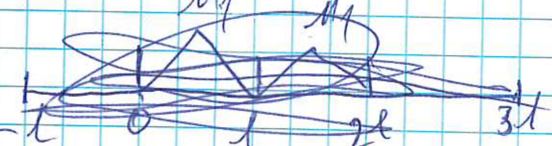
GEOM 2D

Képzeletben

$T_1 \quad T_2$

tükrözési pontok $P_1 \quad P_2$

$u_1(0) = u_2(0) = 0$



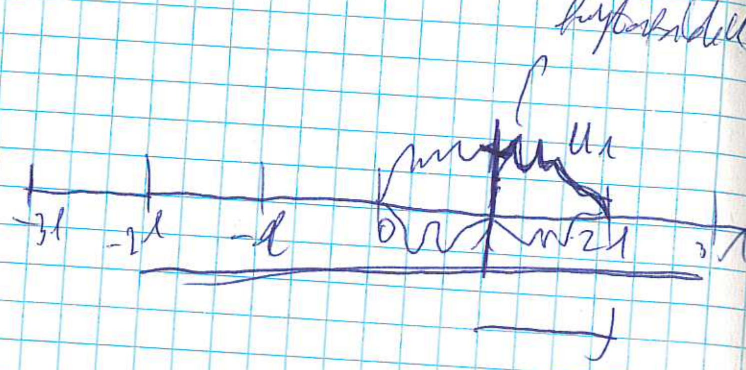
$T_1 : [u_1 \text{ graf}] \to [u_2 \text{ graf}]$

$T_2 : [u_2 \text{ graf}] \to T_2 T_1 [u_1 \text{ graf}] = S(\text{eltol}) [u_1 \text{ graf}]$

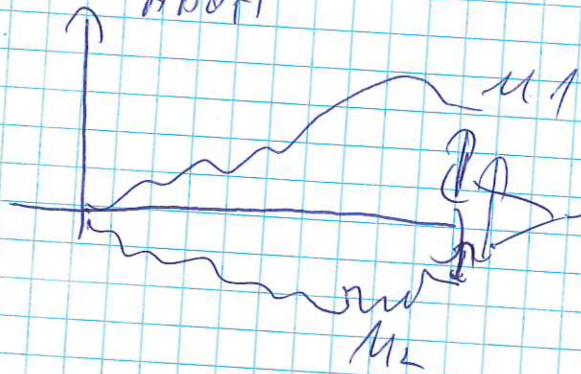
2l-el.

eltolás

u_1 és u_2 2l periodikus !!!!



$u_2 = u_1$ középpontos tükörkép

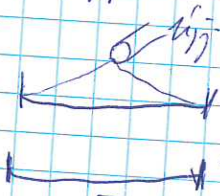


2l periódussal ismételni.

Ez leír minden hullámmozgást 0, l rögzített végpont esetén

Alapeset (1) gitár $u(x,0)$ adott

(2) zongora



0 kezdősebesség

Ott ütés kezdetben

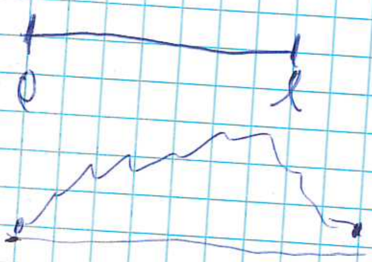
kalapács

$\frac{\partial u}{\partial t}(x,0) \neq 0$

$\frac{\partial u}{\partial t}(x,0) \neq 0$

$\Downarrow$ adott sebesség

11. hét



c hullámsebesség

1, 0. időpontban $u(x,0) = u(x)$

2, sebesség 0. időben $\left.\frac{\partial u}{\partial t}\right|_{t=0} u(x,t) = v(x)$

Alapmodellek

A, gitár $v(x) = 0$ $u(x)$ adott

B, Zongora $u(t) = 0$ $v(t)$ adott

Általános eset: $u(t), v(t) \neq 0$ $\left[u[0] = u(l) = 0 \right.$

$\left. v[0] = v(l) = 0 \right]$

Szuperpozíció elve: → gitár + zongora után

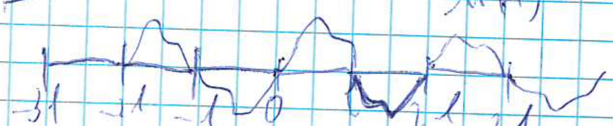
$$u(x,t) = u_{gitár}(x,t) + u_{zongora}(x,t)$$

OK, hogy megoldások $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

← akár zongora, akár gitárt össze adva

~~Teljes ideális hullám egyenlet.~~

Gitár $u(t)$ $\bar{u}(t)$ → kiterjesztés!



2l periodicitás és csak pontokra tükrözés!

Gitár:

$$u(x,t) = \frac{1}{2}\left(\bar{u}(x+ct)\right) + \frac{1}{2}\left(\bar{u}(x-ct)\right)$$

Zongora Superpozíció $u(x,t) = u_1(x+ct) + u_2(x-ct)$

$v(t)$ $u(t)$ 2π szerint periodikus

$$\frac{\partial u}{\partial t}\Big|_{t=0} u(x+1) = v(x)$$

$$u'(x+c\cdot t)\cdot c \Big|_{t=0} - u'(x-ct)(-c)\Big|_{t=0} = v(x)$$

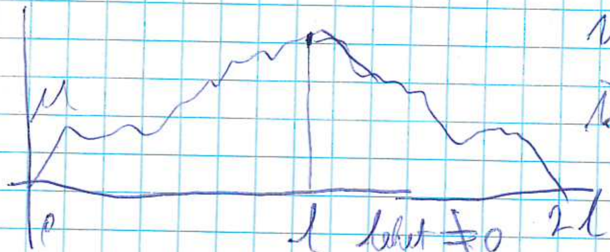
$$u'(x) + u'(x) = \frac{1}{c} v(x)$$

$$u'(x) = \frac{1}{2c} v(x) \qquad u(x) = \int_{s=0}^{*} \frac{1}{2c} v(s)\, ds$$

⇓

v nem akármilyen

$\cancel{u(l)=0}$



Több dim. eset

$\boxed{2D}$ $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u$$

$$\tilde{\Delta} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

1D ——— 2D

$\Omega \subseteq \mathbb{R}^2$

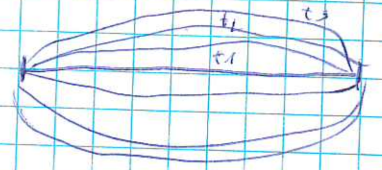
dob

$\boxed{\text{Medence}}$

Fourier módszer:

állóhullámok feltételezése.

$$u = u(x_1, \dots x_n, t) \qquad u = f(t)\cdot u(x_1, \dots, x_n)$$



Szuperpozíció állóhullámokból.

Állóhullámok egyenlete.

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_N^2}$$

$$u(x_1, \dots, x_N, t) = f(t) \cdot U(x_1, \dots, x_N)$$

$$\frac{\partial u}{\partial t} = f'(t) \cdot U(x_1 \dots x_N)$$

$$\frac{\partial^2 u}{\partial t^2} = f''(t) \cdot U(x_1 \dots x_N)$$

$$\frac{\partial u}{\partial x_i} = f(t) \cdot \frac{\partial U}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_i^2} = f(t) \cdot \frac{\partial^2 U}{\partial x_i^2}$$

$$\frac{1}{c^2} f''(t) \cdot U(x_1 \dots x_N) = \sum_{i=1}^{N} f(t) \cdot \frac{\partial^2 U}{\partial x_i^2}(x_1 \dots x_N)$$

TRÜKK $\quad / \, f(t) \, U(x_1 \dots x_N)$

$$\frac{1}{c^2}\frac{f''(t)}{f(t)} = \sum_{i=1}^{N} \frac{\partial^2 U}{\partial x_i^2} \Big/ U \quad (x_1 \dots x_N)$$

↓ függvénye t-nek $\quad = \quad$ függvénye $(x_1, \dots x_N)$

Ilyenkor ez csak közös konstans lehet.

$$\frac{1}{c^2} \cdot \frac{f''(t)}{f(t)} = K \qquad f''(t) = K \cdot c^2 f(t)$$

rezgés megoldása:

↓

sajátértékprobléma

$$\sum_{i=1}^{N} \frac{\partial^2 U}{\partial x_i^2} = K$$

LAPLACE

sin, cos, sinh, cosh

$f(t) = A\sin(\omega t) + B\cdot\cos(\omega t) = \bar{A}\sin(\omega t+\alpha)$

csak időtől eltolt megoldás!

$f''(t) = K\cdot c^2 f(t)$

$\frac{\partial^2 \sin(\omega t)}{\partial t^2} = -\omega^2\cdot\sin(\omega t)$

$\frac{\partial^2 \sin(\omega t)}{\partial t^2} = K\cdot c^2\cdot\sin(\omega t)$

$-\omega^2 \sin(\omega t) = K\cdot c^2\cdot\sin(\omega t)$

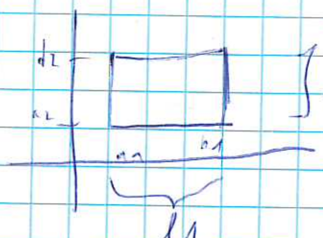
$\omega^2 = -K c^2$ → periódikus a megoldás 1 időegység alatt

$\frac{1\,\sec\cdot\omega}{2\pi}$ megoldás

$\frac{\Delta u}{u} = K$ $\boxed{\Delta u = \frac{-\omega^2}{c^2} u}$

Nézzük meg a ○ □ alapesetekben 2D-ben

Ω



l_2 $u(x,t) = \sin(\omega t)\cdot\underbrace{u(x,y)}_{u_1(x)\,u_2(y)}$

l_1 Változók szeparálása.

$\Delta u = -\frac{\omega^2}{c^2} u$ $\frac{\partial}{\partial x} u = u_1'(x)\, u_2(y)$

$\frac{\partial^2}{\partial x^2} = u_1''(x)\,u_2(y)$

$\frac{\partial^2}{\partial y^2} u = u_1(x)\,u_2''(y)$

$u_1''(x)\,u_2(y) + u_1(x)\cdot u_2''(y) =$

$\underbrace{-\frac{\omega^2}{c^2}}_{K} u_1(x)\,u_2(y)$

$$u_1(x)\, u_2(y)$$

$$\frac{u''_1(x)}{u_1(x)} + \frac{u''_2(y)}{u_2(y)} = R$$

$$\frac{u''_1(x)}{u_1(x)} = R_1$$

$$\frac{u''_1(x)}{u(x)} = -\frac{u''_2(y)}{u(y)} + R$$

$$\frac{u''_2(y)}{u_2(y)} = R_2$$

$$R_1 + R_2 = R = -\frac{w^2}{c^2}$$

$$u''_1(x) = R_1 u_1(x)$$

$$\sin(w_1 x + f_1)$$

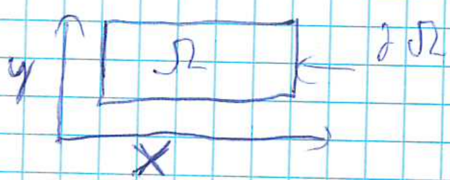
$$u_1(x) = c_1 \sin(w_1 + f_1)$$

$$u_2(y) = c_2 \sin(w_2 + f_2)$$

$$\boxed{w_1^2 + w_2^2 = w^2}$$

$$\frac{u''_1}{u_1} = -\frac{w_1^2}{c^2} \qquad \frac{u''_2}{u_2} = -\frac{w_2^2}{c^2}$$

$$u(x,y,t) = \sin(wt + f_1) \cdot \sin(w_1 t + f_1) \cdot \sin(w_2 t + f_2)$$



Állóhullámok Ω-n

0 (B) ∂Ω

[Hullámtér nedences]

Tanulság $u(x,y,t) \to f(t)\, u_1(x)\, u_2(y) \Rightarrow$

$$\sin(wt + f) \cdot \sin(w_1 x + f_1) \cdot \sin(w_2 y + f_2)$$

$$w^2 = w_1^2 + w_2^2 \quad (\text{Miért??})$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\frac{1}{\lambda^2} \qquad \frac{1}{\lambda_1^2} \qquad \frac{1}{\lambda_2^2}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{1}{c^2}\omega^2 \sin(\omega t + l)\sin(\omega_1 x + l_1)\cdot \sin(\omega_2 y + l_2) =$$

$$\omega_1^2 \sin(\omega_1 x + l_1) + \omega_2^2 \sin(\omega_2 y + l_2)$$

$$\boxed{\frac{\omega^2}{c^2} = \omega_1^2 + \omega_2^2}$$

Általában és N dimenzióban

$$u(x_1, \dots, x_N, t) = f(t)\, u(x_1) \dots u(x_N)$$

$$f(t) = \sin(\omega t + l)$$

$$u_1(x_1) = \sin(\omega_1 x_1 + l_1)$$

$$\frac{\omega^2}{c^2} = \omega_1^2 + \dots + \omega_N^2 = \sum_{i=1}^{N} \omega_i^2$$

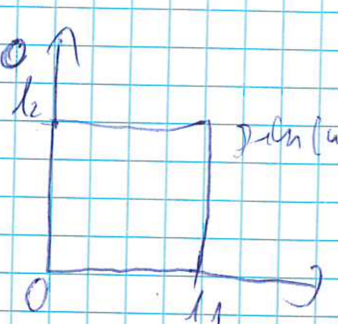
Határfeltételek:

$$u(\partial\Omega(t)) = 0$$ (rögzített a határon)

Kezdő $l=0$

(rögzített dob)

kis dobon mérve



$\sin(\omega t)\sin(\omega_1 x + l_1)\sin(\omega_2 y + l_2) = 0$ ha

ha $(x, y) \in$ 

vagy $(x, y) \in$ 

vagy $(x, y) \in$ 

Anwendung

vagy

Pl:

$\phi(x) = 0$ ha $x \leq 0$

$e^{-1/x}$ ha $x > 0$

Taylor $\phi(x) = \sum_{k=0}^{\infty} 0 \cdot x^n = 0$

$u(x, y, t) = x = 0$

vagy $y = 0$

vagy $x = l_1$

vagy $y = l_2$

$\sin(\omega t) \cdot \underbrace{\sin(w_1 \cdot 0 + l_1)}_{\sin l_1} \cdot \sin(w_2 \cdot 0 + l_2) = 0 \quad \forall x$

$[-1, 1]$-en

$[-1, 1]$-a

~~Ax~~

$\sin l_1 = 0$ $l_1 = k \cdot \pi$ $k \in -1, 1 \ldots -n, n$

$l_1 \in \pi \mathbb{Z}$

Hasonlóan $l_2 \in \pi \cdot \mathbb{Z}$

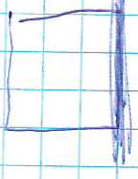
$\sin(w_1 x + k \cdot \pi) =$

$(-1)^k (w_1 x + k \cdot \pi)$

$(-1)^k (w_2 y + k \cdot \pi)$

Végtár: $A \sin(w_1 x) \cdot \sin(w_2 y) \sin(\omega t) = u(x, y, t)$

A helyett $\pm A$?



$0 = u \quad \forall t$

$u(l_1, y, t) = 0 \quad \forall t, y$

$A \sin(w_1 l_1) \sin(w_2 y) \sin(wt) = 0$

tetszőleges $\in [-1, 1]$

$\sin(w_1 l_1) = 0$

$w_1 l = 0, \pm\pi, \pm 2\pi$

$w_1 = \frac{\pi}{l_1}\mathbb{Z}$

$w_1 l_1 \in \pi\mathbb{Z}$

$w_1 = \frac{k_1}{l_1}\pi \qquad \sin(w_1 x + l_1) = \pm\sin\left(\frac{k_1}{l_1}\pi x\right)$

$k_1 \in \mathbb{Z} \neq 0$

Hasonlóan

$w_2 = \frac{k_2}{l_2}\pi$

$\sin(w_2 y + l_2) = \sin\left(\frac{k_2}{l_2}\pi y\right) = \pm\sin\left(\frac{k_2 \pi y}{l_2}\right)$

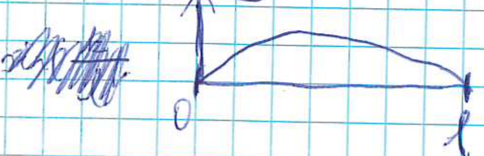
$u(x, y, t) = A \sin(wt) \sin\left(k_1 \pi \frac{x}{l_1}\right) \sin\left(k_2 \pi \frac{y}{l_2}\right)$

$k_1, k_2 \neq 0$

$\frac{w^2}{c^2} = w_1^2 + w_2^2 = \pi^2\left[k_1^2/l_1^2 + k_2^2/l_2^2\right]$

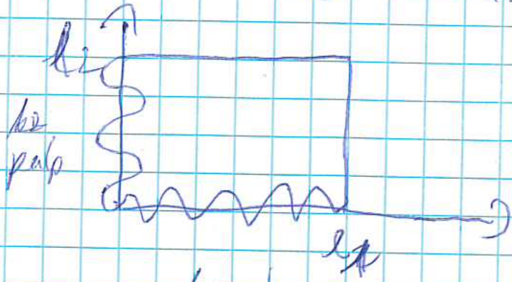
$w = c \cdot \pi \cdot \sqrt{(k_1/l_1)^2 + (k_2/l_2)^2}$

Ábrázolás:

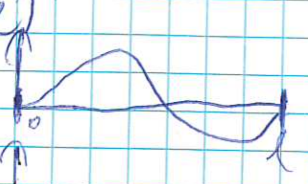


$k = 1$

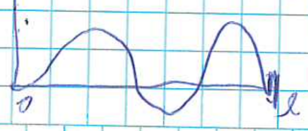
$\sin\left(k \pi \frac{x}{l_1}\right) \sin\left(k_2 \pi \frac{y}{l_2}\right)$



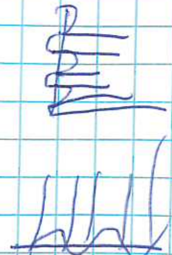
$\sin\left(1 \cdot \pi \frac{x}{l}\right)$



$k = 2$



$k = 2$



hasonlóan rezeg.

$\cos\left(k\pi \frac{x}{l}\right)$

$u(0)=0$ $u(l)=0$ (orgonasíp)

$\sin\left(\left(k+\frac{1}{2}\right)\pi \frac{x}{l}\right)$

(Dirichlet) CARLESON téglalap alakú ▢ -on vagy ⟶ ★ előrehaladó hullám

Fourier sorba fejthető.

Alapfrekvencia:

adott részletek húr $u(x,t) = A\cos(\omega t)\sin\left(k\pi \frac{x}{l}\right)$

$\omega = c\cdot\pi\cdot 1/l = k\cdot\omega_0$

2D ▭

$\omega = c\cdot\pi\sqrt{\left(\frac{k_1}{l_1}\right)^2+\left(\frac{k_2}{l_2}\right)^2}$

meghatározza l_1, l_2-öt

Zn: alapfrekvencia dolog

KÖR ALAKÚ membrán (dob)

álló hullámok:

$u(x,y,t) = \sin(\omega t+\varphi)\cdot u(x,y)$

$u(x,y,t) = \sin(\omega t+\varphi)\cdot u_1(r)\cdot u(\varphi)$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \varphi^2}$$

$$\rightarrow \frac{1}{c^2}(-\omega^2)\sin(\omega t+\varphi)\, u_1(r)\, u_2(\varphi) = \sin(\omega t+\varphi)\cdot\left[u''(r)+\frac{1}{r}u'(r)+\frac{1}{r^2}u(\varphi)\right]$$

$$-\frac{1}{c^2}\cdot\omega^2(\omega t+\varphi) = K$$

$$\cancel{u''(r) + \frac{1}{r}u'(r)}$$

Nem lehet kifejezni sin, cos, log, e

1 segédfüggvénnyel.

$$r^2\left[u''(r)+\frac{1}{r}u'(r)\right] = K_1$$

$$u''(\varphi) = K_2$$

$$K = K_1 + K_2$$

Új speciális fgv. előjönnek: Bessel fgv.

$\boxed{\text{Valszég}}$

KOCKA "cinkelt"

n feldobás

1, kérdés $P(6\text{-os}) \overset{?}{=} \frac{1}{6}$

$$X(\omega) = \begin{cases} 1 & \text{ha 6-ost dobunk} \\ 0 & \text{ha nem 6-os} \end{cases}$$

$X_1 \; X_2$

$\uparrow \;\; \uparrow$

1. 2. dobás

$X_1+\dots+X_n$ = ahányszor 6-ost dobtunk n dobásból

$X_1(\omega)+X_2(\omega)+\dots+X_n(\omega)$

pl: $n=10$ $\Omega_{10} = \{$összes lehetséges 10 dobássor$\}$ = 6^{10}

$\Omega_{10} = \{ \{d_1, d_2, \dots, d_{10}\}\ \cancel{\{d_1, d_2, \dots, d_{10}\}} \}$

$d_i \in \{1 \dots 6\}$

$X_3(\omega) =$ ha $\omega = (6, 3, \boxed{3}, 2, 5, 6, 5, 2, 1, 2)$

3. dobás eredménye.

$\Omega = \Omega_\infty = \{(d_1, d_2, \dots d_\infty)\} \quad (d_i \in 1 \dots 6)$

végtelenségig dobáljuk a kockát.

Kérdés: Mi írja le azt az eseményt, hogy csak párosat dobunk mindig?

Páros $= \{\omega \in \Omega : X_i(\omega) \in \{2,4,6\} \quad i = 1, 2 \dots\}$

$P(\text{páros}) = ?$

$$P(\underbrace{X_1 \in \{2,4,6\}, X_2 \in \{2,4,6\}, \dots X_n \in \{2,4,6\}}_{\substack{\omega : X_1(\omega) \in \{2,4,6\} \\ X_n(\omega) \in \{2,4,6\}}})$$

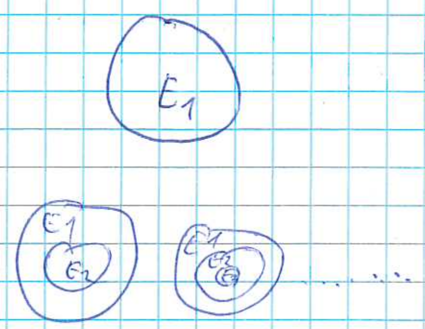
$$= P\left(\bigcap_{i=1}^{n} P(X_i \in \{2,4,6\})\right)$$

dobások függetlenek $= \prod_{i=1}^{n} P(X_i \in \{2,4,6\})$

$= \prod_{i=1}^{n} \frac{1}{2}$ ← [illegible] esetet $= \frac{1}{2^n}$

$$P(\text{páros}) = \lim_{n\to\infty} P(X_1 \text{ páros}, X_2 \text{ páros}, X_n \text{ páros}) = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

$$P_{\text{páros}} = \bigcap_{n=1}^{\infty} \underbrace{(X_1 \text{ páros} \; X_n \text{ páros})}_{E_n}$$



axióma miatt. $P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$

pl: 100 dobás és abból 33 6-os

→ Nem csalt-e? (45) $\frac{100}{6} \approx$ 16-nak kéne lennie

CENTRÁLIS HATÁReloszlás tétele.

~~$X_1 + \dots + X_n - E(X_1 + \dots + X_n)$~~

$$\frac{X_1 + \dots + X_n - n\,E(X)}{D(X)\cdot\sqrt{n}}$$ → tart normális standard eloszlás felé!

$E(X) = \frac{1}{6}$

6 5 4 3 2 1

$\left(1-\frac{1}{6}\right)$ $0 \cdot \frac{1}{6}$ $0 \cdot \frac{1}{6}$ $0 \cdot \frac{1}{6}$

6 5 4 3 2 1

$\frac{1}{6} \cdot 1$ $\frac{1}{6} \cdot 0 \dots$ $\frac{1}{6} \cdot 0$

$$\frac{1}{6}\cdot\left(\frac{5}{6}\right)^2 + \frac{1}{6}\cdot\left(\frac{1}{6}\right)^2 + \frac{1}{6}\cdot\left(\frac{1}{6}\right)^2 + \frac{1}{6}\cdot\left(\frac{1}{6}\right)^2 + \frac{1}{6}\cdot\left(\frac{1}{6}\right)^2 +$$

$$\frac{1}{6}\cdot\left(\frac{1}{6}\right)^2$$

$$\frac{25+1+1+1+1+1+1}{6^3} = \sqrt{\frac{30}{216}} = \frac{\cancel{15}}{\cancel{108}} = \frac{\cancel{5}}{\cancel{36}} = \frac{\sqrt{5}}{6}$$

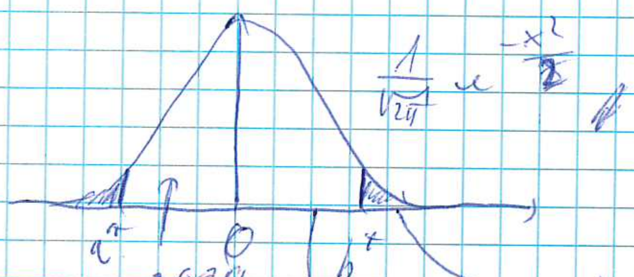
$6^2 \cdot 6$

$6 \cdot 6 = 36 \cdot 6 =$

$$\frac{x_1+\dots x_n - n\frac{1}{6}}{\sqrt{n}\ \sqrt{5}/6} \xrightarrow[\text{konst}]{\text{eloszlásban}}$$ 0 várható értékű 1 szórású normális eloszlás

$$P(a<u_n<b) \to \underbrace{\int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx}_{\text{Gauss görbe}}$$

centrális határeloszlás tétel alapján



$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

u_{100}

0,95 → konvergál

u_n → konvergál!

$n \geq 40$

$$u = \frac{x_1+\dots+x_{100}-100\cdot\frac{1}{6}}{\sqrt{100}\cdot\frac{\sqrt{5}}{6}} = \frac{33-\frac{100}{6}}{10\cdot\sqrt{5}\cdot\frac{1}{6}} = \frac{198-100}{10\cdot\sqrt{5}} =$$

$$\frac{98}{22{,}36} = 4{,}3828$$

Döntés

~~K_{100}~~ →

$u_{100} \notin A^{a}, B^{b}$ által adott a hacka.

„Isten tudja ténylegesen cinkelt-e," de nagyon valószínűtlen, hogy ne legyen cinkelt"

$p < 5\%$, hogy jó legyen a hacka

HA $y = x_1 + \dots + x_n$ ahol $x_1, \dots, x_n$ független és „hasonló" változók

akkor y eloszlása ~ normális eloszlás!

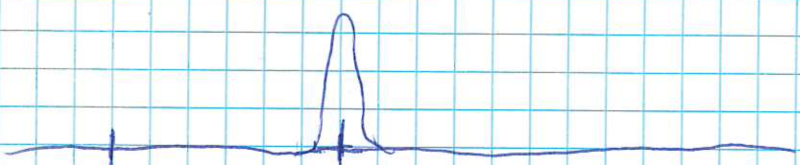
m várható értékkel és

σ szórás mellett.

pl. emberi magasságok $n \sim 86000$

ok → normális eloszlású!

Lehet-e ténylegesen normális eloszlású az emberi magasság?



0

$m = 175$

$\sigma = 4$ cm

normálisnál van esély arra, hogy egy ember −1 centi

Tesztelés és döntés:

U próba

T próba

2 mintás T próba,

χ^2 próba

Mi a kérdés?

1, Van 1 feltevésünk arról, hogy a jelenség (kísérlet.....) X eredményének eloszlása olyan
valószínűségi változó

$P(a<x<b) = \cancel{f(\Theta_1 .. \Theta_k}$

tehát nem tudjuk meg a $f_{\Theta_1 .. \Theta_k}(a, b)$ valamilyen $(\Theta_1 .. \Theta_k)$ Θ_1 és Θ_k mellett.

2) Kérdés adott valamilyen Θ_1, Θ_k konkrét számok, vajon tényleg tényleg $p(a<x<b) = f_{\Theta_1, \Theta_k}(a, b)$ elfogadható-e a mérések alapján.

3, x_1, x_n minta: ugyan olyan eloszlásúak mint X és egymástól függetlenek.

Megadhatunk (ez nehéz), egy olyan $f(x_1 ... x_n, \Theta_1 .. \Theta_k)$ függvényt, ~~[illegible]~~ $p($ és a_*, b_* számokat, hogy annak a valószínűsége $p(a_* < f(x_1, x_n, \Theta_1, \Theta_k) < b_*) = 0,95$

Ezért ha $x_1 ... x_n$ a mért értékek, és ~~[illegible]~~ $a_* < f(x_1, x_n, \Theta_1, \Theta_k) < b_*$ akkor elfogadható, hogy tényleg $\Theta_1 .. \Theta_k$ paraméterek az igaziak.

χ^2 próba kocka dobásokkal

~~Kocka dob~~

1	2	3	4	5	6
15	15	15	15	20	20

100 dobásból

! Cinkelt-e a kocka !

Nem cinkelt esetben n dobásnál $\sim \frac{1}{6}n$ 1,2,3,4,5,6

$$\chi^2_{k,n} = \sum_{i=1}^{n} \frac{(l_i - n p_i)^2}{n \cdot p_i}$$

[db = kimenetény

ismert p_1, p_k valószínűségek]

T: $n \to \infty \Rightarrow \chi^2_{k,n} \xrightarrow{\text{eloszlásban}} \chi^2_k$

↓ 1, 1.6 ↓ k db

$n = l_1 + \dots + l_k$

minták száma.

konkrétan ismert függv. pl. táblázatban van

$$\chi_{6,100} = \sum_{i=1}^{6} \frac{(\#\{i \text{ dobás}\} - 16{,}67)^2}{100 \cdot \frac{1}{6}} \dots$$

$16{,}66 = \frac{100}{6} = 100 \cdot \frac{1}{6}$ → annak valószínűsége, hogy adott számot dobok

dobásszám

$E(\#\{100 \text{ db } 1\text{-es dobásnál}\}) = E(\#\{100 \text{ db } 2\text{-es dobás}\})$

$E(\#\{100 \text{ dobásból } 3\text{-as adnak}\})$

$P_i = P(\text{1 dobásnál } i \text{ jöjjön ki}) = \frac{1}{6}$ jó kockánál

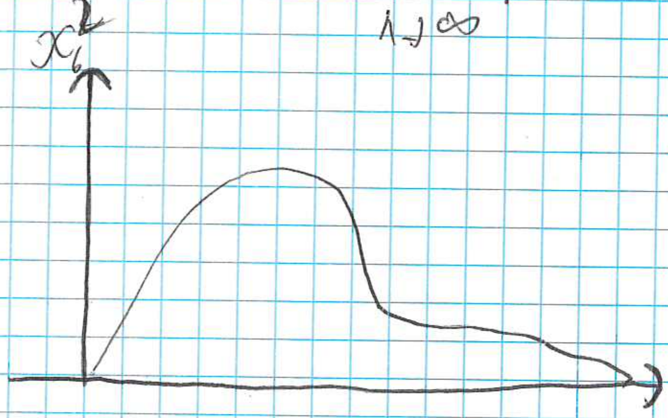
$$\chi^2_{6,n} = \sum_{i=1}^{n} \frac{(\#\{i \text{ dobások száma}\} - n \cdot p_i)^2}{n \cdot p_i} \longrightarrow \chi^2 \xrightarrow{d\text{-ben}} \chi^2_6$$

$(n \to \infty)$

függ $p_1, \ldots, p_6$-tól

$$P(\chi^2_{6,n} \in (a,b)) \xrightarrow[n \to \infty]{} P(\chi^2_6 \in (a,b)) \quad \exists f_6$$

$$P(\chi^2_6 \in (a,b)) = \int_a^b f(t)\,dx$$



f_6 6 db független $N(0,1)$ - eloszlású valószínűségi változó összegének az eloszlása
(független)

X, Y független valószínűségi változók.

$$P(X \in (a,b)) = \int_a^b f(t)\,dx \qquad P(Y \in (a,b)) = \int_a^b g(t)\,dx$$

pl: X $N(0,1)$ eloszlású: Gauss függvény $= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = f(x)$

$X = \chi^2_1$ $N(0,1)$ eloszlás ^{2}-e. $P(X \in (a,b)) = P(G^2 \in (a,b)) =$

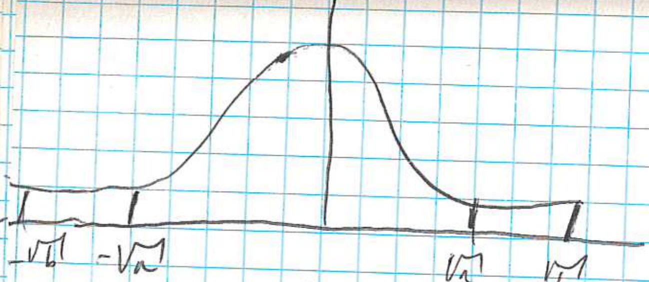
$G \sim N(0,1)$

$G^2 \in (4,9)$

$G \in (2,3) \cup (-2,-3)$ $\quad P(G \in (\sqrt{a}, \sqrt{b}) \cup (-\sqrt{b}, -\sqrt{a}))$

$$P(X \in (a,b)) = P(G \in (\sqrt{a}, \sqrt{b})) + P(G \in (-\sqrt{b}, -\sqrt{a})) =$$

$$\int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx + \int_{-\sqrt{b}}^{-\sqrt{a}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$



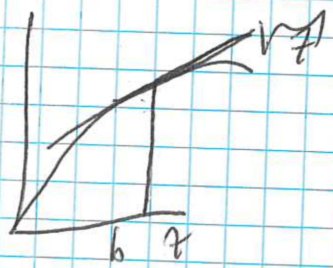
$$\Rightarrow \; 2\int_{\sqrt{a}}^{\sqrt{b}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \int_a^b f(t)\,dt \qquad f = ?$$

$$\int_0^{\sqrt{b}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \int_0^b f(y)\,dy \;\Big/\; \frac{\partial}{\partial b}$$

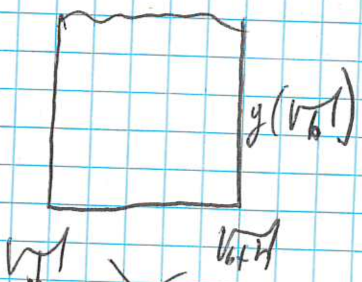
$\int_b f$

$$\frac{\partial}{\partial b}\int_0^{\sqrt{b}} g(x)\,dx = f(b) \approx \lim_{h\to 0} \frac{\int_0^{\sqrt{b+h}} g(t)\,dt - \int_0^{\sqrt{b}} g(t)\,dt}{h} = \frac{1}{h}\int_{\sqrt{b}}^{\sqrt{b+h}} g(t)\,dt$$

$$\sqrt{b+h} \simeq \sqrt{b} + \frac{1}{2\sqrt{b}}h$$



$$t^{\frac{1}{2}} = \frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{b}}$$



$$\simeq \frac{1}{2\sqrt{b}}\cdot h$$

$$\approx \frac{1}{h}\cdot g(\sqrt{b})\,\frac{1}{2\sqrt{b}}\cdot h = \frac{g(\sqrt{b})}{2\sqrt{b}}$$

$$\frac{g(\sqrt{b})}{2\sqrt{b}} = f(b)$$

$\Downarrow$

$$\frac{\frac{1}{\sqrt{2\pi}}\cdot e^{-b/2}}{2\sqrt{b}} = f(b) \qquad P(x\in(a,b)) = \int_a^b \frac{1}{\sqrt{2\pi x}}\, e^{-\frac{x}{2}}\,dx$$

$0 < a < b$

$$f(x) = \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{x}}\, e^{-\frac{x}{2}}$$

χ^2_6 sűrűség fgv-e.

x_1, x_2 független valószínűségi vált.

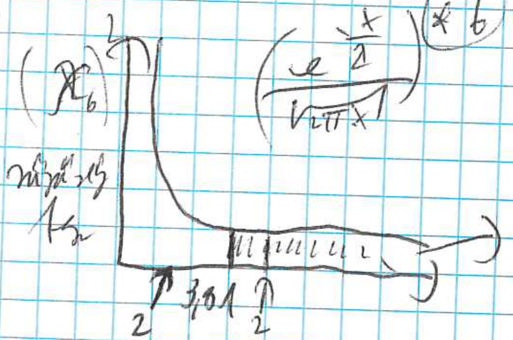
f_1, f_2 sűrűségfgv $\Rightarrow$ $x_1 + x_2$ sűrűségfgv-e $f_1 * f_2$

$\chi^2_6 = f * f * f * f * f * f$ ahol $f(x) = \frac{e^{-\frac{x}{2}}}{\sqrt{2\pi x}}$

$\mathcal{L}\,\chi^2_6 = (\mathcal{L} f)^6 \qquad \chi^2_6 = \mathcal{L}^{-1}\left((\mathcal{L}(f))^6\right)$

Laplace trafo

$\left(\frac{e^{-\frac{x}{2}}}{\sqrt{2\pi x}}\right)^{*6}$ → határoz konvolúcióval.



→ 0,05 → szignifikancia szint

$$\chi^2_{6,100} = \frac{\left(4\cdot\left(15 - \frac{100}{6}\right)\right)^2 + \left(2\cdot\left(20 - \frac{100}{6}\right)\right)^2}{\frac{100}{6}} =$$

~~1,1111~~ +

$$\frac{\frac{100}{3}}{\frac{100}{6}} = 2$$

elfogadjuk. mert $2 < 3,81$

$$\frac{\frac{1}{\sqrt{2\pi}} \cdot e^{-b/2}}{2\sqrt{b}} = f(b)$$

$$P(x \in (a,b)) = \int_a^b \frac{1}{\sqrt{2\pi x}} e^{-\frac{x}{2}}\, dx$$

$0 < a < b$

$$f(t) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{x}} e^{-\frac{x}{2}}$$

χ^2_6 sűrűség fgv-e.

x_1, x_2 független valószínűségi vált.

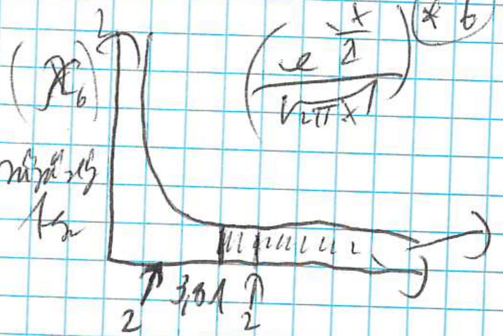
f_1, f_2 sűrűségfgv $\Rightarrow$ $x_1 + x_2$ sűrűségfgv-e $f_1 * f_2$

$\chi^2_6 = f * f * f * f * f * f$ ahol $f(x) = \dfrac{e^{-\frac{x}{2}}}{\sqrt{2\pi x}}$

$\mathcal{L}\chi^2_6 = (\mathcal{L}f)^6$ $\quad \chi^2_6 = \mathcal{L}^{-1}\left((\mathcal{L}f)^6\right)$

Laplace trafo

$\left(\dfrac{e^{-\frac{x}{2}}}{\sqrt{2\pi x}}\right)^{*6} \to$ hatvány konvolúció.



$\to$ 0,05 $\to$ szignifikancia szint

$$\chi^2_{6,100} = \frac{\left(4\cdot\left(15 - \frac{100}{6}\right)\right)^2 + \left(2\cdot\left(20 - \frac{100}{6}\right)\right)^2}{\frac{100}{6}} =$$

~~1,1111~~ +

$$\frac{\frac{100}{3}}{\frac{100}{6}} = 2$$

elfogadjuk. mert $2 < 3{,}81$

Maximum likelihood módszer

Kocka $1, 2, 3, 4 \to 15$

$5, 6 \to 20$

Mit mondhatunk $p_1, \ldots, p_6$-ról ha $p_i = p(\text{dobás} = i)$

TFH. p_1, p_6 valószínűségi változók

$p(1, 2, 3, 4$ 15 ször, $5, 6$ 20 szor$)$

6^{100} sorozat lehet.

$p(1, 5, 3, 6, 1, 1, 2, 4, 4, \ldots, 3$ 100 darabos$) = ?$

$p() = p(1) \cdot p(5) \cdot p(3) \cdot p(6) \cdot \ldots \cdot p(3)$

15 db p_1 — 20 db p_5, p_6

15 db p_2, p_3, p_4

$$p_1^{15} p_2^{15} p_3^{15} p_4^{15} \cdot p_5^{20} p_6^{20}$$

(Melyen p_1, p_6 mellett legvalószínűbb ez.

MAX$_{p_1 \ldots p_6}$)

$$\left.\begin{aligned} \frac{\partial}{\partial p_1} p_1^{15} \cdots p_6^{20} &= 0 \\ \frac{\partial}{\partial p_6} p_1^{15} \cdots p_6^{20} &= 0 \end{aligned}\right\} \quad \begin{aligned} p_1 + \ldots + p_6 &= 1 \\ p_1 \ldots p_6 &\geq 0 \end{aligned}$$

$$p_1 = p_2 = p_3 = p_4 = x$$

$$p_5 = p_6 = \frac{1 - 4x}{2}$$

$x^{60}\cdot\left(\frac{1-4x}{2}\right)^{40} \to max$

$x \in \left[0, \frac{1}{4}\right]$

$\frac{d}{dx}\left(x^{60}\cdot\left(\frac{1-4x}{2}\right)^{40}\right)=0$

$60x^{59}\cdot\left(\frac{1-4x}{2}\right)^{40} - 40x^{60}\left(\frac{1-4x}{2}\right)^{39}\cdot 2 = 0 \;\Big/\; 20x^{59}\cdot\left(\frac{1-4x}{2}\right)^{39}$

$3\cdot\left(\frac{1-4x}{2}\right) - 4x = 0$

$3\left(\frac{1-4x}{2}\right) = 4x$

$\frac{1-4x}{2} = \frac{4x}{3}$

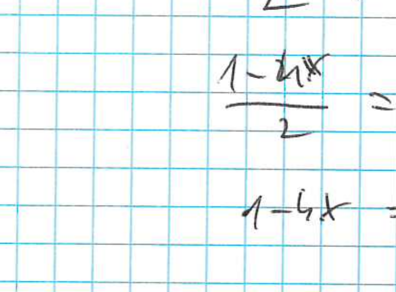
$1-4x = \frac{8x}{3}$

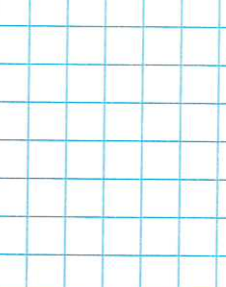
$1 = \frac{20x}{3}$

$3 = 20x$

$x = \frac{3}{20}$ $\quad \frac{4}{20} = 15,16$

$\boxed{\text{14. ábr}}$





$V = \left\{ (R,K,1) : R<K<1,\ R \in (80,100) \right\}$

$R<K<1$

$$x^{60}\cdot\left(\frac{1-4x}{2}\right)^{40} \to \max$$

$$\cancel{60\,x^{59}\ldots\frac{\ldots^{40}}{2^{40}}} = \cancel{\ldots 40\ldots 4^{40}x^{39}\ldots 2^{40}} \cancel{\ldots = 40}$$

$$x \in \left[0, \frac{1}{4}\right]$$

$$\frac{d}{dx}\left(x^{60}\cdot\left(\frac{1-4x}{2}\right)^{40}\right) = 0$$

$$60x^{59}\cdot\left(\frac{1-4x}{2}\right)^{40} - 40x^{60}\left(\frac{1-4x}{2}\right)^{39}\cdot 2 = 0 \quad \Big/\ 20x^{59}\left(\frac{1-4x}{2}\right)^{39}$$

$$3\cdot\left(\frac{1-4x}{2}\right) - 4x = 0$$

$$3\left(\frac{1-4x}{2}\right) = 4x$$

$$\frac{1-4x}{2} = \frac{4x}{3}$$

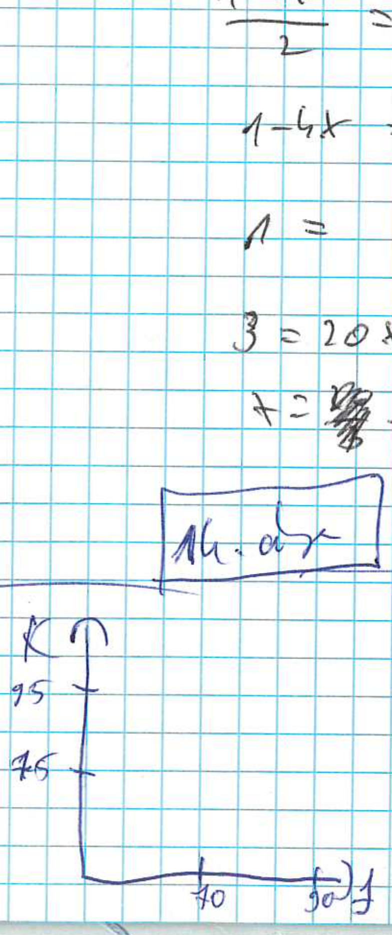
$$1-4x = \frac{8x}{3}$$

$$1 = \frac{20x}{3}$$

$$3 = 20x$$

$$x = \frac{3}{20} \qquad \frac{4}{20} = 15{,}16$$

$\boxed{14.\ dar}$



$$V = \left\{(R,K,t) : R<K<t,\ R<K<t,\ R \in (80,100)\right\}$$

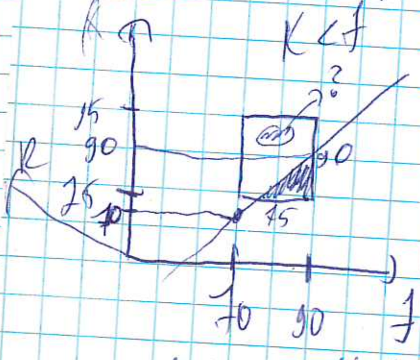
~~$\Omega = \{ \dots \}$~~ $K: 75-95$

$J: 70-90\}$

$$\Omega = \{ (R, K, J) : J \in [70-90],\ K \in [75-95],\ R \in [80-100] \}$$

$$P(R < K < J) = \frac{Vol_3\, V}{Vol_3\, \Omega}$$

Milyen alakú geometriai alakzat a V ??? Térmérete



Feltételes valószínűség:

~~#~~ Tfh. $J = \epsilon_1$ ϵ_1, ϵ_2 adott

$R = \epsilon_2$ e mellett a feltétel mellett mi a valószínűsége, hogy ~~...~~

$J = 85$ $K = 84$

$R \in (80, 84)$ $K = 84,3$ $\epsilon_1 = J > K = \epsilon_2 > R$

$$P = \frac{4}{20} \qquad \frac{4,3}{20}$$

$J = \epsilon_1 = 85$ $R \in [80, 100]$

$K = \epsilon_2 = 80$ $R < 80$

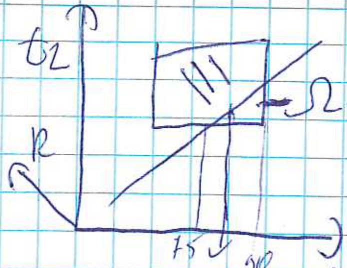
$P = 0$

Ált esetnél.

$$P = \frac{[\epsilon_2 - 80]}{20} \to \text{intervallum hossza}$$

$R \in [80, \epsilon_2]$

$20 \to [80, 100]$-ból adódik.



$\frac{(t_2-80)_+}{20}\cdot 20$

$\frac{1}{20^3}\int\limits_{(t_1,t_2)\in\Omega_2}[t_2-80]_+\,dt_1\,dt_2=\int_{t_1=75}^{90}\int_{t_2=75}^{t_1}[t_2-80]_+\,dt_2$

$Vol_3\,\Omega=20\cdot 20\cdot 20$

$(t_2-80)_+=\max(0,t_2-80)$

$\frac{1}{2}\left[(t_2-80)+|(t_2-80)|\right]$

$>75 \longrightarrow t_1$

$$\int_{t_2=75}^{t_1}[t_2-80]_+\,dt_2=\begin{cases}\int_{80}^{t_1}(t_2-80)\,dt_2=\frac{1}{2}(t_1-80)^2 & t_1\in[80,90]\\ 0 \text{ ha } t_1\le 80\end{cases}$$

$$=\int_{t_1=75}^{90}\begin{bmatrix}0 \text{ ha } t_1\le 80\\ \frac{1}{2}(t_1-80)^2 \quad t_1\ge 80\end{bmatrix}dt_1\cdot\frac{1}{8000}=$$

$$\int_{t_1=80}^{90}\frac{1}{2}(t_1-80)^2\,dt_1\cdot\frac{1}{8000}=\frac{1}{6}(t_1-80)^3\Big|_{80}^{90}\cdot\frac{1}{8000}=$$

$\frac{1}{3}(\;)^3$

$$\frac{1}{6}\cdot\frac{1000}{8000}=\frac{1}{48}$$

p:=(1/20^3)*int(int(max(t2-80,0), t2=75..t1), t1=75..90);

$1/2$ $\quad$ $1/2$ $\qquad$ $\frac{1-\alpha}{2}$ $\quad$ $\frac{1-\alpha}{2}$

Medián $\qquad$ $p=\alpha$

P(x < medián) = P(x > medián)

$\frac{1}{2}$ $\quad$ $\frac{1}{2}$

medián akár lehet => akkor pont a közepe

$x_1, x_2 \ldots x_n$ minta M szám adott

kérdés valószínű-e M x mediánnak $x_1 \ldots x_n \neq$ ~~M~~

Ha M valószínű mediánnak.

akkor $p(x_1 > M) = \frac{1}{2}$ $p(x_n > M) = \frac{1}{2}$

$p(x_2 > M) = \frac{1}{2}$

n_1 $n - n_1$

M

$p(n_1$ db $x_i < M) = ?$

$p(n_1$ db $x_i < M, n - 1$ db $> M)$

~~$n > 4$~~

M

$n = 4$

$\frac{1}{2}$ valószínű

$x_1 \mid x_2 x_3$ $\frac{1}{8}$

$x_2 \mid x_1 x_3$ $\frac{1}{8}$

$x_3 \mid x_1 x_2$ $\frac{1}{8}$

1 | 3

M

16 lehetőség

~~$\frac{3}{4}$~~ 30

$p(x_1 < M) = \frac{1}{2}$

$p(x_2 > M) = \frac{1}{2}$

$p(x_3 > M) = \frac{1}{2}$

az, hogy hogy épp ezek $\frac{1}{8}$

~~113~~ /450

3 eset

$p(1$ db $x_i < M$ és 2 db $x_i > M)) = \frac{3}{8}$

Elfogadható? 95%

$\frac{3}{8} = 0{,}375$ nem elfogadható, ha $P < 0{,}05$

4. feladat.

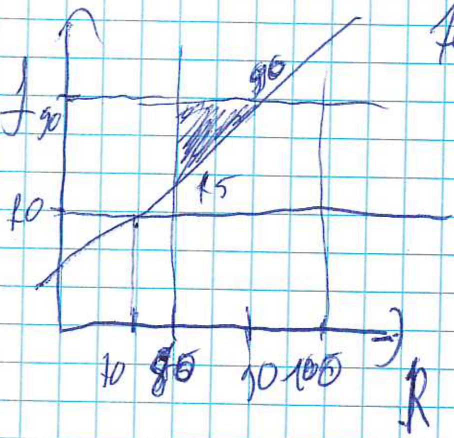
határok adottak

$K = 75 - 95$

$J = 70 - 90$

felbyzálni

$R = 80 - 100$



tornard

K, J, R

$K \in [\cancel{75-95}]$ → 20 a diff.

$75 - 95$

$R = t_1 > J = t_2 > K$

$t_1 = 90$

$t_2 = 85$

$$\frac{85-75}{20} = \frac{1}{2}$$

$$\frac{[t_2 - 75]_+}{20}$$

$$\int_{t_1=75}^{90} \int_{t_2=75}^{t_1} [t_2 - 75]_+ \, dt_2 \, dt_1$$

$t_1 > 75$

$$\int_{t_2=75}^{t_1} \left[\begin{array}{l} 0 \text{ ha } t_2 \le 75 \\ \int_{t_2=75}^{t_1} (t_2 - 75) \, dt_2 \end{array} \right. = \begin{array}{l} 0 \\ \frac{1}{2}[t_1 - 75]^2 \cdot \frac{1}{20^3} \end{array}$$

$$\int_{t_1=75}^{90} \frac{1}{2}(t_1 - 75)^2 = \frac{1}{20^3} \cdot \frac{1}{6} (t_1 - 75)^3 \Big|_{75}^{90} = \frac{1}{20^3} \cdot \frac{1}{6} \cdot (15)^3$$